

A FRACTIONAL ORDER SEIR MODEL WITH DENSITY DEPENDENT DEATH RATE

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Abstract

In this paper, we introduce a fractional order SEIR epidemic model with vertical transmission, where the death rate of the population is density dependent, i.e., dependent on the population size. It is also assumed that there exists an infection related death rate. We show the existence of nonnegative solutions of the model, and also give a detailed stability analysis of disease free and positive fixed points. A numerical example is also presented.

Keywords: Fractional derivative, Initial value problem, SEIR model, Stability, Numerical solution.

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1. Introduction

Mathematical modeling in epidemiology provides new aspects in understanding the spread of diseases, and it suggests control strategies [3]. One of the early models in epidemiology was introduced in 1927 [9] to predict the spreading behaviour of a disease. Since then, many epidemic models have been derived [8]. In [7] a detailed analysis for integer order SEIR models with vertical transmission within a constant population can be found. There are also several papers [12, 13] about epidemic models within a nonconstant population, which is more realistic.

Although a large number of works has been done on modeling the dynamics of epidemiological diseases, it has been restricted to integer order (delay) differential equations. In recent years, it has turned out that many phenomena in different fields can be described very successfully by the models using fractional order differential equations [1,2,4,6].

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In this paper, we first introduce a fractional order SEIR model with vertical transmission in a population with density dependent death rate. This gives a logistic growth in the population for a specific, more realistic death rate. We show the nonnegativeness of the solutions and also give a detailed stability analysis. Finally, numerical simulations are presented to illustrate the obtained results.

We begin by giving the definitions of fractional order integrals and derivatives [15]. For fractional order differentiation, we will use Caputo's definition, due to its convenience for initial conditions of the differential equations.

1.1. Definition. The fractional integral of order $\alpha > 0$ for a function $f : R^+ \rightarrow R$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau,$$

and the Caputo fractional derivative of order $\alpha \in (n - 1, n)$ of $f(t)$ is defined by

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t),$$

with $n - 1$ being the integer part of α and $D = d/dt$.

Here and elsewhere, Γ denotes the Gamma function.

Note that under natural conditions on the function $f(t)$, for $\alpha \rightarrow n$ the Caputo derivative becomes the conventional derivative [15].

2. Model derivation

Many infectious diseases in nature have both horizontal and vertical transmission routes. These include such human diseases as Rubella, Herpes Simplex, Hepatitis B, Chagas, and the HIV/AIDS. Horizontal transmission of diseases among humans and animals, occurs through physical contact with hosts or through disease vectors like mosquitos, flies, etc. Vertical transmission is the transmission of an infection from parents to child during the perinatal period.

The model that we study in this paper is a fractional order SEIR epidemic model with vertical transmission. The total host population $N(t)$ is partitioned into four compartments which are *susceptible*, *exposed*, *infectious* and *recovered*, with sizes denoted by $S(t)$, $E(t)$, $I(t)$ and $R(t)$, respectively. Let b denote the natural birth rate of the population. The horizontal transmission of the disease is assumed to take place with direct contact between infectious and susceptible hosts with a transmission rate r , which means $r \frac{SI}{N}$ is the number of infections caused by all infected individuals per unit of time. For the vertical transmission of the disease, we assume that the offsprings of exposed and infectious classes are born into the exposed class with probabilities of p and q , respectively. We also assume that the natural death rate $d(N)$ depends on the size of the population. For convenience, d is assumed to be a continuous and non decreasing function on R^+ , which does not contradict with biological phenomena. Also we assume that there exists a positive constant K which represents the carrying capacity of the population, and such that $d(K) = b$. We shall note that if $d(N)$ is a linear function then the population has a logistic growth.

These assumptions lead to the following system of differential equations of order α , with $\beta, \gamma > 0$ being the rate that exposed individuals become infectious and the recovery

rate, respectively, and $\theta \geq 0$ is the infection related death rate:

$$\begin{aligned}
 D^\alpha S &= bN - pbE - qbI - r\frac{SI}{N} - d(N)S, \\
 (2.1) \quad D^\alpha E &= pbE + qbI + r\frac{SI}{N} - \beta E - d(N)E, \\
 D^\alpha I &= \beta E - \theta I - \gamma I - d(N)I, \\
 D^\alpha R &= \gamma I - d(N)R, \\
 (2.2) \quad S(0) &= S_0, E(0) = E_0, I(0) = I_0, R(0) = R_0,
 \end{aligned}$$

where $0 < \alpha \leq 1$, $N = S + E + I + R$, $(S, E, I, R) \in R_+^4$. The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also, using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. We should note that the system (2.1) can be reduced to an integer order system by setting $\alpha = 1$.

