

RELATIVE METRIC SPACES

M. Malziri* and M. R. Molaei*[†]

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Abstract

In this paper the notion of a *relative metric space*, as a mathematical model compatible with a physical phenomena, is considered. The notion of relative topological entropy for relative semi-dynamical systems on a relative metric space is studied. It is proved that *observational topological entropy* is an invariant object up to a relative conjugate relation.

Keywords: Observer, Relative structure, Molecular lattice, Relative topology, Observational topological entropy.

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1. Introduction

The theory of fuzzy systems [16] was the reason for considering new theories of uncertainty [2, 3]. The recent mathematical results of fuzzy theory [16] in topology [1, 4, 10, 11, 12, 13, 14], and geometry [7] created a new approach to considering *space*. An *Observer*, as one of the main objects which determines the uncertainty of a space X , can be considered as a fuzzy set $\mu : X \rightarrow [0, 1]$. Any mathematical model according to the viewpoint of an observer μ is called a *relative model* [4, 5, 6, 8].

There is a space description using fuzzy theory which is called *fuzzy metric spaces* [1, 10]. In this paper relative metric spaces are introduced as another approach for *considering space* by using an observer. A method for constructing relative topologies via a relative metric space is presented. The notion of relative entropy for relative semi-dynamical systems created by a relative continuous map on a relative metric space is considered. The relative entropy for the iteration of a relative continuous map is studied.

*Department of Mathematics, University of Kerman (Shahid Bahonar), 76169-14111, Kerman, Iran. E-mail: (M. R. Molaei) mrmolaei@mail.uk.ac.ir

[†]Corresponding Author.

2. Metric spaces from an observer viewpoint

To introduce the notion of relative metric spaces we use of the following definition of a continuous t -norm, which is a special case of the usual continuous t -norm [9]. In fact a binary operation

$$* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$$

is called a continuous t -norm if $*$ satisfies the following conditions;

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0, 1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
5. If $a * b = a * c$, and $a, b, c \in (0, 1]$ then $b = c$.

Properties 4 and 5 of a continuous t -norm imply that if $a * b \leq a * c$ and $a, b, c \in (0, 1]$, then $b \leq c$.

2.1. Definition. Let X be a nonempty set and $\mu : X \rightarrow (0, 1]$ a non-vanishing observer. Moreover let $*$ be a t -norm and $M : X^2 \times (0, \infty) \rightarrow (0, 1]$ a function. Then $(X, M, *, \mu)$ is called a *relative metric space* if the following axioms hold:

- (1) $M(x, y, t) = \mu(y)$ if and only if $\mu(x) = \mu(y)$.
- (2) $M(x, y, t) * M(y, z, s) \leq \mu(y) * M(x, z, t + s)$.
- (3) $\mu(x) * M(x, y, t) = \mu(y) * M(y, x, t)$.
- (4) For given $x, y \in X$, $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is a continuous map.

In the definition of relative metric space we try to present a mathematical model for detecting a distance in a space from the viewpoint of an observer. As we know, different observers determine different distances when they are considering the distance between two points using their eyes. The observer is the main object in physical theories, but before the paper [6] there was no mathematical model for it.

Part (1) of Definition 2.1 implies that the observer μ can see two points x and y as similar if the distance detected is exactly equal to the seeing power of μ at one of their points. In other words, μ cannot distinguish two points which are closer than its seeing power.

The condition (2) of Definition 2.1 means that μ can detect the triangle inequality up to its power of seeing. Condition (3) means that the detected distance between (x, y) when the observer looks to x is equal to the detected distance between (y, x) when it looks to y . One must pay attention to the point that $M(x, y, t)$ determines the detected distance between x and y in the level t .

2.2. Example. Let $X = R$, $a * b = ab$, and $M(x, y, t) = \mu(y) * [2^{\frac{|\mu(y) - \mu(x)|}{t}}]^{-1}$. Then we show that $(X, M, *, \mu)$ is a relative metric space.

