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Orientable small covers over the product of 2-cube with n-gon

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Abstract

We calculate the number of D-J equivalence classes and equivariant homeomorphism classes of all orientable small covers over the product of 2-cube with n-gon.

Keywords: Small cover; D-J equivalence; Equivariant homeomorphism

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1. Introduction

As defined by Davis and Januszkiewicz [5], a small cover is a smooth closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$−action such that its orbit space is a simple convex polytope. For instance, the real projective space $\mathbb{R}P^n$ with a natural $(\mathbb{Z}_2)^n$−action is a small cover over an n-simplex. This gives a direct connection between equivariant topology and combinatorics, making research on the topology of small covers possible through the combinatorial structure of quotient spaces.

Lü and Masuda [7] showed that the equivariant homeomorphism class of a small cover over a simple convex polytope P^n agrees with the equivalence class of its corresponding $(\mathbb{Z}_2)^n$−coloring under the action of the automorphism group of the face poset of P^n. This finding also holds true for orientable small covers by the orientability condition in [8] (see Theoerem 2.5). However, general formulas for calculating the number of equivariant homeomorphism classes of (orientable) small covers over an arbitrary simple convex polytope do not exist.

In recent years, several studies have attempted to enumerate the number of equivalence classes of all small covers over a specific polytope. Garrison and Scott [6] used a computer program to calculate the number of homeomorphism classes of all small covers over a dodecahedron. Cai, Chen and Lü [2] calculated the

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number of equivariant homeomorphism classes of small covers over prisms (an n-sided prism is the product of 1-cube and n-gon). Choi [3] determined the number of equivariant homeomorphism classes of small covers over cubes. However, little is known about orientable small covers. Choi [4] calculated the number of D-J equivalence classes of orientable small covers over cubes. This paper aims to determine the number of D-J equivalence classes and equivariant homeomorphism classes of all orientable small covers over $I^2 \times P_n$ (see Theorem 3.1 and Theorem 4.1), where I^2 and P_n denote 2-cube and n-gon, respectively.

The paper is organized as follows. In Section 2, we review the basic theory on orientable small covers and calculate the automorphism group of the face poset of $I^2 \times P_n$. In Section 3, we determine the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$. In Section 4, we obtain a formula for the number of equivariant homeomorphism classes of orientable small covers over $I^2 \times P_n$.

2. Preliminaries

A convex polytope P^n of dimension n is simple if every vertex of P^n is the intersection of n facets (i.e., faces of dimension $(n-1)$) [9]. An n-dimensional smooth closed manifold M^n is a small cover if it admits a smooth $(\mathbb{Z}_2)^n$−action such that the action is locally isomorphic to a standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ and the orbit space $M^n/(\mathbb{Z}_2)^n$ is a simple convex polytope of dimension n.

Let P^n be a simple convex polytope of dimension n and $\mathcal{F}(P^n) = \{F_1, \cdots, F_\ell\}$ be the set of facets of P^n. Assuming that $\pi : M^n \to P^n$ is a small cover over P^n, then there are ℓ connected submanifolds $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_\ell)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a $\mathbb{Z}_2$−subgroup $\mathbb{Z}_2(F_i)$ of $(\mathbb{Z}_2)^n$. Obviously, the $\mathbb{Z}_2$−subgroup $\mathbb{Z}_2(F_i)$ agrees with an element ν_i in $(\mathbb{Z}_2)^n$ as a vector space. For each face F of codimension u, given that P^n is simple, there are u facets $F_{i_1}, \cdots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}), \cdots, \pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbb{Z}_2(F)$ of $\pi^{-1}(F)$ is a subtorus of rank u generated by $\mathbb{Z}_2(F_{i_1}), \cdots, \mathbb{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1}, \cdots, \nu_{i_u}$ in $(\mathbb{Z}_2)^n$). This gives a characteristic function [5]

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

which is defined by $\lambda(F_i) = \nu_i$ such that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_u}$ is non-empty, $\lambda(F_{i_1}), \cdots, \lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$. Assuming that each nonzero vector of $(\mathbb{Z}_2)^n$ is a color, then the characteristic function λ means that each facet is colored. Hence, we also call λ a $(\mathbb{Z}_2)^n$-coloring on P^n.

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, we denote the minimal face containing p in its relative interior by $F(p)$. Assuming that $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbb{Z}_2(F(p)) = \bigoplus_{j=1}^{u} \mathbb{Z}_2(F_{i_j})$, then $\mathbb{Z}_2(F(p))$ is a u-dimensional subgroup of $(\mathbb{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbb{Z}_2)^n / \sim$, where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^n$ on $P^n \times (\mathbb{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient P^n. Thus, $M(\lambda)$ is a small cover over P^n [5].

Two small covers M_1 and M_2 over P^n are called weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t\cdot x) = \varphi(t)\cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are equivariantly homeomorphic. Following [5], two small covers M_1 and M_2 over P^n are called Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P^n.

By $\Lambda(P^n)$, we denote the set of all $(\mathbb{Z}_2)^n$-colorings on P^n. We have

2.1. Theorem. ([5]) *All small covers over P^n are given by $\{M(\lambda)|\lambda \in \Lambda(P^n)\}$, i.e., for each small cover M^n over P^n, there is a $(\mathbb{Z}_2)^n$-coloring λ with an equivariant homeomorphism $M(\lambda) \longrightarrow M^n$ covering the identity on P^n.*

Nakayama and Nishimura [8] found an orientability condition for a small cover.

2.2. Theorem. *For a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ is defined by $\varepsilon(e_i) = 1 (i = 1, \cdots, n)$. A small cover $M(\lambda)$ over a simple convex polytope P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.*

A $(\mathbb{Z}_2)^n$-coloring that satisfies the orientability condition in Theorem 2.2 is an orientable coloring of P^n. We know that there exists an orientable small cover over every simple convex 3-polytope [8]. Similarly, we know the existence of orientable small cover over $I^2 \times P_n$ by the existence of orientable colorings and determine the number of D-J equivalence classes and equivariant homeomorphism classes.

