ON DERIVATIONS OF SUBTRACTION ALGEBRAS

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Abstract

The aim of this paper is to introduce the notion of derivations of subtraction algebras. We define a derivation of a subtraction algebra $X$ as a function $d$ on $X$ satisfying $d(x - y) = (d(x) - y) \land (x - d(y))$ for all $x, y \in X$. Then it is characterized as a function $d$ satisfying $d(x - y) = d(x) - y$ for all $x, y \in X$. Also we define a simple derivation as a function $d_a$ on $X$ satisfying $d_a(x) = x - a$ for all $x \in X$. Then every simple derivation is a derivation and every derivation can be partially a simple derivation on intervals. For any derivation $d$ of a subtraction algebra $X$, $\text{Ker}(d)$ and $\text{Im}(d)$ are ideals of $X$, and $X/\text{Ker}(d) \cong \text{Im}(d)$ and $X/\text{Im}(d) \cong \text{Ker}(d)$. Finally, we show that every subtraction algebra $X$ is embedded in $\text{Im}(d) \times \text{Ker}(d)$ for any derivation $d$ of $X$.

Keywords: Subtraction algebra, Derivation, Simple derivation, Non-expansive map, Dual closure operator, Boolean algebra.

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1. Introduction

B. M. Schein [2] considered systems of the form $(\Phi; \circ , \setminus)$, where $\Phi$ is a set of functions closed under the composition “$\circ$” of functions (and hence $(\Phi; \circ)$ is a function semigroup) and set theoretic subtraction “$\setminus$” (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [4] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called atomic subtraction algebras. The notion of derivation of lattices was introduced and studied in [3].

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In this paper, we define a derivation of a subtraction algebra and introduce the notion of derivations. In Section 2, we introduce some basic results of subtraction algebras. In Section 3, we define a derivation as a function $d$ on $X$ satisfying $d(x - y) = (d(x) - y) \land (x - d(y))$ for all $x, y \in X$, and characterize it as a function $d$ satisfying $d(x - y) = d(x) - y$ for all $x, y \in X$. Also we define a simple derivation as a function $d_a$ on $X$ satisfying $d_a(x) = x - a$ for all $x \in X$, and we show that every simple derivation is a derivation and conversely, every derivation is partially a simple derivation on intervals. In Section 4 we show that for any derivation $d$ of a subtraction algebra $X$, $\text{Ker}(d)$ and $\text{Im}(d)$ are ideals of $X$ and $X/\text{Ker}(d) \cong \text{Im}(d)$ and $X/\text{Im}(d) \cong \text{Ker}(d)$. Also the map $\mu : x \mapsto x - d(x)$ is a derivation of $X$, hence the sequence of derivations and subtraction algebras:

$$0 \rightarrow \text{Im}(d) \xrightarrow{i} X \xrightarrow{\mu} \text{Ker}(d) \rightarrow 0$$

is similar to a split exact sequence. Finally, we show that every subtraction algebra $X$ is embedded in $\text{Im}(d) \times \text{Ker}(d)$ for any derivation $d$ of $X$.

2. Subtraction algebras

We first recall some basic concepts which are used to present the paper.

By a subtraction algebra we mean an algebra $(X; -)$ with a single binary operation "-" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$.

The subtraction determines an order relation on $X$: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \land b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

\[ b \lor c = (b' \land c')' = a - ((a - b) \land (a - c)) \]
\[ = a - ((a - b) - ((a - b) - (a - c))). \]

In a subtraction algebra, the following are true:

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $x - y \leq x$.
(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.
(p6) $x - (x - (x - y)) = x - y$.
(p7) $(x - y) - (z - y) \leq x - z$.
(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(p10) $x, y \leq z$ implies $x - y = x \land (z - y)$.
(p11) $(x \land y) - (x \land z) \leq x \land (y - z)$.
(p12) $(x - y) - z = (x - z) - (y - z)$.

Let $X$ and $Y$ be subtraction algebras. A mapping $f$ from $X$ to $Y$ is called a homomorphism if $f(x - y) = f(x) - f(y)$ for all $x, y \in X$. Especially, $f$ is monomorphism (resp. epimorphism) if $f$ is one-to-one (resp. onto) homomorphism, and $f$ is an isomorphism if
f is a monomorphism and epimorphism. In this case, we say X is isomorphic to Y, and denote this by \( X \cong Y \).