Adding up the equations given in (2.1), we have

$$(2.3) \quad D^\alpha N = N(b - d(N)) - \theta I,$$

which means the population size is not constant.

3. Non-negative solutions

Let $R_+^4 = \{X \in R^4 : X \geq 0\}$ and $X(t) = (S(t), E(t), I(t), N(t))^T$. For the proof of the theorem about non-negative solutions we shall need the following Lemma [14]:

3.1. Lemma. (Generalized Mean Value Theorem) *Let $f(x) \in C[a, b]$ and $D^\alpha f(x) \in C(a, b]$ for $0 < \alpha \leq 1$. Then we have*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\xi) (x - a)^\alpha$$

with $0 \leq \xi \leq x, \forall x \in (a, b]$. □

3.2. Remark. Suppose $f(x) \in C[0, b]$ and $D^\alpha f(x) \in C(0, b]$ for $0 < \alpha \leq 1$. It is clear from the Lemma 3.1 that if $D^\alpha f(x) \geq 0, \forall x \in (0, b)$, then the function f is nondecreasing, and if $D^\alpha f(x) \leq 0, \forall x \in (0, b)$, then the function f is nonincreasing for all $x \in [0, b]$.

3.3. Theorem. *There is a unique solution for the initial value problem given by (2.1)-(2.2), and the solution remains in R_+^4 .*

Proof. The existence and uniqueness of the solution of (2.1)-(2.2) in $(0, \infty)$ can be obtained from [10, Theorem 3.1 and Remark 3.2]. We need to show that the domain R_+^4 is positively invariant. Since

$$\begin{aligned}
 D^\alpha S|_{S=0} &= bN - pbE - qbI \geq 0, \\
 D^\alpha E|_{E=0} &= qbI + r\frac{SI}{N} \geq 0, \\
 D^\alpha I|_{I=0} &= \beta E \geq 0, \\
 D^\alpha R|_{R=0} &= \gamma I \geq 0,
 \end{aligned}$$

on each hyperplane bounding the nonnegative orthant, the vector field points into R_+^4 . □

It is clear that $N(t)$ also remains nonnegative. For convenience in calculations we consider the following system, which can be obtained from (2.1) and (2.3):

$$(3.1) \quad \begin{aligned} D^\alpha S &= bN - pbE - qbI - r\frac{SI}{N} - d(N)S, \\ D^\alpha E &= pbE + qbI + r\frac{SI}{N} - \beta E - d(N)E, \\ D^\alpha I &= \beta E - \theta I - \gamma I - d(N)I, \\ D^\alpha N &= N(b - d(N)) - \theta I, \end{aligned}$$

with initial conditions

$$(3.2) \quad S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad N(0) = N_0.$$

4. Equilibrium points and stability

Consider the initial value problem (3.1)-(3.2) with α satisfying $0 < \alpha \leq 1$. To evaluate the equilibrium points of (3.1), let

$$\begin{cases} D^\alpha S = 0, \\ D^\alpha E = 0, \\ D^\alpha I = 0, \\ D^\alpha N = 0. \end{cases}$$

Then the equilibrium points are $F_0 = (K, 0, 0, K)$ and $F_1 = (S^*, E^*, I^*, N^*)$, where

$$(4.1) \quad S^* = \frac{(d(N^*) + \beta - pb)(d(N^*) + \theta + \gamma) - qb\beta}{r\beta} N^*$$

$$(4.2) \quad E^* = \frac{(d(N^*) + \theta + \gamma)(b - d(N^*))}{\beta\theta} N^*$$

$$(4.3) \quad I^* = \frac{b - d(N^*)}{\theta} N^*.$$

Here $m := d(N^*)$ is the positive root of the following equation:

$$(4.4) \quad (r - \theta)d^3(N^*) + [(r - \theta)(\beta + \gamma + \theta) - b(r - p\theta)]d^2(N^*) - rb\beta\gamma + [\beta(r - \theta)(\theta + \gamma) - b(r - p\theta)(\theta + \gamma) - b\beta(r - q\theta)]d(N^*) = 0.$$

Indeed, for $r > \theta$ one can show that (4.4) has only one positive root using Descartes' rule of signs.