By $O(P^n)$, we denote the set of all orientable colorings on P^n. There is a natural action of $GL(n, \mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$, and the action on $O(P^n)$ is free. We assume that $F_1, \cdots, F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1, \cdots, e_n$ be the standard basis of $(\mathbb{Z}_2)^n$ and $B(P^n) = \{\lambda \in O(P^n)|\lambda(F_i) = e_i, i = 1, \cdots, n\}$. Then $B(P^n)$ is the orbit space of $O(P^n)$ under the action of $GL(n, \mathbb{Z}_2)$.

2.3. Remark. We have $B(P^n) = \{\lambda \in O(P^n)|\lambda(F_i) = e_i, i = 1, \cdots, n$ and for $n+1 \le j \le \ell, \lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}, 1 \le j_1 < j_2 < \cdots < j_{2h_j+1} \le n\}$. Below, we show that $\lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}$ for $n + 1 \le j \le \ell$. If $\lambda \in O(P^n)$, there exists a basis $\{e'_1, \cdots, e'_n\}$ of $(\mathbb{Z}_2)^n$ such that for $1 \le i \le \ell$, $\lambda(F_i) = e'_{i_1} + \cdots + e'_{i_{2f_i+1}}, 1 \le i_1 < \cdots < i_{2f_i+1} \le n$. Given that $\lambda(F_i) = e_i, i = 1, \cdots, n$, then $e_i = e'_{i_1} + \cdots + e'_{i_{2f_i+1}}$. Thus, for $n + 1 \le j \le \ell$, $\lambda(F_j)$ is not of the form $e_{j_1} + \cdots + e_{j_{2k}}, 1 \le j_1 < \cdots < j_{2k} \le n$.

Given that $B(P^n)$ is the orbit space of $O(P^n)$, then we have

2.4. Lemma. $|O(P^n)| = |B(P^n)| \times |GL(n, \mathbb{Z}_2)|$.

Note that $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} \left(2^n - 2^{k-1}\right)$ [1]. Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. Thus the number of D-J equivalence classes of orientable small covers over P^n is $|B(P^n)|$.

Let P^n be a simple convex polytope of dimension n. All faces of P^n form a poset (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^n)$ is a bijection from $\mathcal{F}(P^n)$ to itself that preserves the poset structure of all faces of P^n. By $Aut(\mathcal{F}(P^n))$, we denote the group of automorphisms of $\mathcal{F}(P^n)$. We define the right action of $Aut(\mathcal{F}(P^n))$ on $O(P^n)$ by $\lambda \times h \longmapsto \lambda \circ h$, where $\lambda \in O(P^n)$ and $h \in Aut(\mathcal{F}(P^n))$. By improving the classifying result on unoriented small covers in [7], we have

2.5. Theorem. *Two orientable small covers over an n-dimensional simple convex polytope P^n are equivariantly homeomorphic if and only if there is $h \in Aut(\mathcal{F}(P^n))$ such that $\lambda_1 = \lambda_2 \circ h$, where λ_1 and λ_2 are their corresponding orientable colorings on P^n.*

Proof. Theorem 2.5 is proven true by combining Lemma 5.4 in [7] with Theorem 2.2. □

According to Theorem 2.5, the number of orbits of $O(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$ is the number of equivariant homeomorphism classes of orientable small covers over P^n. Thus, we count the number of orbits. Burnside Lemma is very useful in enumerating the number of orbits.

Burnside Lemma *Let G be a finite group acting on a set X. Then the number of orbits X under the action of G equals* $\frac{1}{|G|}\sum_{g\in G}|X_g|$, *where* $X_g = \{x \in X | gx = x\}$.

Burnside Lemma suggests that, to determine the number of the orbits of $O(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$, the structure of $Aut(\mathcal{F}(P^n))$ should first be understood. We shall particularly be concerned when the simple convex polytope is $I^2 \times P_n$.

For convenience, we introduce the following marks. By F_1', F_2', F_3', and F_4' we denote four edges of the 2-cube I^2 in their general order (here I^2 is considered as a 4-gon). Similarly, by $F_5', F_6', \cdots$, and F_{n+4}', we denote all edges of n-gon P_n in their general order. Set $\mathcal{F}' = \{F_i = F_i' \times P_n | 1 \le i \le 4\}$, and $\mathcal{F}'' = \{F_i = I^2 \times F_i' | 5 \le i \le n+4\}$. Then $\mathcal{F}(I^2 \times P_n) = \mathcal{F}' \bigcup \mathcal{F}''$.

Next, we determine the automorphism group of face poset of $I^2 \times P_n$.

2.6. Lemma. *When n=4, the automorphism group $Aut(\mathcal{F}(I^2\times P_n))$ is isomorphic to $(\mathbb{Z}_2)^4 \times S_4$, where S_4 is the symmetric group on four symbols. When $n \neq 4$, $Aut(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $D_4 \times D_n$, where D_n is the dihedral group of order $2n$.*

Proof. When n=4, $I^2 \times P_n$ is a 4-cube I^4. Obviously, the automorphism group $Aut(\mathcal{F}(I^4))$ contains a symmetric group S_4 because there is exactly one automorphism for each permutation of the four pairs of opposite sides of I^4. All elements of $Aut(\mathcal{F}(I^4))$ can be written in a simple form as $\chi_1^{e_1}\chi_2^{e_2}\chi_3^{e_3}\chi_4^{e_4} \cdot u$, where $e_1, e_2, e_3, e_4 \in \mathbb{Z}_2$, with reflections $\chi_1, \chi_2, \chi_3, \chi_4$ and $u \in S_4$. Thus, the automorphism group $Aut(\mathcal{F}(I^4))$ is isomorphic to $(\mathbb{Z}_2)^4 \times S_4$.