A function \( f \) of a semilattice (\( \land \)-semilattice) \( X \) into itself is a dual closure if \( f \) is monotone, non-expansive (i.e., \( f(x) \leq x \) for all \( x \in X \)) and idempotent (i.e., \( f \circ f = f \)).

3. Derivations and simple derivations

3.1. Definition. Let \( X \) be a subtraction algebra. By a derivation of \( X \) we mean a self-map \( d \) of \( X \) satisfying the identity \( d(x - y) = (d(x) - y) \land (x - d(y)) \) for all \( x, y \in X \).

3.2. Example. (1) Let \( X = \{0, a, b, 1\} \) in which “−” is defined by

\[
\begin{array}{cccc}
- & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & b & 0 \\
1 & 1 & b & a \\
\end{array}
\]

It is easy to check that \( (X; -) \) is a subtraction algebra. Define a map \( d : X \rightarrow X \) by

\[
d(x) = \begin{cases} 
0 & \text{if } x = 0, \ a, \\
b & \text{if } x = b, \ 1.
\end{cases}
\]

Then \( d \) is a derivation of the subtraction algebra \( X \).

Figure 1. The Hasse diagram of Example 3.2 (1)

(2) Let \( X = \{0, a, b\} \) be a subtraction algebra with the following Cayley table

\[
\begin{array}{ccc}
- & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & b & 0 \\
\end{array}
\]

Define a map \( d : X \rightarrow X \) by

\[
d(x) = \begin{cases} 
0 & \text{if } x = 0, \ b, \\
b & \text{if } x = a.
\end{cases}
\]

Then it is easily checked that \( d \) is a derivation of subtraction algebra \( X \).

3.3. Example. Let \( X \) be a subtraction algebra. We define a function \( d \) by \( d(x) = 0 \) for all \( x \in X \). Then \( d \) is a derivation on \( X \), which is called the zero derivation.

3.4. Example. Let \( d \) be the identity function on a subtraction algebra \( X \). Then \( d \) is a derivation on \( X \), which is called the identity derivation.

3.5. Proposition. Let \( d \) be a derivation of a subtraction algebra \( X \). Then \( d(0) = 0 \).
Proof. Let $d$ be a derivation of a subtraction algebra of $X$. Then
\[ d(0) = d(0 - x) = (d(0) - x) \land (0 - d(x)) = (d(0) - x) \land 0 = 0. \]
\[ \square \]

3.6. Proposition. Let $d$ be a derivation of a subtraction algebra $X$. Then $d(x - d(x)) = 0$ for every $x \in X$.

Proof. Let $d$ be a derivation of a subtraction algebra of $X$ and let $x \in X$. Then
\[ d(x - d(x)) = (d(x) - d(x)) \land (x - d(d(x))) = 0 \land (x - d(d(x))) = 0. \]
\[ \square \]

3.7. Proposition. Let $d$ be a derivation of a subtraction algebra $X$. Then we have $d(x) = d(x) \land x$.

Proof. Let $d$ be a derivation of $X$. Then
\[ d(x) = d(x - 0) = (d(x) - 0) \land (x - 0) = d(x) \land (x - 0) = d(x) \land x. \]
\[ \square \]

3.8. Corollary. Let $d$ be a derivation of subtraction algebra $X$. Then we have $d(x) \leq x$.

Proof. Then $d(x) \leq x$.

3.9. Theorem. Let $d$ be a derivation of a subtraction algebra $X$. If $x \leq y$ for $x, y \in X$, then $d(x) \leq d(y)$.