1. If $M(x, y, t) = \mu(y)$ then $\mu(y) = \mu(y) * [2^{\frac{|\mu(y) - \mu(x)|}{t}}]^{-1}$. This implies

$$[2^{\frac{|\mu(y) - \mu(x)|}{t}}]^{-1} = 1.$$

Thus $\mu(x) = \mu(y)$. Conversely if $\mu(x) = \mu(y)$, then $[2^{\frac{|\mu(y) - \mu(x)|}{t}}]^{-1} = 1$. So $M(x, y, t) = \mu(y)$.

2. For all $x, y \in X$ and $t, s \in (0, \infty)$ we have

$$\frac{|\mu(x) - \mu(z)|}{t + s} \leq \frac{|\mu(x) - \mu(y)|}{t} + \frac{|\mu(y) - \mu(z)|}{s}.$$

So

$$\left[2^{\frac{|\mu(x)-\mu(z)|}{t+s}}\right]^{-1} \geq \left[2^{\frac{|\mu(x)-\mu(y)|}{t}}\right]^{-1} \left[2^{\frac{|\mu(y)-\mu(z)|}{s}}\right]^{-1}.$$

Thus

$$\left[2^{\frac{|\mu(x)-\mu(z)|}{t+s}}\right]^{-1} \geq \left[2^{\frac{|\mu(x)-\mu(y)|}{t}}\right]^{-1} \left[2^{\frac{|\mu(y)-\mu(z)|}{s}}\right]^{-1}.$$

Since

$$\mu(y) * \mu(z) * \left[2^{\frac{|\mu(x)-\mu(z)|}{t+s}}\right]^{-1} \geq \mu(y) * \mu(z) * \left[2^{\frac{|\mu(x)-\mu(y)|}{t}}\right]^{-1} \left[2^{\frac{|\mu(y)-\mu(z)|}{s}}\right]^{-1},$$

then $\mu(y) * M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$.

The other two properties can be easily deduced.

2.3. Example. If X is a nonempty set and $M(x, y, t) = \mu(y) * \frac{t}{t+|\mu(y)-\mu(x)|}$, with $a * b = ab$, then $(X, M, \mu, *)$ is a relative metric space. We only prove the condition 2.

Since

$$\frac{|\mu(x) - \mu(z)|}{t + s} \leq \frac{|\mu(x) - \mu(y)|}{t} + \frac{|\mu(y) - \mu(z)|}{s},$$

then

$$1 + \frac{|\mu(x) - \mu(z)|}{t + s} \leq 1 + \frac{|\mu(x) - \mu(y)|}{t} + \frac{|\mu(y) - \mu(z)|}{s}.$$

Thus

$$\frac{t + s}{t + s + |\mu(x) - \mu(z)|} \geq \frac{ts}{ts + s|\mu(x) - \mu(y)| + t|\mu(y) - \mu(z)|}.$$

Hence

$$\begin{aligned} & \frac{ts}{(t + |\mu(x) - \mu(y)|)(s + |\mu(y) - \mu(z)|)} \\ & \leq \frac{ts}{ts + s|\mu(x) - \mu(y)| + t|\mu(y) - \mu(z)|} \\ & \leq \frac{t + s}{t + s + |\mu(x) - \mu(z)|}. \end{aligned}$$

So

$$\left(\frac{t}{t + |\mu(x) - \mu(y)|}\right) \left(\frac{s}{s + |\mu(y) - \mu(z)|}\right) \leq \frac{t + s}{t + s + |\mu(x) - \mu(z)|}.$$

Thus

$$\begin{aligned} & \left(\frac{t}{t + |\mu(x) - \mu(y)|}\right) \left(\frac{s}{s + |\mu(y) - \mu(z)|}\right) \mu(y) \mu(z) \\ & \leq \frac{t + s}{t + s + |\mu(x) - \mu(z)|} \mu(y) \mu(z). \end{aligned}$$

So $M(x, y, t) * M(y, z, s) \leq \mu(y) * M(x, z, t+s)$.