When$n \neq 4$, the facets of $\mathcal{F}'$ and $\mathcal{F}''$ are mapped to $\mathcal{F}'$ and $\mathcal{F}''$, respectively, under the automorphisms of $Aut(\mathcal{F}(I^2\times P_n))$. Given that the automorphism group $Aut(\mathcal{F}(I^2))$ is isomorphic to D_4 and $Aut(\mathcal{F}(P_n))$ is isomorphic to D_n, $Aut(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $D_4 \times D_n$. □

2.7. Remark. Let x, y, x', y' be the four automorphisms of $Aut(\mathcal{F}(I^2 \times P_n))$ with the following properties:

(a) $x(F_i) = F_{i+1}(1 \le i \le 3), x(F_4) = F_1, x(F_j) = F_j, 5 \le j \le n+4;$

(b) $y(F_i) = F_{5-i}(1 \le i \le 4), y(F_j) = F_j, 5 \le j \le n+4;$

(c) $x'(F_i) = F_i(1 \le i \le 4), x'(F_j) = F_{j+1}(5 \le j \le n+3), x'(F_{n+4}) = F_5;$

(d) $y'(F_i) = F_i(1 \le i \le 4), y'(F_j) = F_{n+9-j}, 5 \le j \le n+4.$

Then, when $n \neq 4$, all automorphisms of $Aut(\mathcal{F}(I^2 \times P_n))$ can be written in a simple form as follows:

(1) $\qquad x^u y^v x'^{u'} y'^{v'},\ u \in \mathbb{Z}_4, u' \in \mathbb{Z}_n, v, v' \in \mathbb{Z}_2$

with $x^4 = y^2 = x'^n = y'^2 = 1, x^u y = y x^{4-u}$, and $x'^{u'} y' = y' x'^{n-u'}$.

3. Orientable colorings on $I^2 \times P_n$

This section is devoted to calculating the number of all orientable colorings on $I^2 \times P_n$. We also determine the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$.

3.1. Theorem. *By $\mathbb{N}$, we denote the set of natural numbers. Let a, b, c be the functions from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:*

(1) $a(j) = 2a(j-1) + 8a(j-2)$ *with* $a(1) = 1,\ a(2) = 2$;

(2) $b(j) = b(j-1) + 4b(j-2)$ *with* $b(1) = b(2) = 1$;

(3) $c(j) = 2c(j-1)+4c(j-2)-6c(j-3)-3c(j-4)+4c(j-5)$ *with* $c(1) = c(2) = 1$, $c(3) = 3,\ c(4) = 7,\ c(5) = 17$.

Then, the number of all orientable colorings on $I^2 \times P_n$ is

$$|O(I^2 \times P_n)| = \prod_{k=1}^{4} \left(2^4 - 2^{k-1}\right) \cdot [a(n-1) + 4b(n-1) + 2c(n-1) + 5 \cdot \tfrac{1+(-1)^n}{2}].$$

Proof. Let e_1, e_2, e_3, e_4 be the standard basis of $(\mathbb{Z}_2)^4$, then $(\mathbb{Z}_2)^4$ contains 15 nonzero elements (or 15 colors). We choose F_1, F_2 from $\mathcal{F}'$ and F_5, F_6 from $\mathcal{F}''$ such that F_1, F_2, F_5, F_6 meet at one vertex of $I^2 \times P_n$. Then

$$B(I^2 \times P_n) = \{\lambda \in O(I^2 \times P_n) | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \lambda(F_5) = e_3, \lambda(F_6) = e_4\}.$$

By Lemma 2.4, we have

$$|O(I^2 \times P_n)| = |B(I^2 \times P_n)| \times |GL(4, \mathbb{Z}_2)| = \prod_{k=1}^{4} \left(2^4 - 2^{k-1}\right) \cdot |B(I^2 \times P_n)|.$$

Write

$B_0(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, e_1 + e_3 + e_4\}$,

$B_1(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_2 + e_3, e_1 + e_2 + e_4\}$.

By the definition of $B(P^n)$ and Remark 2.3, we have $|B(I^2 \times P_n)| = |B_0(I^2 \times P_n)| + |B_1(I^2 \times P_n)|$. Then, our argument proceeds as follows.

(I) Calculation of $|B_0(I^2 \times P_n)|$.

In this case, no matter which value of $\lambda(F_3)$ is chosen, $\lambda(F_4) = e_2, e_2 + e_1 + e_3, e_2 + e_1 + e_4, e_2 + e_3 + e_4$. Write

$B_0^0(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2\}$,

$B_0^1(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_3\}$,
$B_0^2(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_4\}$,
$B_0^3(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_3 + e_4\}$,
$B_0^4(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2\}$,
$B_0^5(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_3\}$,
$B_0^6(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_4\}$,
$B_0^7(I^2 \times P_n) = \{\lambda \in B(I^2 \times P_n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}$.

By the definition of $B_0(I^2 \times P_n)$ and Remark 2.3, we have $|B_0(I^2 \times P_n)| = \sum_{i=0}^{7} |B_0^i(I^2 \times P_n)|$. Then, our argument is divided into the following cases.

Case 1. Calculation of $|B_0^0(I^2 \times P_n)|$.

By the definition of $B(P^n)$ and Remark 2.3, we have $\lambda(F_{n+4}) = e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4 + e_2 + e_3$. Set $B_0^{0,0}(I^2 \times P_n) = \{\lambda \in B_0^0(I^2 \times P_n) | \lambda(F_{n+3}) = e_3, e_1 + e_2 + e_3\}$ and $B_0^{0,1}(I^2 \times P_n) = B_0^0(I^2 \times P_n) - B_0^{0,0}(I^2 \times P_n)$. Take an orientable coloring λ in $B_0^{0,0}(I^2 \times P_n)$. Then, $\lambda(F_{n+2}), \lambda(F_{n+4}) \in \{e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4 + e_2 + e_3\}$. In this case, the values of λ restricted to F_{n+3} and F_{n+4} have eight possible choices. Thus, $|B_0^{0,0}(I^2 \times P_n)| = 8|B_0^0(I^2 \times P_{n-2})|$. Take an orientable coloring λ in $B_0^{0,1}(I^2 \times P_n)$. Then, $\lambda(F_{n+3}) = e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4 + e_2 + e_3$. If we fix any value of $\lambda(F_{n+3})$, then $\lambda(F_{n+4})$ has only two possible values. Thus, $|B_0^{0,1}(I^2 \times P_n)| = 2|B_0^0(I^2 \times P_{n-1})|$. Furthermore, we have that

$$|B_0^0(I^2 \times P_n)| = 2|B_0^0(I^2 \times P_{n-1})| + 8|B_0^0(I^2 \times P_{n-2})|.$$

A direct observation shows that $|B_0^0(I^2 \times P_2)| = 1$ and $|B_0^0(I^2 \times P_3)| = 2$. Thus, $|B_0^0(I^2 \times P_n)| = a(n-1)$.