Proof. Let $x \leq y$ for $x, y \in X$. Then by (p8), $x = y - w$ for some $w \in X$. Hence we have
\[ d(x) = d(y - w) = (d(y) - w) \land (y - d(w)) \leq d(y) - w \leq d(y). \]
\[ \square \]

3.10. Theorem. Let $d$ be a derivation of a subtraction algebra $X$. Then we have $d^2 = d \circ d = d$.

Proof. Let $d$ be a derivation of $X$. Then by definition of the derivation $d$ and Proposition 3.6, we have
\[ d^2(x) = d(d(x)) = d(x \land d(x)) \]
\[ = d(x - (x - d(x))) \]
\[ = (d(x) - (x - d(x))) \land (x - d(x) - d(x)) \]
\[ = d(x) \land (x - 0) \]
\[ = d(x) \land x \]
\[ = d(x) \]
\[ \square \]

3.11. Corollary. Let $d$ be a derivation of a subtraction algebra $X$. Then $d$ is a dual closure operator on $X$.

Proof. Clear from Corollary 3.8 and Theorems 3.9 and 3.10.
\[ \square \]

3.12. Proposition. Let $f$ be a non-expansive map on a subtraction algebra $X$, i.e., $f(x) \leq x$ for all $x \in X$. Then $f(x) - y \leq x - f(y)$ for all $x, y \in X$.

Proof. Suppose that $f$ is a non-expansive map on $X$ and $x, y \in X$. Then $f(x) \leq x$ and $f(y) \leq y$. Hence $f(x) - y \leq x - y$ and $x - y \leq x - f(y)$ by (p9). It follows that $f(x) - y \leq x - f(y)$.
3.13. **Theorem.** Let $d$ be a map on a subtraction algebra $X$. Then the following are equivalent:

1. $d$ is a derivation of $X$;
2. $d(x - y) = d(x) - y$ for all $x, y \in X$.

**Proof.** Suppose that $d$ is a derivation of $X$. Then $d$ is non-expansive by Corollary 3.8. Hence for any $x, y \in X$, $d(x) - y \leq x - d(y)$ by Proposition 3.12, and

$$d(x - y) = (d(x) - y) \wedge (x - d(y)) = d(x) - y.$$  

Suppose that $d$ is a map satisfying $d(x - y) = d(x) - y$ for all $x, y \in X$. Then $d(0) = d(0 - 0) = d(0) - d(0) = 0$, hence we have

$$0 = d(0) = d(x - x) = d(x) - x$$  

for any $x \in X$. It follows that $d(x) \leq x$ for any $x \in X$. That is, $d$ is non-expansive. Hence by Proposition 3.12, $d(x) - y \leq x - d(y)$ and

$$d(x - y) = d(x) - y = (d(x) - y) \wedge (x - d(y))$$  

for any $x, y \in X$. \qed

3.14. **Theorem.** Let $X$ be a subtraction algebra. The every derivation of $X$ is an homomorphism.

**Proof.** Suppose that $d$ is a derivation of $X$ and $x, y \in X$. Then $d(y) \leq y$. It implies

$$d(x - y) = d(x) - y \leq d(x) - d(y)$$

by (p9). Also we have

$$d(x - y) - (d(x) - y) = (d(x) - d(y)) - (d(x) - y) = (d(x) - (d(x) - y)) - d(y) = d(x) - (d(x) - y) - d(y) \leq d(y) - d(y) - d(y) = 0.$$  

It follows that $(d(x) - d(y)) - (d(x) - y) = 0$ and $d(x) - d(y) \leq d(x) - y = d(x - y)$. Hence $d(x) - d(y) = d(x - y)$. \qed

The converse of Theorem 3.14 is not true in general.

3.15. **Example.** Let $X = \{0, a, b, 1\}$ be the subtraction algebra of Example 3.2(1). Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a, \\ 1 & \text{if } x = b, 1. \end{cases}$$

Then $f$ is an endomorphism of $X$ which is not a derivation because of $f(b - a) = f(b) = 1 \neq b = 1 - a = f(b) - a$.

Let $X$ be a subtraction algebra. Then, for each $a \in X$, we will define a map $d_a : X \to X$ by

$$d_a(x) = x - a$$  

for all $x \in X$. 

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**On Derivations of Subtraction Algebras**

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3.16. Proposition. Let $X$ be a subtraction algebra. Then for each $a \in X$, the map $d_a$ is a derivation of $X$.

Proof. Suppose that $d_a$ is the map defined by $d_a(x) = x - a$ for each $x \in X$. Then for any $x, y \in X$, we have
\[
d_a(x - y) = (x - y) - a = (x - a) - y = d_a(x) - y
\]
by (S3). Hence $d_a$ is a derivation of $X$ by Theorem 3.13. □

We will call the derivation $d_a$ of Proposition 3.16 a simple derivation.