The next theorem implies that: in a relative metric space $(X, M, *, \mu)$ the relative metric M is an observable object according to the viewpoint of the observer μ .

2.4. Theorem. If $(X, M, *, \mu)$ is a relative metric space, then $M(x, y, t) \leq \mu(y)$ for each $x, y \in X$ and $t \in (0, \infty)$.

Proof. Definition 2.1 implies $M(x, y, t) * M(y, x, t) \leq \mu(y) * M(x, x, 2t)$. Since $M(x, x, 2t) = \mu(x)$ then $M(x, y, t) * M(y, x, t) \leq \mu(y) * \mu(x)$. So

$$M(x, y, t) * \mu(y) * M(y, x, t) \leq \mu(y) * \mu(y) * \mu(x).$$

Since $\mu(x) * M(x, y, t) = \mu(y) * M(y, x, t)$, then

$$M(x, y, t) * \mu(x) * M(x, y, t) \leq \mu(y) * \mu(y) * \mu(x).$$

So $M(x, y, t) * M(x, y, t) \leq \mu(y) * \mu(y)$. Thus $M(x, y, t) \leq \mu(y)$. \square

3. Relative topologies created by a relative metric space and relative entropy

Relative topological spaces are a special case of topological molecular lattices [15] which are compatible with physical models [5]. Let us to recall the definition of a relative topology [6]. If X is a nonempty set and $\mu : X \rightarrow (0, 1]$ is an observer of X , then a family τ_μ of fuzzy subsets of μ is called a μ -relative topology if it satisfies the following conditions:

- i. $\mu, \chi_\emptyset \in \tau_\mu$;
- ii. If $\lambda_1, \lambda_2 \in \tau_\mu$ then $\lambda_1 \cap \lambda_2 \in \tau_\mu$;
- iii. If $\{\lambda_i : i \in \Gamma\} \subseteq \tau_\mu$ then $\bigcup_{i \in \Gamma} \lambda_i \in \tau_\mu$.

We now would like to construct some relative topologies via a relative metric space $(X, M, *, \mu)$. For given $x_0 \in X$ we can define $\lambda_t^{x_0} : X \rightarrow (0, 1]$ by $\lambda_t^{x_0}(y) = M(x_0, y, t)$. Theorem 2.4 implies $\lambda_t^{x_0} \leq \mu$, for all $t \in (0, \infty)$.

3.1. Theorem. *If $t < s$ then $\lambda_t^{x_0}(y) \leq \lambda_s^{x_0}(y)$, for all $y \in X$.*

Proof. Since $M(x_0, y, t) * M(y, y, s - t) \leq \mu(y) * M(x_0, y, s)$, then $M(x_0, y, t) * \mu(y) \leq \mu(y) * M(x_0, y, s)$. So $M(x_0, y, t) \leq M(x_0, y, s)$. Thus, $\lambda_t^{x_0}(y) \leq \lambda_s^{x_0}(y)$. \square

Let $D \subset (0, \infty)$ be a set such that if $C \subset D$ then $\sup C \in D$. Then

$$\tau_\mu^{x_0} = \{\lambda_t^{x_0} : t \in D\} \cup \{\mu, \chi_\emptyset\}$$

is a μ -relative topology.

If $\alpha \in (0, 1]$, and $\lambda \in \tau_\mu^{x_0}$, then λ_α is the set $\{x \in X : \lambda(x) > \alpha\}$. With this notation the set

$$(\tau_\mu^{x_0})_\alpha = \{\lambda_\alpha : \lambda \in \tau_\mu^{x_0}\}$$

is a topology for the set μ_α . If this space is a compact Hausdorff space then $(X, \tau_\mu^{x_0})$ is called a *compact (α, μ) -Hausdorff space*. In the rest of this section we assume that α is fixed and that $(X, \tau_\mu^{x_0})$ is a compact (α, μ) -Hausdorff space. Moreover we assume that $\Theta = \{\lambda_\alpha^i : \lambda^i \in \tau_\mu^{x_0}, i = 1, \dots, n\}$ is an open cover for μ_α . Then an open cover Σ is called a subcover of Θ if $\Sigma \subset \Theta$.

