Continuous Dependence of Solutions to Fourth-Order Nonlinear Wave Equation

İpek Güleç*, Şevket Gür**

* Department of Mathematics, Hacettepe University, Ankara, Turkey
** Department of Mathematics, Sakarya University, Sakarya, Turkey

**Tel Number +90 2642955979 Fax Number: +90 264 2955950, Email: sgur@sakarya.edu.tr

Abstract

This paper gives a priori estimates and continuous dependence of the solutions to fourth-order nonlinear wave equation.

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1 Introduction

We consider the following initial boundary value problem

\[ u_{tt} - \alpha \Delta u - \beta \Delta u_t - \gamma \Delta u_{tt} = f(u) \quad (1) \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega \quad (2) \]

\[ u = 0, \quad x \in \partial \Omega, \quad t > 0, \quad (3) \]
where $\Omega \subset \mathbb{R}^n$ is bounded region with smooth boundary $\partial \Omega$; $\alpha, \beta$ and $\gamma$ are positive constants. $f(u)$ is a given nonlinear function which satisfies

$$f \in C^1(R), \ |f'(u)| \leq c \left(1 + |u|^{p-1}\right), p \geq 1, (n-2)p \leq n$$

and

$$\limsup_{u \to \infty} \frac{f(u)}{u} < \alpha \lambda_1$$

where $\lambda_1$ is the first eigenvalue of the Laplace operator with the homogeneous Dirichlet boundary condition.

Continuous dependence of solutions on coefficients of equations is a type of structural stability, which reflects the effect of small changes in coefficients of equations on the solutions. Many results of this type can be found in [1].

In [2], authors studied asymptotic behaviour of solution to initial value problem of fourth order wave equation with dispersive and dissipative terms by taking coefficients $\alpha = \beta = \gamma = 1$ in (1). They proved that the global strong solution of the problem decays to zero exponentially as $t \to \infty$. The authors Guo-wang Chen and Chang-Shun Hou, in article [3], studied the following initial value problem for a class of fourth order nonlinear wave equations,

$$v_{tt} - a_1 v_{xx} - a_2 v_{xxt} - a_3 v_{xxx} = f(v_x)_x, x \in R, t > 0$$

$$v(x, 0) = v_0(x), \ v_t(x, 0) = v_1(x), x \in R$$

where $a_1, a_2, a_3$ are positive constants. They gave also the blow up results for this problem.

In [4], Shang studied the initial boundary value problem

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), x \in \Omega, t > 0 \quad (1')$$

$$u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), x \in \Omega \quad (2')$$

$$u = 0, x \in \partial \Omega, t > 0, \quad (3')$$

Under the assumptions that $n = 1, 2, 3; f \in C^1, f'(u)$ is bounded above and satisfies

(i) $|f'(u)| \leq A|u|^p + B, 0 < p < \infty$ if $n = 2; 0 < p \leq \frac{2}{n-2}$ if $n = 3; u_i(x) \in$
The aim of this paper is to prove the continuous dependence of solutions to the problem (1)-(3) on coefficients $\alpha$, $\beta$ and $\gamma$.

Throughout this paper, we use the notation $\|\cdot\|_p$ for the norm in $L^p(\Omega)$. We use $\|\cdot\|$ instead of $\|\cdot\|_2$.

2 A Priori Estimates

In this section, we obtain a priori estimates for the problem (1)-(3).

Theorem 1 Assume that the conditions (4) and (5) hold. Then for $u_0, u_1 \in H^1_0(\Omega)$ the solution $u$ of problem (1)-(3) satisfies the following estimates:

$$\|\nabla u\|^2 + \|\nabla u_t\|^2 \leq D_1$$

(6)

and

$$\int_0^t \|\nabla u_{ss}\|^2 \, ds \leq D_2 t$$

(7)

for any $t > 0$. Here $D_1 > 0$ and $D_2 > 0$ depend on initial data and the parameters of (1).
Proof. First, by taking the inner product of (1) by $u_t$ in $L^2(\Omega)$ and integrating by parts, we get
\[ \frac{d}{dt} \left[ \frac{1}{2} ||u_t||^2 + \frac{\alpha}{2} ||\nabla u||^2 + \frac{\gamma}{2} ||\nabla u_t||^2 - \int_{\Omega} F(u) \, dx \right] + \beta ||\nabla u||^2 = 0 \quad (8) \]
and
\[ E(t) \leq E(0) \quad (9) \]
where $F(u) = \int_{0}^{u} f(s) \, ds$ and $E(t) = \frac{1}{2} ||u_t||^2 + \frac{\alpha}{2} ||\nabla u||^2 + \frac{\gamma}{2} ||\nabla u_t||^2 - \int_{\Omega} F(u) \, dx$.