3.17. Proposition. Let $d$ be a derivation of a subtraction algebra $X$. Then for each $x \in X$, there exists a unique $\hat{x} \in [0, x]$ such that $d(x) = x - \hat{x}$ and $d(\hat{x}) = 0$.

Proof. Suppose that $d$ is a derivation of $X$ and $x \in X$. Then $d(x) \leq x$ since $d$ is non-expansive.

Let $\hat{x} = x - d(x)$. Then $\hat{x} \in [0, x]$ and $d(\hat{x}) = 0$ by Proposition 3.6, and we have
\[
x - \hat{x} = x - (x - d(x)) = x \wedge d(x) = d(x).
\]
If $x - \hat{x} = d(x) = x - w'$ for some $w' \in [0, x]$, then
\[
\hat{x} - w' = (x \wedge \hat{x}) - w' = (x - (x - \hat{x})) - w' = (x - w') - (x - \hat{x}) \quad \text{(by (S3))}
\]
\[
= 0.
\]
It follows that $\hat{x} \leq w'$. Similarly, we can show that $w' \leq \hat{x}$. Hence $\hat{x} = w'$, and $\hat{x}$ is the unique element in $[0, x]$ such that $d(x) = x - \hat{x}$. □

3.18. Lemma. Let $d$ be a derivation of a subtraction algebra $X$. Then $\ker(d) = \{\hat{x} \mid x \in X\}$.

Proof. It is clear that $\{\hat{x} \mid x \in X\} \subseteq \ker(d)$ by Theorem 3.17.

If $x \in \ker(d)$, then $x = x - 0 = x - d(x) = \hat{x}$. It implies $\ker(d) \subseteq \{\hat{x} \mid x \in X\}$. □

3.19. Theorem. Let $d$ be a derivation of a subtraction algebra $X$. The for each interval $[0, a]$ in $X$,
\[
d(x) = d_a(x)
\]
for all $x \in [0, a]$, that is, the restriction $d|_{[0, a]} : [0, a] \to X$ of $d$ is a simple derivation $d_a$, where $\hat{a} \in [0, a]$ is the unique element of Theorem 3.17.

Proof. Suppose that $d$ is a derivation of $X$ and $a \in X$. Then by Theorem 3.17 there is a unique $\hat{a} \in [0, a]$ such that $d(a) = a - \hat{a}$, and for any $x \in [0, a]$ we have
\[
d(x) = d(a \wedge x) = d(a - (a - x)) = d(a) - (a - x) = (a - \hat{a}) - (a - x)
\]
\[
= (a - (a - x)) - \hat{a} = (a \wedge x) - \hat{a} = x - \hat{a}.
\]
Hence $d(x) = x - \hat{a} = d_a(x)$ for all $x \in [0, a]$. □

3.20. Corollary. Let $X$ be a subtraction algebra with greatest element 1. Then every derivation $d$ of $X$ is a simple derivation $d_1$. 
Proof. Suppose that 1 ∈ X and d is a derivation of X. Then X = [0, 1] and by Theorem 3.19,
\[ d(x) = x - \hat{1} = d_1(x) \]
for all x ∈ [0, 1] = X. Hence d is the simple derivation d_1.
□

There can be a derivation on a subtraction algebra which is not simple.

3.21. Example. Let X = {0, a, b, c, e, f} be a subtraction algebra with “−” defined by

\[
\begin{array}{cccccc}
- & 0 & a & b & c & e & f \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 & a & a \\
b & b & b & 0 & b & 0 & 0 \\
c & c & c & c & 0 & c & 0 \\
ed & e & b & a & e & 0 & a \\
f & f & f & c & b & c & 0 \\
\end{array}
\]

Figure 2. The Hasse diagram of Example 3.21

Define a map \( d : X \to X \) by
\[ d(x) = \begin{cases} 
0 & \text{if } x = 0, a, c, \\
b & \text{if } x = b, c, f. 
\end{cases} \]

Then d is a derivation of X which is not simple, because there is no x ∈ X satisfying either \( d(e) = b = e - x \) or \( d(f) = b = f - x \). For the interval \( A = [0, e] \) and \( B = [0, f] \), \( \hat{e} = e - d(e) = e - b = a \) and \( \hat{f} = c \). Hence the restrictions \( d|_A \) and \( d|_B \) are simple, being given by
\[ d|_A(x) = x - a = d(x) \ (x \in A) \text{ and } d|_B(x) = x - c = d(x) \ (x \in B), \]
respectively.