The relative topological entropy [8] of the cover Θ with level α is $H_\alpha(\Theta) = \log N(\Theta)$, where $N(\Theta)$ is the smallest number of sets which can be used in a subcover of Θ .

Now let $f : X \rightarrow X$ be a mapping and $(f, X, \tau_\mu^{x_0})$ a relative semi-dynamical system [6] i.e. $f^{-1}(\lambda) \cap \mu \in \tau_\mu$ for all $\lambda \in \tau_\mu$, where $f^{-1}(\lambda)(x) = \lambda(f(x))$.

The following example implies μ can also create a relative semi-dynamical system.

3.2. Example. Let $X = [0, 1]$ and let $\mu : X \rightarrow [0, 1]$ be defined by

$$\mu(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

If $M(x, y, t) = \mu(y) \frac{t}{t + |\mu(x_0) - \mu(y)|}$ and $x_0 \in X$ then

$$\begin{aligned} \mu^{-1}(\lambda_t^{x_0}(y)) &= \lambda_t^{x_0}(\mu(y)) \\ &= \mu(\mu(y)) \frac{t}{t + |\mu(x_0) - \mu(\mu(y))|} \\ &= \mu(y) \frac{t}{t + |\mu(x_0) - \mu(y)|} \\ &= \lambda_t^{x_0}(y). \end{aligned}$$

So $(\mu, X, \tau_\mu^{x_0})$ is a relative semi-dynamical system.

If Θ_1 and Θ_2 are two covers of μ_α , then their join $\Theta_1 \vee \Theta_2$ is the open cover by all sets of the form $\theta_1 \cap \theta_2$, where $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$.

The relative topological entropy for $(f, X, \tau_\mu^{x_0})$ with level α is defined by

$$\begin{aligned} h_\alpha^{x_0}(f) &= \sup\{h_\alpha(f, \Theta) : \Theta \text{ is a finite cover of } \mu_\alpha\}, \text{ where} \\ h_\alpha(f, \Theta) &= \lim_{n \rightarrow \infty} \frac{1}{n} H_\alpha \left(\bigvee_{i=0}^{n-1} f^{-i} \Theta \right). \end{aligned}$$

The next theorem implies that the nonconstant observers have the main role in $h_\alpha^{x_0}(f)$.

3.3. Theorem. *If μ is a constant map then $h_\alpha^{x_0}(f) = 0$, for all $x_0 \in X$ and $\alpha \in (0, 1]$.*

Proof. Since μ is a constant map then for each $x_0 \in X$ and $t > 0$, $\lambda_t^{x_0} = \mu$. Hence $\tau_\mu^{x_0} = \{\mu, \chi_\emptyset\}$. Thus $\{\mu_\alpha\}$ is the only open cover for $(\tau_\mu^{x_0})_\alpha$. So $h_\alpha^{x_0}(f) = 0$. \square

3.4. Theorem. *If $f : X \rightarrow X$ is a $\tau_\mu^{x_0}$ continuous map then $h_\alpha^{x_0}(f^m) \leq mh_\alpha^{x_0}(f)$, for all $m \in \mathbb{N}$.*

Proof. For a given finite open cover Δ for $(\tau_\mu^{x_0})_\alpha$ we have $\bigvee_{i=0}^{n-1} f^{-mi} \Delta \subset \bigvee_{i=0}^{mn-1} f^{-i} \Delta$. So,

$$H_\alpha \left(\bigvee_{i=0}^{n-1} f^{-mi} \Delta \right) \leq H_\alpha \left(\bigvee_{i=0}^{mn-1} f^{-i} \Delta \right).$$

Thus

$$\begin{aligned} h_\alpha(f^m, \Delta) &= \lim_{n \rightarrow \infty} \frac{1}{n} H_\alpha \left(\bigvee_{i=0}^{n-1} f^{-mi} \Delta \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{m}{mn} H_\alpha \left(\bigvee_{i=0}^{mn-1} f^{-i} \Delta \right) \\ &\leq mh_\alpha(f, \Delta). \end{aligned}$$