The Jacobian matrix $J(F_0)$ for the system given by (3.1), evaluated at the disease free equilibrium is as follows:

$$(4.5) \quad J(F_0) = \begin{pmatrix} -b & -pb & -qb - r & b - d'(K)K \\ 0 & (p-1)b - \beta & qb + r & 0 \\ 0 & \beta & -b - \theta - \gamma & 0 \\ 0 & 0 & -\theta & -d'(K)K \end{pmatrix}$$

4.1. Theorem. *Disease free equilibrium of the system (3.1) is asymptotically stable if*

$$\frac{\beta(qb + r)}{(b + \beta - bp)(\gamma + \theta + b)} < 1.$$

Proof. Disease free equilibrium is asymptotically stable if all of the eigenvalues, λ_i , $i = 1, 2, 3, 4$, of $J(F_0)$ satisfy the following conditions [2,11]:

$$(4.6) \quad |\arg \lambda_i| > \alpha \frac{\pi}{2}.$$

These eigenvalues can be determined by solving the characteristic equation

$$\det(J(F_0) - \lambda I) = 0.$$

Thus, we have the following algebraic equation:

$$(\lambda + b)(\lambda + kd'(k))[\lambda^2 + (A + B)\lambda + AB - C] = 0$$

where

$$\begin{aligned} A &= b + \theta + \gamma, \\ B &= b - pb + \beta, \\ C &= \beta(qb + r). \end{aligned}$$

If $AB > C$, then the condition given by (4.6) is satisfied. □

4.2. Remark. Consider the threshold value

$$R_0 = \frac{\beta(qb + r)}{(b + \beta - bp)(\gamma + \theta + b)},$$

which is the basic reproduction number of the system (3.1). The biological interpretation of R_0 is that the disease will take off if R_0 exceeds 1, and will die out if R_0 is less than 1.

We now discuss the asymptotic stability of the endemic (positive) equilibrium of the system given by (3.1). The Jacobian matrix $J(F_1)$ evaluated at the endemic equilibrium is given by:

$$J(F_1) = \begin{pmatrix} -m - rB_1 & -pb & -rB_2B_3 & b + (B_2B_3 - B_4)(rB_1 - d'(N^*)N^*) \\ rB_1 & -rB_3 & rB_2B_3 & -rB_1(B_2B_3 - B_4) - B_1B_2d'(N^*)N^* \\ 0 & \beta & -\beta B_2 & -B_1d'(N^*)N^* \\ 0 & 0 & -\theta & b - m - d'(N^*)N^* \end{pmatrix}$$

where

$$B_1 = \frac{b - m}{\theta}, \quad B_2 = \frac{m + \theta + \gamma}{\beta}, \quad B_3 = \frac{m + \beta - pb}{r}, \quad B_4 = \frac{qb}{r}.$$

So, the characteristic equation of the linearized system is of the form

$$(4.7) \quad \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

with a_1, a_2, a_3 and a_4 being

$$\begin{aligned} a_1 &= -b + 2m + rB_1 + rB_3 + \beta B_2 + d'(N^*)N^* \\ a_2 &= (mr - rb + bpr)B_1 + \beta(2m - b)B_2 + r(2m - b)B_3 \\ &\quad + [m + (r - \theta)B_1 + rB_3 + \beta B_2]d'(N^*)N^* \\ &\quad + r^2B_1B_3 - bm + m^2 + r\beta B_1B_2 \\ a_3 &= (mrB_3 + bprB_1 + r^2B_1B_3 + \beta B_2(m + rB_2) \\ &\quad - \theta B_1(m - rB_1) - \beta\theta B_1B_2)d'(N^*)N^* \\ (4.8) \quad &\quad - \theta B_1(prbB_1 + m\beta B_2 + mrB_3 + r^2B_1B_3) \\ &\quad + r(m\beta - b\beta + bp)B_1B_2 + r\theta B_1(\beta B_4 + B_3) \\ &\quad + r\beta(r - \theta)B_1B_2B_3 \\ a_4 &= br\beta\theta B_1 - \theta b\beta prB_1^2B_2 + r\beta(mr - br - m\theta)B_1B_2B_3 \\ &\quad + mr\beta\theta B_1B_4[-bpr\theta B_1^2 + \beta(bpr - m\theta - r\theta B_1)B_1B_2 \\ &\quad + r\theta(m - rB_1)B_1B_3 + r\beta\theta B_1B_4 \\ &\quad + r\beta(r - \theta)B_1B_2B_3]d'(N^*)N^* \end{aligned}$$

4.3. Theorem. *Let a_4 be as given in (4.8). If $a_4 < 0$ then the positive equilibrium point F_1 of the system (3.1) is unstable.*