4. Derivations and ideals of subtraction algebras

A nonempty subset \( I \) of a subtraction algebra \( X \) is called an ideal of \( X \) if it satisfies
(I1) 0 ∈ I,
(I2) for any \( x, y \in X \), \( y \in I \) and \( x - y \in I \) implies \( x \in I \).

For an ideal \( I \) of a subtraction algebra \( X \), it is clear that \( x \leq y \) and \( y \in I \) imply \( x \in I \) for any \( x, y \in X \).

4.1. Proposition. Let \( d \) be a derivation of a subtraction algebra \( X \). Then \( \text{Kerd} = \{ x \in X \mid d(x) = 0 \} \) is an ideal of \( X \).
Proof. Let $y \in \text{Ker} d$ and $x \in X$ with $x - y \in \text{Ker} d$. Then $d(y) = 0$ implies
\[ d(x) = d(x) - 0 = d(x) - d(y) = d(x - y) = 0. \]
Hence $x \in \text{Ker} d$. \qed

4.2. Proposition. Let $d$ be a derivation of a subtraction algebra $X$. If $\text{Ker} d = \{0\}$, then $d$ is the identity derivation.

Proof. Let $x \in X$. Then $d(x) \leq x$, and $x - d(x) \in \text{Ker} d = \{0\}$ by Proposition 3.6. It implies $x - d(x) = 0$ and $x \leq d(x)$. Hence $d(x) = x$. \qed

Let $X$ be a subtraction algebra and $A$ a non-empty subset of $X$. Then we will write $A^* = \{x \in X \mid x \wedge a = 0 \text{ for all } a \in A\}$.

4.3. Proposition. Let $X$ be a subtraction algebra and $A$ non-empty subset of $X$. Then $A^*$ is an ideal of $X$.

Proof. Let $y \in A^*$ and $x - y \in A^*$ for any $x \in X$. Then $y \wedge a = 0$ and $(x - y) \wedge a = 0$ for all $a \in A$. By (p11), we have
\[ x \wedge a = (x \wedge a) - 0 = (x \wedge a) - (y \wedge a) \leq (x - y) \wedge a = 0 \]
for all $a \in A$. It implies $x \wedge a = 0$ for all $a \in A$, and $x \in A^*$. Hence $A^*$ is an ideal of $X$. \qed

In particular, for any singleton subset $A = \{a\}$ of a subtraction algebra $X$, $\{a\}^* = A^* = \{x \in X \mid x \wedge a = 0\}$ is an ideal of $X$.

4.4. Proposition. Let $X$ be a subtraction algebra and $d_y$ a simple derivation with $y \in X$. Then $d_y(x) = x$ if and only if $x \in \{y\}^*$.

Proof. Suppose that $x, y \in X$ and $d_y(x) = x$. Then $x \wedge y = x - (x - y) = x - d_y(x) = x - x = 0$. Hence $x \in \{y\}^*$.

Conversely, suppose that $x \in \{y\}^*$. Then $y - (y - x) = x - (x - y) = x \wedge y = 0$. Hence we have
\[ d_y(x) = x - y = (x - y) - (y - x) \quad \text{(by (p5))} \]
\[ = (x - (y - x)) - (y - (y - x)) \quad \text{(by (p12))} \]
\[ = x - 0 \quad \text{(by (S1))} \]
\[ = x. \] \qed

4.5. Corollary. Let $X$ be a subtraction algebra and $d_y$ a simple derivation with respect to $y \in X$. Then $d_y(X) = \{y\}^*$, that is, $\text{Im}(d_y)$ is an ideal of $X$.