From (5) and definition of limsup we obtain
\[ F(u) \leq c + \frac{\alpha \lambda_1}{2} u^2 - \frac{\varepsilon}{2} u^2 \quad (10) \]
Using (10) and Poincare’s inequality from (9) we find (6).

Next we multiply (1) by $u_{tt}$ in $L^2(\Omega)$ to get
\[ \frac{d}{dt} \frac{\beta}{2} ||\nabla u_t||^2 + \gamma ||\nabla u_{tt}||^2 + ||u_{tt}||^2 + \alpha \int_{\Omega} \nabla u \nabla u_{tt} \, dx = \int_{\Omega} f(u) u_{tt} \, dx \quad (11) \]
Using Cauchy-Schwarz inequality, $\varepsilon$-Cauchy inequality and from (4), we take,
\[ (\gamma - \frac{\varepsilon}{2}) ||\nabla u_{tt}||^2 + \frac{d}{dt} \frac{\beta}{2} ||\nabla u_t||^2 \leq c_2 + \frac{|\alpha|^2}{2 \varepsilon} ||\nabla u||^2 + \frac{c_2}{2} \int_{\Omega} |u|^{2p} \, dx \quad (12) \]
where $c_1, c_2$ are constants and $\varepsilon$ is sufficiently small and positive. Using Sobolev inequality and (6) we have
\[ \int_{\Omega} |u|^{2p} \, dx = ||u||_{2p}^{2p} \leq c_3 ||\nabla u||^{2p} \leq c_4 \quad (13) \]
where $c_3$ is a Sobolev constant and $c_4 = c_4(\alpha, \gamma, u_0, u_1)$. From (12) and (13) we obtain
\[ (\gamma - \frac{\varepsilon}{2}) ||\nabla u_{tt}||^2 + \frac{d}{dt} \frac{\beta}{2} ||\nabla u_t||^2 \leq c_5 \quad (14) \]
where $c_5$ depends on initial data and the parameters of (1). Now, we integrate (14) from (0,t), then we obtain
\[ \int_{0}^{t} ||\nabla u_{ss}||^2 \, ds \leq c_{6} t \quad (15) \]
where \( c_6 \) depends on initial data and the parameters of (1). Hence, (7) follows from (15).

## 3 Continuous Dependence on the Coefficients

In this section, we prove that the solution of the problem (1)-(3) depends continuously on the coefficients \( \alpha, \beta \) and \( \gamma \) in \( H^1(\Omega) \).

We consider the problem

\[
\begin{align*}
  u_{tt} - \alpha_1 \Delta u - \beta_1 \Delta u_t - \gamma_1 \Delta u_{tt} &= f(u) \\
  u(x,0) &= 0, u_t(x,0) = 0 \\
  u|_{\partial \Omega} &= 0
\end{align*}
\]

(16)

(17)

(18)

and

\[
\begin{align*}
  v_{tt} - \alpha_2 \Delta v - \beta_2 \Delta v_t - \gamma_2 \Delta v_{tt} &= f(v) \\
  v(x,0) &= 0, v_t(x,0) = 0 \\
  v|_{\partial \Omega} &= 0
\end{align*}
\]

(19)

(20)

(21)

Let us define the difference variables \( w, \alpha, \beta \) and \( \gamma \) by \( w = u - v \), \( \alpha = \alpha_1 - \alpha_2 \), \( \beta = \beta_1 - \beta_2 \) and \( \gamma = \gamma_1 - \gamma_2 \) then \( w \) satisfy the following the initial boundary value problem:

\[
\begin{align*}
  w_{tt} - \alpha_1 \Delta w - \alpha \Delta v - \beta_1 \Delta w_t - \beta \Delta v_t - \gamma_1 \Delta w_{tt} - \gamma \Delta v_{tt} &= f(u) - f(v) \\
  w(x,0) &= 0, w_t(x,0) = 0 \\
  w|_{\partial \Omega} &= 0
\end{align*}
\]

(22)

(23)

(24)

The main result of this section is the following theorem.