Proof. If $a_4 < 0$, from Descartes' rule of signs it is clear that the characteristic equation (4.7) has at least one positive real root. So, the equilibrium point F_1 of the system (3.1) is unstable. \square

5. Numerical methods and simulations

For the numerical solutions of a system of fractional differential equations, using an Adam's-type predictor corrector method is appropriate [5]. For the parameters

$$(5.1) \quad \begin{aligned} b &= 0.001555, \quad p = 0.8, \quad q = 0.95, \quad r = 0.05, \quad \beta = 0.05, \\ \theta &= 0.002, \quad \gamma = 0.003, \quad d(N) = 0.00001 + 0.000007 N \end{aligned}$$

with the initial conditions

$$(5.2) \quad S(0) = 140, \quad E(0) = 0.01, \quad I(0) = 0.02, \quad N(0) = 141,$$

there exists a positive fixed point

$$S^* = 14.1268, \quad E^* = 4.38274, \quad I^* = 36.0006, \quad N^* = 153.864$$

for IVP (3.1)–(3.2). The approximate solutions $S(t)$, $E(t)$, $I(t)$ and $N(t)$ are displayed in Figures 1–4 for $\alpha = 1, 0.95, 0.9$.

6. Concluding Remarks

We have formulated and analysed an SEIR model with vertical transmission within a population having a density dependent death rate. We have obtained a stability condition for disease free equilibrium, and a nonstability condition for positive equilibrium. We have also given a numerical example and verified our results.

Figure 1. Size of the suspended class over time for system (3.1) with parameters (5.1) and initial conditions (5.2) for different values of α

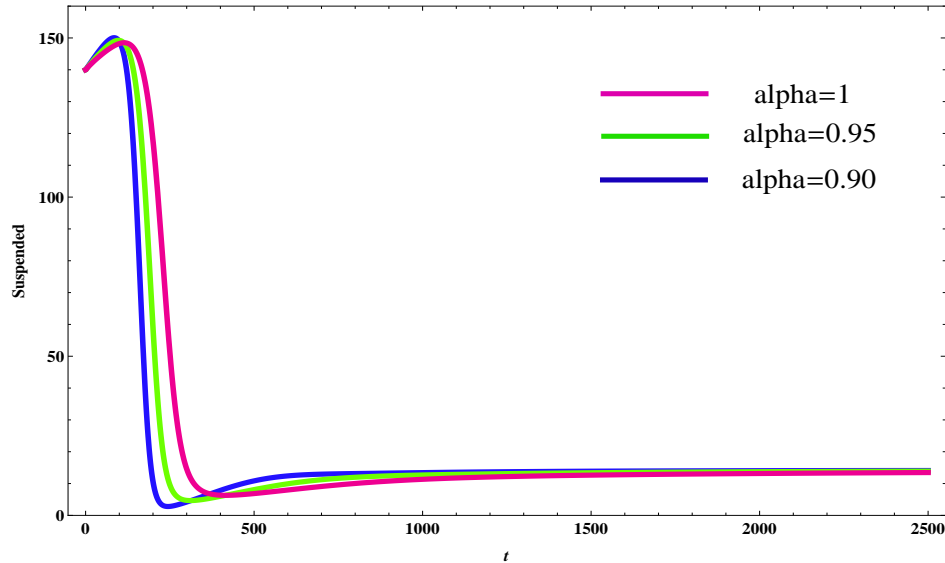


Figure 2. Size of the exposed class over time for system (3.1) with parameters (5.1) and initial conditions (5.2) for different values of α