Proof. Let $x \in d_y(X)$. Then $x = d_y(z)$ for some $z \in X$, and by Theorem 3.10
\[ x = d_y(z) = d_y(d_y(z)) = d_y(x). \]
It implies $x \in \{y\}^*$ by Proposition 4.4. Hence $d_y(X) \subseteq \{y\}^*$. Also it is clear that $\{y\}^* \subseteq d_y(X)$ from Proposition 4.4. \qed

4.6. Proposition. Let $d$ be a derivation of a subtraction algebra $X$. If $I$ is an ideal of $X$, then we have $d(I) \subseteq I$.

Proof. For all $x \in I$, we have $d(x) \leq x$, and $d(x) = x - w$ for some $w \in X$ by (p8). Hence by the definition of an ideal, we have $d(x) \in I$. \qed
4.7. Theorem. Let \( d \) be a derivation of a subtraction algebra \( X \). Then \( d(X) = \text{Im}(d) \) is an ideal of \( X \).

Proof. Let \( y \in d(X) \) and \( x - y \in d(X) \) with \( x \in X \). Then \( d(y) = y \) and \( d(x - y) = x - y \) by Theorem 3.10, there exists \( \hat{x} \in [0, x] \) satisfying \( d(x) = \hat{x} \) and \( d(\hat{x}) = 0 \), and \( d(\hat{z}) = d(z) \) for all \( z \in [0, x] \) by Theorems 3.17 and 3.19. Since \( x - y \leq x \), we have
\[
d(\hat{x}) = d(x - y) = x - y.
\]
It implies \( x - y \in \{\hat{x}\} \) by Proposition 4.4, i.e., \( (x - y) \wedge \hat{x} = 0 \). Since \( \hat{x} \leq x \), we have
\[
\hat{x} - y = (x \wedge \hat{x}) - y = (x - (x - \hat{x})) - y = (x - y) - ((x - \hat{x}) - y) \quad \text{(by (p12))}
\]
\[
= (x - y) - ((x - y) - \hat{x}) = (x - y) \wedge \hat{x} = 0.
\]
Hence \( \hat{x} \leq y \) and we have
\[
\hat{x} = y \wedge \hat{x} = y - (y - \hat{x}) = y - d(y - \hat{x}) = y - (d(y) - d(\hat{x})) \quad \text{(by Theorem 3.14)}
\]
\[
= y - (d(y) - 0) = y - y = 0.
\]
It implies \( x = x - 0 = x - \hat{x} = d(x) \in d(X) \), and so \( d(X) \) is an ideal of \( X \). \( \square \)

Let \( X \) be a subtraction algebra and \( I \) an ideal of \( X \). If \( \sim_I \) is the binary relation on \( X \) given by
\[
x \sim_I y \text{ if and only if } x - y \in I \text{ and } y - x \in I,
\]
then \( \sim_I \) is a congruence relation and the quotient set \( X/I \) is a subtraction algebra with the binary operation defined by
\[
[x] - [y] = [x - y]
\]
for all \( [x], [y] \in X/I \), where \( [x] \) is an equivalence class of \( x \) with respect to \( \sim_I \).

4.8. Theorem. Let \( d \) be a derivation of a subtraction algebra \( X \). Then there exists a monomorphism \( \tilde{d} : X/\text{Ker}(d) \to X \) such that \( \tilde{d}([x]) = d(x) \). Hence \( X/\text{Ker}(d) \) is isomorphic to \( \text{Im}(\tilde{d}) = \text{Im}(d) \).

Proof. Suppose that \( d \) is a derivation on \( X \). Then \( d \) is a homomorphism of \( X \) by Theorem 3.14.

Define a map \( \tilde{d} : X/\text{Ker}(d) \to X \) by \( \tilde{d}([x]) = d(x) \) for all \( [x] \in X/\text{Ker}(d) \). If \( [x] = [y] \), then \( x \sim_{\text{Ker}(d)} y \) implies \( x - y, y - x \in \text{Ker}(d) \). Hence we have
\[
d(x) - d(y) = d(x - y) = 0 \text{ and } d(y) - d(x) = d(x - y) = 0.
\]
It follow that \( d(x) \leq d(y) \) and \( d(y) \leq d(x) \), that is, \( \tilde{d}([x]) = d(x) = d(y) = \tilde{d}([y]) \).
Therefore \( \tilde{d} \) is well-defined.
Let \([x], [y] \in X/\text{Ker}(d)\). Then we have\[d([x] - [y]) = d([x - y]) = d(x - y) = d(x) - d(y) = \tilde{d}([x]) - \tilde{d}([y]).\]
Hence \(\tilde{d}\) is a homomorphism.