Case 2. Calculation of $|B_0^1(I^2 \times P_n)|$.

Set $B_0^{1,0}(I^2 \times P_n) = \{\lambda \in B_0^1(I^2 \times P_n) | \lambda(F_{n+3}) = e_3\}$ and $B_0^{1,1}(I^2 \times P_n) = B_0^1(I^2 \times P_n) - B_0^{1,0}(I^2 \times P_n)$. Take an orientable coloring λ in $B_0^{1,0}(I^2 \times P_n)$. Then, $\lambda(F_{n+2}), \lambda(F_{n+4}) \in \{e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4 + e_2 + e_3\}$, so $|B_0^{1,0}(I^2 \times P_n)| = 4|B_0^1(I^2 \times P_{n-2})|$. Take an orientable coloring λ in $B_0^{1,1}(I^2 \times P_n)$. Then, $\lambda(F_{n+3}) = e_4, e_4 + e_1 + e_2, e_4 + e_1 + e_3, e_4 + e_2 + e_3$. However, $\lambda(F_{n+4})$ has only one possible value whichever of the four possible values of $\lambda(F_{n+3})$ is chosen. Thus, $|B_0^{1,1}(I^2 \times P_n)| = |B_0^1(I^2 \times P_{n-1})|$. We easily determine that $|B_0^1(I^2 \times P_2)| = |B_0^1(I^2 \times P_3)| = 1$. Thus, $|B_0^1(I^2 \times P_n)| = b(n-1)$.

Case 3. Calculation of $|B_0^2(I^2 \times P_n)|$.

If we interchange e_3 and e_4, then the problem is reduced to Case 2. Thus, $|B_0^2(I^2 \times P_n)| = b(n-1)$.

Case 4. Calculation of $|B_0^3(I^2 \times P_n)|$.

In this case, $\lambda(F_{n+4}) = e_4, e_4 + e_1 + e_3$. Set $B_0^{3,0}(I^2 \times P_n) = \{\lambda \in B_0^3(I^2 \times P_n) | \lambda(F_{n+3}) = e_3\}$, $B_0^{3,1}(I^2 \times P_n) = \{\lambda \in B_0^3(I^2 \times P_n) | \lambda(F_{n+3}) = e_4, e_4 + e_1 + e_3\}$, and $B_0^{3,2}(I^2 \times P_n) = \{\lambda \in B_0^3(I^2 \times P_n) | \lambda(F_{n+3}) = e_1 + e_2 + e_3, e_1 + e_2 + e_4\}$.

Then, $|B_0^3(I^2\times P_n)| = |B_0^{3,0}(I^2\times P_n)| + |B_0^{3,1}(I^2\times P_n)| + |B_0^{3,2}(I^2\times P_n)|$. An easy argument shows that $|B_0^{3,0}(I^2\times P_n)| = 2|B_0^3(I^2\times P_{n-2})|$ and $|B_0^{3,1}(I^2\times P_n)| = |B_0^3(I^2\times P_{n-1})|$. Thus,

(2) $\quad |B_0^3(I^2\times P_n)| = |B_0^3(I^2\times P_{n-1})| + 2|B_0^3(I^2\times P_{n-2})| + |B_0^{3,2}(I^2\times P_n)|.$

Set $B(n) = \{\lambda\in B_0^{3,2}(I^2\times P_n)|\lambda(F_{n+2}) = e_1+e_3+e_4\}$. Then,

(3) $\quad |B_0^{3,2}(I^2\times P_n)| = |B_0^{3,2}(I^2\times P_{n-1})| + |B(n)|$

and

(4) $\quad |B(n)| = 2|B_0^3(I^2\times P_{n-4})| + 2|B_0^3(I^2\times P_{n-5})| + |B(n-2)| + 2|B_0^{3,2}(I^2\times P_{n-2})|.$

Combining Eqs. (2), (3) and (4), we obtain

$$|B_0^3(I^2\times P_n)| = 2|B_0^3(I^2\times P_{n-1})| + 4|B_0^3(I^2\times P_{n-2})| - 6|B_0^3(I^2\times P_{n-3})| - 3|B_0^3(I^2\times P_{n-4})| + 4|B_0^3(I^2\times P_{n-5})|.$$

A direct observation shows that $|B_0^3(I^2\times P_2)| = |B_0^3(I^2\times P_3)| = 1, |B_0^3(I^2\times P_4)| = 3, |B_0^3(I^2\times P_5)| = 7$, and $|B_0^3(I^2\times P_6)| = 17$. Thus, $|B_0^3(I^2\times P_n)| = c(n-1)$.

Case 5. Calculation of $|B_0^4(I^2\times P_n)|$.

If we interchange e_1 and e_2, then the problem is reduced to Case 4; thus, $|B_0^4(I^2\times P_n)| = c(n-1)$.

Case 6. Calculation of $|B_0^5(I^2\times P_n)|$.

In this case, $\lambda(F_7) = e_3, \lambda(F_8) = e_4, \cdots, \lambda(F_{7+2i}) = e_3, \lambda(F_{7+2i+1}) = e_4, \cdots$. Thus, $|B_0^5(I^2\times P_n)| = \frac{1+(-1)^n}{2}$.

Case 7. Calculation of $|B_0^6(I^2\times P_n)|$.

Similar to Case 6, we have $|B_0^6(I^2\times P_n)| = \frac{1+(-1)^n}{2}$.