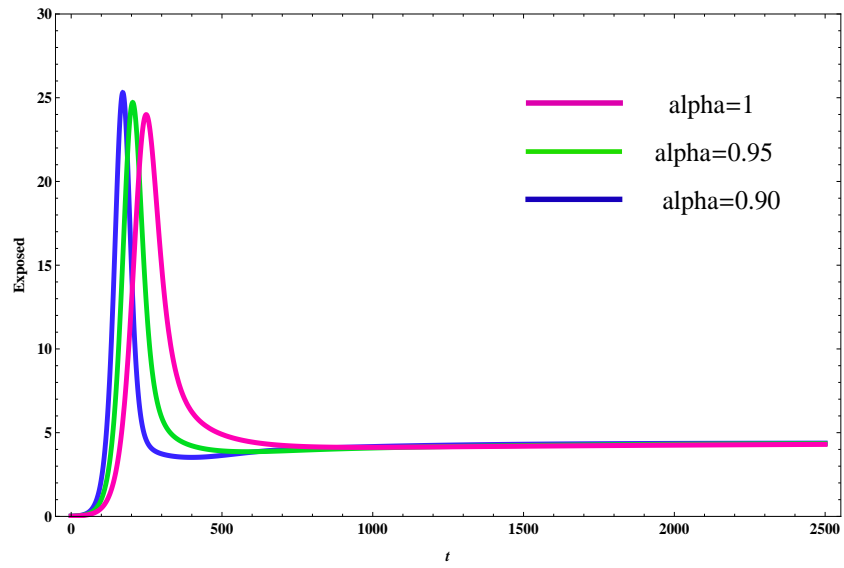


Figure 3. Size of the infectious class over time for system (3.1) with parameters (5.1) and initial conditions (5.2) for different values of α

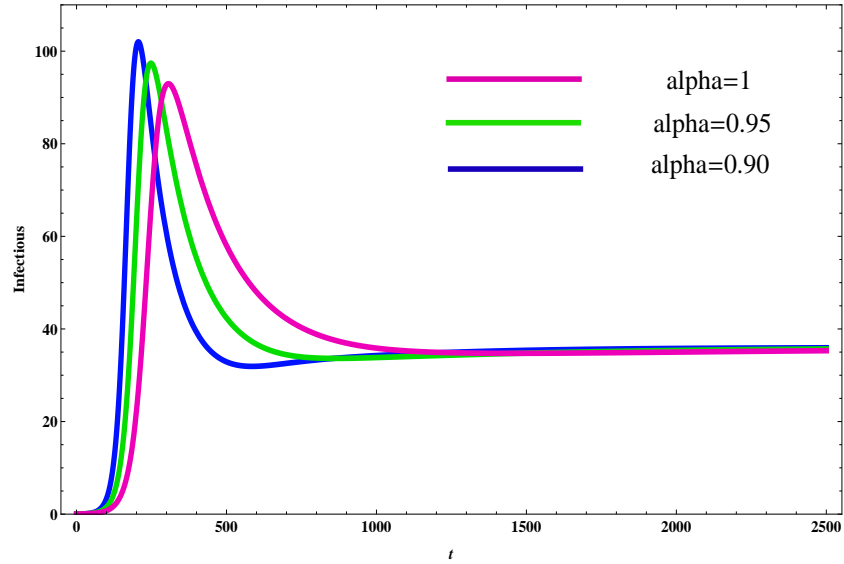
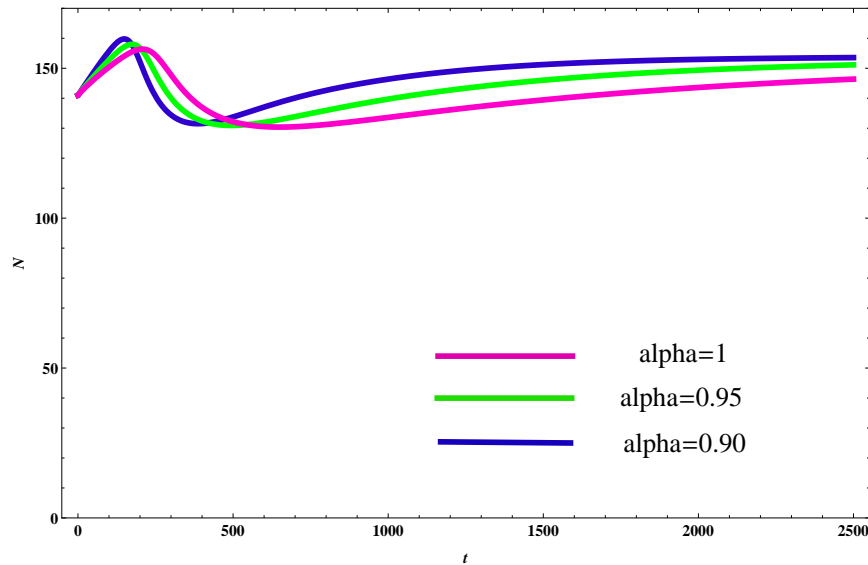


Figure 4. Size of the total population over time for system (3.1) with parameters (5.1) and initial conditions (5.2) for different values of α



We should note that although the equilibrium points are the same for both integer order and fractional order models, the solution of the fractional order model tends to the fixed point over a longer period of time. We also need to mention that when dealing with real life problems, the order of the system can be determined by using the collected data.

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