**Theorem 2** Let \( w \) be the solution of the problem (22)-(24). If

\[
|f(u) - f(v)| \leq c_7 \left( 1 + |u|^{p-1} + |v|^{p-1} \right) |u - v|
\]

(25)
holds, then $w$ satisfies the estimate

$$\|w_t\|^2 + \|\nabla w\|^2 + \|\nabla w_t\|^2 \leq e^{Mt}K \left[ (\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 \right] t$$

where $M$ and $K$ are positive constants depending on initial data and the parameters of (1).

**Proof.** Let us take the inner product of (22) with $w_t$ in $L^2(\Omega)$; we have

$$\frac{d}{dt} \left[ \frac{1}{2} \|w_t\|^2 + \frac{\alpha_1}{2} \|\nabla w\|^2 + \frac{\gamma_1}{2} \|\nabla w_t\|^2 \right] + \beta_1 \|\nabla w_t\|^2 +$$

$$\alpha \int_\Omega \nabla v \nabla w_t dx + \beta \int_\Omega \nabla v_t \nabla w_t dx + \gamma \int_\Omega \nabla v_{tt} \nabla w_t dx = \int_\Omega |f(u) - f(v)| w_t dx \quad (26)$$

From (26) we obtain

$$\frac{d}{dt} E_1(t) + \beta_1 \|\nabla w_t\|^2 \leq |\alpha| \|\nabla w_t\| \|\nabla v\| + |\beta| \|\nabla w_t\| \|\nabla v_t\| +$$

$$|\gamma| \|\nabla w_t\| \|\nabla v_{tt}\| + \int_\Omega |f(u) - f(v)| w_t dx \quad (27)$$

where $E_1(t) = \frac{1}{2} \|w_t\|^2 + \frac{\alpha_1}{2} \|\nabla w\|^2 + \frac{\gamma_1}{2} \|\nabla w_t\|^2$.

Using the Holder, Sobolev, Cauchy-Schwarz inequalities and (25) we obtain the estimate

$$\int_\Omega |f(u) - f(v)| w_t dx \leq c_7 \int_\Omega \left( 1 + |u|^{p-1} + |v|^{p-1} \right) |w| w_t dx$$

$$\leq c_8 \left( 1 + \|\nabla u\|^{p-1} + \|\nabla v|^{p-1}\| \right) \|w\| \|w_t\|$$

$$\leq C \left( \|\nabla w\|^2 + \|w_t\|^2 \right) \quad (28)$$

where $c_7, c_8$ are constants and $C = C(c_7, c_8)$. Using Cauchy-Schwarz inequality and (28), from (27), we get

$$\frac{d}{dt} E_1(t) + (\beta_1 - \varepsilon) \|\nabla w_t\|^2 \leq \frac{3}{4\varepsilon} |\alpha|^2 \|\nabla v\|^2 + \frac{3}{4\varepsilon} |\beta|^2 \|\nabla v_t\|^2 +$$

$$\frac{3}{4\varepsilon} |\gamma|^2 \|\nabla v_{tt}\|^2 + c_9 \left( \|\nabla w\|^2 + \|w_t\|^2 \right) \quad (29)$$
\[
\frac{d}{dt} E_1(t) \leq \frac{3}{4\varepsilon} \left( |\alpha|^2 \| \nabla v \|^2 + |\beta|^2 \| \nabla v_t \|^2 + |\gamma|^2 \| \nabla v_{tt} \|^2 \right) + ME_1(t) \tag{30}
\]

where \( M = \frac{2C(1+\alpha_1)}{\alpha_1} \). Applying Gronwall’s inequality with (6) and (7), we get

\[
E_1(t) \leq e^{Mt} K \left( |\alpha|^2 + |\beta|^2 + |\gamma|^2 \right) t \tag{31}
\]

Hence proof is completed.

References


[5] Liu Yacheng, Li Xiaoyuan, Some remarks on the equation \( u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u) \), Journal of Natural Science of Heilongjiang University 21(3) (2004) 1-6.