Case 8. Calculation of $|B_0^7(I^2\times P_n)|$.

Similar to Case 6, we have $|B_0^7(I^2\times P_n)| = \frac{1+(-1)^n}{2}$.

Thus, $|B_0(I^2\times P_n)| = a(n-1) + 2b(n-1) + 2c(n-1) + 3\cdot\frac{1+(-1)^n}{2}$.

(II) Calculation of $|B_1(I^2\times P_n)|$.

In this case, no matter which value of $\lambda(F_3)$ is chosen, $\lambda(F_4) = e_2, e_2+e_3+e_4$. Write

$B_1^0(I^2\times P_n) = \{\lambda\in B(I^2\times P_n)|\lambda(F_3) = e_1+e_2+e_3, \lambda(F_4) = e_2\}$,

$B_1^1(I^2\times P_n) = \{\lambda\in B(I^2\times P_n)|\lambda(F_3) = e_1+e_2+e_3, \lambda(F_4) = e_2+e_3+e_4\}$,

$B_1^2(I^2\times P_n) = \{\lambda\in B(I^2\times P_n)|\lambda(F_3) = e_1+e_2+e_4, \lambda(F_4) = e_2\}$,

$B_1^3(I^2\times P_n) = \{\lambda\in B(I^2\times P_n)|\lambda(F_3) = e_1+e_2+e_4, \lambda(F_4) = e_2+e_3+e_4\}$.

By the definition of $B_1(I^2\times P_n)$ and Remark 2.3, we have $|B_1(I^2\times P_n)| = \sum_{i=0}^{3}|B_1^i(I^2\times P_n)|$. Then, our argument is divided into the following cases.

Case 1. Calculation of $|B_1^0(I^2\times P_n)|$.

If we interchange e_1 and e_2, then the problem is reduced to Case 2 in (I); thus, $|B_1^0(I^2 \times P_n)| = b(n-1)$.

Case 2. Calculation of $|B_1^1(I^2 \times P_n)|$.

Similar to Case 6 in (I), we have $|B_1^1(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Case 3. Calculation of $|B_1^2(I^2 \times P_n)|$.

If we interchange e_1 and e_2, then the problem is reduced to Case 3 in (I); thus $|B_1^2(I^2 \times P_n)| = b(n-1)$.

Case 4. Calculation of $|B_1^3(I^2 \times P_n)|$.

Similar to Case 6 in (I), we have $|B_1^3(I^2 \times P_n)| = \frac{1+(-1)^n}{2}$.

Thus, $|B_1(I^2 \times P_n)| = 2b(n-1) + 1 + (-1)^n$. □

3.2. Remark. By using the above method, we prove that

$$|O(P_2 \times P_n)| = \prod_{k=1}^{4} \left(2^4 - 2^{k-1}\right) \cdot a(n-1).$$

Based on Theorem 3.1, we know that the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$ is $a(n-1) + 4b(n-1) + 2c(n-1) + 5 \cdot \frac{1+(-1)^n}{2}$.

4. Number of equivariant homeomorphism classes

In this section, we determine the number of equivariant homeomorphism classes of all orientable small covers over $I^2 \times P_n$.

Let φ denote the Euler's totient function, i.e., $\varphi(1) = 1$, $\varphi(N)$ for a positive integer N $(N \geq 2)$ is the number of positive integers both less than N and coprime to N. We have

4.1. Theorem. *Let $E_o(I^2 \times P_n)$ denote the number of equivariant homeomorphism classes of orientable small covers over $I^2 \times P_n$. Then, $E_o(I^2 \times P_n)$ is equal to*

(1) $\frac{1}{16n}\{\sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|] + 40320 \sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[a(t'-1) + 2b(t'-1) + c(t'-1)]\}$ *for n odd,*

(2) $\frac{1}{16n}\{\sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|] + 40320 \sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[a(t'-1) + 2b(t'-1) + c(t'-1)] + 40320n[\tilde{a}(n) + \tilde{c}(n) + \tilde{d}(n) + \tilde{e}(n) + \frac{5}{4}]\}$ *for n even and $n \neq 4$,*

(3) *12180 for $n = 4$,*

where $\tilde{a}(j), \tilde{b}(j), \tilde{c}(j), \tilde{d}(j)$, and $\tilde{e}(j)$ are defined as follows

$$\tilde{a}(j) = \begin{cases} 0, & j \text{ odd}, \\ 1, & j = 2, \\ 4, & j = 4, \\ 2\tilde{a}(j-2) + 8\tilde{a}(j-4), & j \text{ even and } j \geq 6, \end{cases}$$

$$\tilde{b}(j)=\begin{cases}4, & j=6,\\ 8, & j=8,\\ \tilde{b}(j-2)+4\tilde{b}(j-4), & j\ even\ and\ j\geq 10,\\ 0, & otherwise,\end{cases}$$

$$\tilde{c}(j)=\begin{cases}0, & j\ \ odd,\\ 1, & j=2,\\ 2, & j=4,\\ 6, & j=6,\\ \tilde{b}(j)+\tilde{b}(j-2)+\tilde{c}(j-4), & j\ even\ and\ j\geq 8,\end{cases}$$

$$\tilde{d}(j)=\begin{cases}0, & j\ \ odd,\\ 1, & j=2,\\ 4, & j=4,\\ \tilde{d}(j-2)+4\tilde{d}(j-4), & j\ even\ and\ j\geq 6,\end{cases}$$

and

$$\tilde{e}(j)=\begin{cases}0, & j\ \ odd,\\ 1, & j=2,\\ 2, & j=4,\\ 6, & j=6,\\ 14, & j=8,\\ 38, & j=10,\\ 2\tilde{e}(j-2)+4\tilde{e}(j-4)-6\tilde{e}(j-6)-3\tilde{e}(j-8)+4\tilde{e}(j-10), & \\ & j\ even\ and\ j\geq 12.\end{cases}$$

Proof. Based on Theorem 2.5, Burnside Lemma, and Lemma 2.6, we have

$$E_o(I^2\times P_n)=\begin{cases}\frac{1}{16n}\sum_{g\in Aut(\mathcal{F}(I^2\times P_n))}|\Lambda_g|, & n\neq 4,\\ \frac{1}{384}\sum_{g\in Aut(\mathcal{F}(I^4))}|\Lambda_g|, & n=4,\end{cases}$$

where $\Lambda_g=\{\lambda\in O(I^2\times P_n)|\lambda=\lambda\circ g\}$.