To show that \(\tilde{d}\) is a monomorphism, let \(d(x) = d(y)\). Then \(d(x - y) = d(x) - d(y) = 0\) and \(d(y - x) = d(y) - d(x) = 0\). Hence \(x - y, y - x \in \text{Ker}(d)\). It follows that \(x \sim_{\text{Ker}(d)} y\), and \([x] = [y]\). Therefore \(\tilde{d}\) is a monomorphism.

**4.9. Theorem.** Let \(X\) be a subtraction algebra and \(d\) a derivation of \(X\). If \(\mu : X \to X\) is the map defined by\[\mu(x) = \hat{x} = x - d(x)\]
for all \(x \in X\), then \(\mu\) is a derivation with \(\text{Ker}(\mu) = \text{Im}(d)\).

**Proof.** Suppose that \(\mu : X \to X\) is the map defined by \(\mu(x) = \hat{x} = x - d(x)\) for all \(x \in X\).

Since \(\hat{x} = x - d(x)\) is unique for each \(x \in X\), \(\mu\) is well-defined.

Let \(x, y \in X\). The\[
\begin{align*}
\mu(x - y) & = (x - y) - d(x - y) \\
& = (x - y) - (d(x) - d(y)) \\
& = (x - d(x)) - y \quad \text{(by (p12))} \\
& = \mu(x) - y.
\end{align*}
\]
Hence \(\mu\) is a derivation.

If \(d(x) \in \text{Im}(d)\), then \(\mu(d(x)) = d(x) - d(x) = 0\), and \(d(x) \in \text{Ker}(\mu)\), hence \(\text{Im}(d) \subseteq \text{Ker}(\mu)\). If \(x \in \text{Ker}(\mu)\), then \(0 = \mu(x) = x - d(x)\), and \(x = x - 0 = x - (x - d(x)) = x \wedge d(x) = d(x) \in \text{Im}(d)\), and so \(\text{Ker}(\mu) \subseteq \text{Im}(d)\). Hence it follows that \(\text{Ker}(\mu) = \text{Im}(d)\). \(\square\)

**4.10. Corollary.** Let \(X\) be a subtraction algebra and \(d\) a derivation of \(X\). Then the corestriction \(\mu^\circ : X \to \text{Ker}(d)\) of \(\mu\) is an epimorphism.

**Proof.** By Theorem 4.9, \(\mu : X \to X\) is a derivation, hence \(\mu\) is a homomorphism, and it is clear that \(\text{Im}(\mu) = \text{Ker}(d)\) by Lemma 3.18. \(\square\)

**4.11. Theorem.** Let \(X\) be a subtraction algebra and \(d\) a derivation of \(X\). If \(\bar{\mu} : X/\text{Im}(d) \to X\) is the map defined by\[\bar{\mu}([x]) = \mu(x)\]
for all \([x] \in X/\text{Im}(d)\), then \(\bar{\mu}\) is a monomorphism. In particular, \(X/\text{Im}(d) \cong \text{Ker}(d)\).

**Proof.** Suppose that \(\bar{\mu} : X/\text{Im}(d) \to X\) is the map defined by\[\bar{\mu}([x]) = \mu(x)\]
for all \([x] \in X/\text{Im}(d)\). If \([x] = [y]\), then \(x \sim_{\text{Im}(d)} y\), which implies \(x - y, y - x \in \text{Im}(d)\), hence \(d(x - y) = x - y\) and \(d(y - x) = y - x\). It follows that\[\bar{\mu}([x]) - \bar{\mu}([y]) = \mu(x) - \mu(y) = \mu(x - y) = (x - y) - d(x - y) = 0,\]
and \(\bar{\mu}([y]) - \bar{\mu}([x]) = 0\) in a similar way. Hence \(\bar{\mu}([x]) = \bar{\mu}([y])\), and \(\bar{\mu}\) is well-defined.