So $h_\alpha(f^m, \Delta) \leq mh_\alpha(f, \Delta)$ for all finite open covers Δ . Thus $h_\alpha^{x_0}(f^m) \leq mh_\alpha^{x_0}(f)$. \square

Now we define the *observational topological entropy* of f up to the observer μ with level α by:

$$h_\alpha(f) = \sup_{x_0 \in X} h_\mu^{x_0}(f).$$

3.5. Corollary. *$h_\mu(f^m) \leq mh_\mu(f)$ for all $m \in \mathbb{N}$.* \square

Two relative semi-dynamical systems $(f, X, \tau_\mu^{x_0})$ and $(g, X, \tau_\mu^{x_0})$ are called μ -conjugate at $x_0 \in X$ if there exists a bijection $\phi : X \rightarrow X$ such that $(\phi, X, \tau_\mu^{x_0})$ and $(\phi^{-1}, X, \tau_\mu^{x_0})$ are relative semi-dynamical systems and $\phi \circ f = g \circ \phi$.

3.6. Theorem. *If $f : X \rightarrow X$ and $g : X \rightarrow X$ are μ -conjugate at each $x_0 \in X$ then $h_\alpha(f) = h_\alpha(g)$.*

Proof. [8, Theorem 5.5] implies $h_\alpha^{x_0}(f) = h_\alpha^{x_0}(g)$ for all $x_0 \in X$. Thus

$$h_\alpha(f) = \sup_{x_0 \in X} h_\alpha^{x_0}(f) = \sup_{x_0 \in X} h_\alpha^{x_0}(g) = h_\alpha(g) \quad \square$$

4. Conclusion

In this section we assume that: X is a relative metric space with an observer μ , and that $f : X \rightarrow X$ is a mapping. We also assume that α is a fixed number in $(0, 1]$.

We say that f is a *minimal map on X with level α* if there exists $x_0 \in X$ such that $(f, X, \tau_\mu^{x_0})$ is a relative semi-dynamical system with the following properties:

- i) $f(\mu_\alpha) \subset \mu_\alpha$.
- ii) $\{f^n(x) : n = 0, 1, 2, \dots\}$ is a dense subset of μ_α for all $x \in \mu_\alpha$.

For example let X be an arbitrary set and $x_0 \in X$. Moreover let $\mu : X \rightarrow X$ be a map such that $\sup\{\mu(x) : x \in X\} = \mu(x_0)$ and let $M(x, y, t)$ be an arbitrary relative metric. For all $t \in D$, we have $M(x_0, x, t) \leq \mu(x)$. Since $M(x_0, x_0, t) = \mu(x_0)$, then $\sup_{x \in X} \lambda_t^{x_0}(x) = \mu(x_0)$. So for all $0 < \alpha < 1$, $x_0 \in \mu_\alpha$.

Let $f : X \rightarrow X$ be a function such that $f(x) = x_0$. Then $f(\mu_\alpha) \subset \mu_\alpha$. Since for every $n \geq 1$, $f^n(x) = x_0$ then $\lim_{n \rightarrow \infty} f^n(x) = x_0$. Thus for each $x \in X$ we have

$$\sup\{\lambda_t^{x_0}(f^n(x)) : n \geq 1\} = \mu(x_0).$$

So, $\{f^n(x) : n = 0, 1, 2, \dots\}$ is a dense subset of μ_α , for all $x \in \mu_\alpha$. Moreover,

$$f^{-1}(\lambda_t^{x_0})(x) = \lambda_t^{x_0}(f(x)) = \lambda_t^{x_0}(x_0) = \mu(x_0).$$

So $f^{-1}(\lambda_t^{x_0}) \cap \mu = \mu$. Thus f is a minimal map on X with level α .

The consideration of minimal mappings on relative metric spaces is a topic for further research.

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