The argument is divided into three cases: (I) n odd, (II) n even and $n\neq 4$, (III) $n=4$.

(I) n odd

Given that n is odd, by Remark 2.7, each automorphism g of $Aut(\mathcal{F}(I^2\times P_n))$ can be written as $x^u y^v x'^{u'} y'^{v'}$.

Case 1. $g=x^u x'^{u'}$.

Let $t = gcd(u, 4)$ (the greatest common divisor of u and 4) and $t' = gcd(u', n)$. Then all facets of $\mathcal{F}'$ are divided into t orbits under the action of g, and each orbit contains $\frac{4}{t}$ facets. Thus, each orientable coloring of Λ_g gives the same coloring on all $\frac{4}{t}$ facets of each orbit. Similarly, all facets of $\mathcal{F}''$ are divided into t' orbits under the action of g, and each orbit contains $\frac{n}{t'}$ facets. Thus, each orientable coloring of Λ_g gives the same coloring on all $\frac{n}{t'}$ facets of each orbit. Hence, if $t \neq 1$ and $t' \neq 1$, then $|\Lambda_g| = |O(P_t \times P_{t'})|$. If t=1 (or $t' = 1$), then all facets of $\mathcal{F}'$ (or $\mathcal{F}''$) have the same coloring, which is impossible by the definition of orientable colorings. For every $t > 1$, there are exactly $\varphi(\frac{4}{t})$ automorphisms of the form x^u, each of which divides all facets of $\mathcal{F}'$ into t orbits. Similarly, for every $t' > 1$, there are exactly $\varphi(\frac{n}{t'})$ automorphisms of the form $x'^{u'}$, each of which divides all facets of $\mathcal{F}''$ into t' orbits. Thus, when $g = x^u x'^{u'}$,

$$\begin{aligned}\sum_{g=x^u x'^{u'}} |\Lambda_g| &= \sum_{t,t'>1,t|4,t'|n} \varphi(\tfrac{4}{t})\varphi(\tfrac{n}{t'})|O(P_t \times P_{t'}| \\ &= \sum_{t'>1,t'|n} \varphi(\tfrac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|].\end{aligned}$$

Case 2. $g = x^u x'^{u'} y'$ or $x^u y x'^{u'} y'$.

Given that n is odd, each automorphism always gives an interchange between two neighborly facets of $\mathcal{F}''$. Thus, the two neighborly facets have the same coloring, which contradicts the definition of orientable colorings. Hence, Λ_g is empty.

Case 3. $g = x^u y x'^{u'}$ with u even.

Let $l = \frac{4-u}{2}$. Such an automorphism gives an interchange between two neighborly facets F_l and F_{l+1}. Hence, both facets F_l and F_{l+1} have the same coloring, which contradicts the definition of orientable colorings. Thus, in this case Λ_g is also empty.

Case 4. $g = x^u y x'^{u'}$ with u odd.

Let $t' = gcd(u', n)$. All facets of $\mathcal{F}''$ are divided into t' orbits under the action of g, and each orbit contains $\frac{n}{t'}$ facets. Hence, each orientable coloring of Λ_g gives the same coloring on all $\frac{n}{t'}$ facets of each orbit. If we choose an arbitrary facet from each orbit, it suffices to color t' chosen facets for $\mathcal{F}''$. Moreover, given that each automorphism $g = x^u y x'^{u'}$ contains y as its factor and u is odd, it suffices to color only three neighborly facets of $\mathcal{F}'$ for $\mathcal{F}'$. In fact, it suffices to consider the case $g = xyx'^{u'}$ because there is no essential difference between this case and other cases. Based on the argument of Theorem 3.1, we have

$$|\Lambda_g| = 20160[a(t'-1) + 2b(t'-1) + c(t'-1)],$$

where $a(t'-1), b(t'-1)$ and $c(t'-1)$ are stated as in Theorem 3.1. Given that u is odd and $u \in \mathbb{Z}_4$, u=1, 3. For every $t' > 1$, there are exactly $\varphi(\frac{n}{t'})$ automorphisms of the form $x'^{u'}$, each of which divides all facets of $\mathcal{F}''$ into t' orbits. Thus, when $g = x^u y x'^{u'}$,

$$\sum_{g=x^u y x'^{u'}} |\Lambda_g| = 2 \sum_{t'>1,t'|n} \varphi(\tfrac{n}{t'}) 20160[a(t'-1) + 2b(t'-1) + c(t'-1)].$$

Combining Cases 1 to 4, we complete the proof in (I).

(II) n **even and** $n \neq 4$

Given that $n \neq 4$, by Remark 2.7, each automorphism g of $Aut(\mathcal{F}(I^2 \times P_n))$ can be written as $x^u y^v x'^{u'} y'^{v'}$.

Case 1. $g = x^u x'^{u'}$.

Similar to Case 1 in (I), we have $\sum_{g=x^u x'^{u'}} |\Lambda_g| = \sum_{t'>1, t'|n} \varphi(\frac{n}{t'})[|O(P_2 \times P_{t'})| + |O(P_4 \times P_{t'})|]$.

Case 2. $g = x^u y x'^{u'}$ with u even.

Similar to Case 3 in (I), Λ_g is empty.

Case 3. $g = x^u y x'^{u'}$ with u odd.

Similar to Case 4 in (I), $\sum_{g=x^u y x'^{u'}} |\Lambda_g| = 2 \sum_{t'>1, t'|n} \varphi(\frac{n}{t'}) 20160[a(t'-1) + 2b(t'-1) + c(t'-1)]$.