Let \([x], [y] \in X/\text{Im}(d)\). Then we have\[\bar{\mu}([x] - [y]) = \bar{\mu}([x - y]) = \mu(x - y) = \mu(x) - \mu(y) = \bar{\mu}([x]) - \bar{\mu}([y]),\]
and \(\bar{\mu}\) is a homomorphism.
To show that $\bar{\mu}$ is a monomorphism, let $\bar{\mu}(x) = \bar{\mu}(y)$. Then $\mu(x) = \mu(y)$, and

$$0 = \mu(x) - \mu(y) = \mu(x - y) = (x - y) - d(x - y),$$

$$0 = \mu(y) - \mu(x) = \mu(y - x) = (y - x) - d(y - x),$$

hence $x - y \leq d(x - y)$ and $y - x \leq d(y - x)$. Since $d$ is non-expansive, $x - y = d(x - y) \in \text{Im}(d)$ and $y - x = d(y - x) \in \text{Im}(d)$. Therefore, $x \sim_{\text{Im}(d)} y$. This implies $[x] = [y]$. Hence $\bar{\mu}$ is a monomorphism.

It is clear that $\text{Im}(\bar{\mu}) = \text{Im}(\mu)$, and $\text{Im}(\mu) = \text{Ker}(d)$ by Corollary 4.10. Hence $X/\text{Im}(d) \cong \text{Ker}(d)$. □

Now consider the sequence

$$0 \rightarrow \text{Im}(d) \rightarrow X \xrightarrow{\mu} \text{Ker}(d) \rightarrow 0,$$

of homomorphisms of subtraction algebras, where $i$ is the inclusion map. We note that it is similar to a split exact sequence, since $i$ is a monomorphism, $\mu^*$ is an epimorphism and $\text{Ker}(\mu^*) = \text{Im}(i)$ by Corollary 4.10 and Theorem 4.9.

4.12. Proposition. Let $d$ be a derivation of a subtraction algebra $X$. Then for each $x \in X$, $x = d(x) \lor \hat{x}$ with $d(x) \in \text{Im}(d)$ and $\hat{x} \in \text{Ker}(d)$.

Proof. Let $X$ be a subtraction algebra and $x \in X$. Then the interval $[0, x]$ is a Boolean algebra with respect to the induced partial order and $\hat{x} = x - d(x)$ is the complement of $d(x)$ in $[0, x]$. Hence $d(x) \lor \hat{x} = d(x) \lor (x - d(x)) = x$. □

Let $d$ be a derivation of a subtraction algebra $X$. Then $\text{Im}(d)$ and $\text{Ker}(d)$ are subtraction subalgebras. Hence $\text{Im}(d) \times \text{Ker}(d)$ is also a subtraction algebra with the binary operation $\ast$ defined by

$$(x_1, y_1) \ast (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in \text{Im}(d) \times \text{Ker}(d)$.

4.13. Theorem. Let $d$ be a derivation of a subtraction algebra $X$. If $\phi = (d, \mu) : X \rightarrow \text{Im}(d) \times \text{Ker}(d)$ is the map defined by

$$\phi(x) = (d(x), \mu(x))$$

for all $x \in X$, then $\phi$ is a monomorphism.

Proof. Suppose that $\phi = (d, \mu) : X \rightarrow \text{Im}(d) \times \text{Ker}(d)$ is the map defined by $\phi(x) = (d(x), \mu(x))$ for all $x \in X$. Then for any $x, y \in X$ we have

$$\phi(x - y) = (d(x - y), \mu(x - y))$$

$$= (d(x) - d(y), \mu(x) - \mu(y))$$

$$= (d(x), \mu(x)) - (d(y), \mu(y))$$

$$= \phi(x) - \phi(y).$$

If $\phi(x) = \phi(y)$, then $(d(x), \mu(x)) = (d(y), \mu(y))$, and by Proposition 4.12,

$$x = d(x) \lor \hat{x} = d(x) \lor \mu(x) = d(y) \lor \mu(y) = d(y) \lor \hat{y} = y.$$

Hence $\phi$ is a monomorphism. □

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References