Case 4. $g = x^u x'^{u'} y'$ with u' even.

Similar to Case 3 in (I), Λ_g is also empty.

Case 5. $g = x^u x'^{u'} y'$ with u' odd.

Let $t = gcd(u, 4)$. Then, all facets of $\mathcal{F}'$ are divided into t orbits under the action of g, and each orbit contains $\frac{4}{t}$ facets. Thus, each orientable coloring of Λ_g gives the same coloring on all $\frac{4}{t}$ facets of each orbit. If we choose an arbitrary facet from each orbit, it suffices to color t chosen facets for $\mathcal{F}'$. When t=1 (i.e., u=1, 3), all facets of $\mathcal{F}'$ have the same coloring, which is impossible by the definition of orientable colorings. Moreover, given that each automorphism $g = x^u x'^{u'} y'$ contains y' as its factor and u' is odd, it suffices to color only $\frac{n}{2} + 1$ neighborly facets of $\mathcal{F}''$ for $\mathcal{F}''$. First, we consider the case t=4 (i.e., u=4).

The argument of Theorem 3.1 can still be carried out. It suffices to consider the case $g = x'y'$ because no essential difference exists between this case and other cases. Set

$$C(n) = \{\lambda \in \Lambda_g | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \lambda(F_5) = e_3, \lambda(F_6) = e_4\}.$$

We have $|\Lambda_g| = 20160|C(n)|$. Write

$C_0(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, e_1 + e_3 + e_4\}$,

$C_1(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_3, e_1 + e_2 + e_4\}$.

Based on the definition of $B(P^n)$ and Remark 2.3, we have $|C(n)| = |C_0(n)| + |C_1(n)|$. Next, we calculate $|C_0(n)|$ and $|C_1(n)|$.

(5.1). Calculation of $|C_0(n)|$.

Write

$C_0^0(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2\}$,

$C_0^1(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_3\}$,
$C_0^2(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_1 + e_4\}$,
$C_0^3(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1, \lambda(F_4) = e_2 + e_3 + e_4\}$,
$C_0^4(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2\}$,
$C_0^5(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_3\}$,
$C_0^6(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_1 + e_4\}$,
$C_0^7(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_3 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}$.

By the definition of $C_0(n)$ and Remark 2.3, we have $|C_0(n)| = \sum_{i=0}^{7} |C_0^i(n)|$. Then, our argument proceeds as follows.

(5.1.1). Calculation of $|C_0^0(n)|$.

Using a similar argument of Case 1 in (I) of Theorem 3.1, we have $|C_0^0(n)| = 2|C_0^0(n-2)| + 8|C_0^0(n-4)|$ with initial values of $|C_0^0(2)| = 1$ and $|C_0^0(4)| = 4$. Thus, $|C_0^0(n)| = \tilde{a}(n)$, where $\tilde{a}(n)$ is stated in Theorem 4.1.

(5.1.2). Calculation of $|C_0^1(n)|$.

In this case, $\lambda(F_7) = e_3, e_3+e_1+e_4$. Set $C_0^{1,0}(n) = \{\lambda \in C_0^1(n) | \lambda(F_7) = e_3\}$ and $C_0^{1,1}(n) = C_0^1(n) - C_0^{1,0}(n)$. Using a similar argument of Case 2 in (I) of Theorem 3.1, when $n \geq 10$, $|C_0^{1,0}(n)| = |C_0^{1,0}(n-2)| + 4|C_0^{1,0}(n-4)|$ with initial values of $|C_0^{1,0}(6)| = 4$ and $|C_0^{1,0}(8)| = 8$. Thus, $|C_0^{1,0}(n)| = \tilde{b}(n)$ for $n \geq 6$, where $\tilde{b}(n)$ is stated in Theorem 4.1.

Take an orientable coloring λ in $C_0^{1,1}(n)$. Then $\lambda(F_8) = e_3, e_4$ and $|C_0^{1,1}(n)| = \tilde{b}(n-2) + |C_0^1(n-4)|$ for $n \geq 8$. Therefore, when $n \geq 8, |C_0^1(n)| = \tilde{b}(n) + \tilde{b}(n-2) + |C_0^1(n-4)|$ with initial values of $|C_0^1(2)| = 1, |C_0^1(4)| = 2$ and $|C_0^1(6)| = 6$. Thus, $|C_0^1(n)| = \tilde{c}(n)$.

(5.1.3). Calculation of $|C_0^2(n)|$.

Similar to Case 2 in (I) of Theorem 3.1, we have $|C_0^2(n)| = |C_0^2(n-2)| + 4|C_0^2(n-4)|$ with initial values of $|C_0^2(2)| = 1$ and $|C_0^2(4)| = 4$. Thus, $|C_0^2(n)| = \tilde{d}(n)$.

(5.1.4). Calculation of $|C_0^3(n)|$.

Similar to Case 4 in (I) of Theorem 3.1, we have $|C_0^3(n)| = 2|C_0^3(n-2)| + 4|C_0^3(n-4)| - 6|C_0^3(n-6)| - 3|C_0^3(n-8)| + 4|C_0^3(n-10)|$. A direct observation shows that $|C_0^3(2)| = 1, |C_0^3(4)| = 2, |C_0^3(6)| = 6, |C_0^3(8)| = 14$, and $|C_0^3(10)| = 38$. Thus, $|C_0^3(n)| = \tilde{e}(n)$.

(5.1.5). Calculation of $|C_0^4(n)|$.

If we interchange e_1 and e_2, then the problem is reduced to (5.1.4). Thus, $|C_0^4(n)| = \tilde{e}(n)$.

(5.1.6). Calculation of $|C_0^5(n)|$.

In this case, $\lambda(F_7) = e_3, \lambda(F_8) = e_4, \cdots, \lambda(F_{\frac{n+10}{2}}) = \begin{cases} e_3, & n = 4k, \\ e_4, & n = 4k+2. \end{cases}$

Thus, $|C_0^5(n)| = 1$.

(5.1.7). Calculation of $|C_0^6(n)|$.

Similar to (5.1.6), $|C_0^6(n)| = 1$.

(5.1.8). Calculation of $|C_0^7(n)|$.

Similar to (5.1.6), $|C_0^7(n)| = 1$.

Thus, $|C_0(n)| = \tilde{a}(n) + \tilde{c}(n) + \tilde{d}(n) + 2\tilde{e}(n) + 3$.

(5.2). Calculation of $|C_1(n)|$.

Set

$C_1^0(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_3, \lambda(F_4) = e_2\}$,
$C_1^1(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_3, \lambda(F_4) = e_2 + e_3 + e_4\}$,
$C_1^2(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_4, \lambda(F_4) = e_2\}$,
$C_1^3(n) = \{\lambda \in C(n) | \lambda(F_3) = e_1 + e_2 + e_4, \lambda(F_4) = e_2 + e_3 + e_4\}$.

Based on the definition of $C_1(n)$ and Remark 2.3, we have $|C_1(n)| = \sum_{i=0}^{3} |C_1^i(n)|$. Then, the argument proceeds as follows.

(5.2.1). Calculation of $|C_1^0(n)|$.

If we interchange e_1 and e_2, then the problem is reduced to (5.1.2). Thus, $|C_1^0(n)| = \tilde{c}(n)$.

(5.2.2). Calculation of $|C_1^1(n)|$.

Similar to (5.1.6), $|C_1^1(n)| = 1$.

(5.2.3). Calculation of $|C_1^2(n)|$.

If we interchange e_1 and e_2, then the problem is reduced to (5.1.3). Thus, $|C_1^2(n)| = \tilde{d}(n)$.

(5.2.4). Calculation of $|C_1^3(n)|$.

Similar to (5.1.6), $|C_1^3(n)| = 1$.

Thus, $|C_1(n)| = \tilde{c}(n) + \tilde{d}(n) + 2$.

Hence, the number of all orientable colorings in Λ_g is just

$$|\Lambda_g| = 20160[\tilde{a}(n) + 2\tilde{c}(n) + 2\tilde{d}(n) + 2\tilde{e}(n) + 5].$$

There are exactly $\frac{n}{2}$ such automorphisms $g = x'^{u'}y'$ because n is even and u' is odd. Thus,

$$\sum_{g=x'^{u'}y'} |\Lambda_g| = 20160 \cdot \frac{n}{2}[\tilde{a}(n) + 2\tilde{c}(n) + 2\tilde{d}(n) + 2\tilde{e}(n) + 5].$$

When t=2 (i.e., u=2), we have

$$\sum_{g=x^2x'^{u'}y'} |\Lambda_g| = 20160 \cdot \frac{n}{2}\tilde{a}(n).$$

Thus, $\sum\limits_{g=x^u x'^{u'} y'} |\Lambda_g| = 20160[n\tilde{a}(n) + n\tilde{c}(n) + n\tilde{d}(n) + n\tilde{e}(n) + \frac{5}{2}n]$.

Case 6. $g = x^u y x'^{u'} y'$ with u even or u' even.

Similar to Case 3 in (I), Λ_g is empty.

Case 7. $g = x^u y x'^{u'} y'$ with u odd and u' odd.

Similar to Case 5 , we have

$$\sum_{g=x^u y x'^{u'} y'} |\Lambda_g| = 20160n[\tilde{a}(n) + \tilde{c}(n) + \tilde{d}(n) + \tilde{e}(n)].$$

Combining Cases 1 to 7, we complete the proof in (II).

(III) n**=4**

When n=4, $I^2 \times P_n$ is a 4-cube I^4, and the automorphism group $Aut(\mathcal{F}(I^4))$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_4$. As before, let χ_1, χ_2, χ_3, and χ_4 denote generators of the first subgroup $\mathbb{Z}_2$, the second subgroup $\mathbb{Z}_2$, the third subgroup $\mathbb{Z}_2$, and the fourth subgroup $\mathbb{Z}_2$ of $Aut(\mathcal{F}(I^4))$ respectively. If $g = \chi_1$ and $\lambda \in \Lambda_g$, then $\lambda(F_1) = \lambda(F_3)$. Based on Theorem 3.1, we have $|\Lambda_g| = 20160[a(3) + 2b(3) + c(3)]$. Similarly, we also have $|\Lambda_g| = 20160[a(3) + 2b(3) + c(3)]$ for $g = \chi_2$, χ_3 or χ_4. If g=$\chi_1\chi_2$ and $\lambda \in \Lambda_g$, then $\lambda(F_1) = \lambda(F_3)$ and $\lambda(F_2) = \lambda(F_4)$. Based on Case 1 in (I) of Theorem 3.1, we obtain $|\Lambda_g| = 20160a(3)$. Similarly, we also obtain $|\Lambda_g| = 20160a(3)$ for $g = \chi_1\chi_3, \chi_1\chi_4, \chi_2\chi_3, \chi_2\chi_4$ or $\chi_3\chi_4$. If g=$\chi_1\chi_2\chi_3$ and $\lambda \in \Lambda_g$, then $\lambda(F_i) = \lambda(F_{i+2})$ for $i = 1, 2, 5$. We obtain $|\Lambda_g| = 20160 \cdot 4$. Similarly, we also obtain $|\Lambda_g| = 20160 \cdot 4$ for $g = \chi_1\chi_2\chi_4, \chi_1\chi_3\chi_4$ or $\chi_2\chi_3\chi_4$. If g=$\chi_1\chi_2\chi_3\chi_4$ and $\lambda \in \Lambda_g$, then $\lambda(F_i) = \lambda(F_{i+2})$ for $i = 1, 2, 5, 6$. We obtain $|\Lambda_g| = 20160$. Thus

$$\begin{aligned} E_o(I^4) = \tfrac{1}{384}\{&20160 \cdot 4[a(3) + 2b(3) + c(3)] + 20160 \cdot 6a(3) + 20160 \cdot 16 + 20160+ \\ &20160[a(3) + 4b(3) + 2c(3) + 5]\} \\ = 12180.& \end{aligned}$$

□

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