NEW COMMON FIXED POINT
THEOREMS OF GREGUŠ TYPE
FOR R-WEAKLY COMMUTING
MAPPINGS IN 2-METRIC SPACES

P. P. Murthy* and Kenan Taş†‡

Received 06 : 03 : 2009 : Accepted 07 : 10 : 2009

Abstract
In this paper we extend and generalize a theorem of M. R. Singh, L. S. Singh and P. P. Murthy (Common fixed points of set valued mappings, Int. J. Math. Sci., 25 (6), 411–415, 2001) in a 2-metric space with a Greguš type condition, and give some common fixed point theorems of set-valued maps in 2-metric spaces.

Keywords: Contraction, Fixed point, Greguš condition, 2-metric space.

2000 AMS Classification: 54H25.

1. Introduction
The concept of 2-metric spaces was introduced and studied initially by Gahler [7, 8, 9]. After Gahler there was a flood of new results obtained by many authors in these spaces [3, 11, 12, 13, 15]. Military applications of fixed point theory in 2-metric spaces can be found, as well as applications in Medicine and Economics [1, 2, 18].

Dhage [4] introduced the concept of $D$-metric space as follows:
Let $X$ be a non-empty set and $\mathbb{R}^+$ the set of non-negative real numbers. If the real-valued mapping $D : X \times X \times X \to \mathbb{R}^+$ satisfies the following properties:

$(D_1)$ $D(x_1, x_2, x_3) \geq 0$ for every $x_1, x_2, x_3 \in X$ and $D(x_1, x_2, x_3) = 0$ if and only if $x_1 = x_2 = x_3$;

$(D_2)$ $D(x_1, x_2, x_3) = D(x_1, x_3, x_2) = D(x_3, x_2, x_1) = D(x_2, x_1, x_3) = D(x_3, x_1, x_2) = D(x_2, x_3, x_1)$ (symmetric) for all $x_1, x_2, x_3 \in X$;

*Department of Pure and Applied Mathematics, Guru Ghasidas University, Koni, Bilaspur (C.G.), 495 009, India. E-mail: ppmurthy@gmail.com

†Çankaya University, Department of Mathematics and Computer Science, Ankara, Turkey. E-mail: kenan@cankaya.edu.tr

‡Corresponding Author.
(D3) \( D(x_1, x_2, x_3) \leq d(x_1, x_2, u) + d(x_1, u, x_3) + d(u, x_2, x_3) \) for all \( x_1, x_2, x_3, u \in X \) (rectangle inequality),

then the pair \((X, D)\) is called a D-metric space.

Gahler defined a 2-metric space as follows:

A 2-metric on a set \( X \) with at least three points is a non-negative real-valued mapping \( d: X \times X \times X \to \mathbb{R}^+ \) satisfying the following properties:

\[(G1)\] To each pair of points \( a, b \) with \( a \neq b \) in \( X \) there is a point \( c \in X \) such that \( d(a, b, c) \neq 0; \)

\[(G2)\] \( d(a, b, c) = 0 \), if at least two of the points are equal;

\[(G3)\] \( d(a, b, c) = d(b, c, a) = d(a, c, b); \)

\[(G4)\] \( d(a, b, c) \leq d(a, b, u) + d(a, u, c) + d(u, b, c) \) for all \( a, b, c, u \in X \).

The pair \((X, d)\) is then called a 2-metric space.

Geometrically the value of a 2-metric \( d(x, y, c) \) represents the area of a triangle with vertices \( x, y \) and \( c \), whereas, the value of a D-metric \( D(x, y, c) \) represents the perimeter of the triangle with vertices \( x, y \) and \( c \).

Throughout this note \((X, D)\) stands for a D-metric space, \((X, d)\) is a 2-metric space and \( B(X) \) the class of all non-empty bounded subsets of \( X \).

Let \( A, B, C \) be non-empty sets in \( B(X) \). We define

\[
\delta(A, B, C) = \sup\{d(a, b, c) : a \in A, b \in B, c \in C\}
\]

\[
D(A, B, C) = \inf\{d(a, b, c) : a \in A, b \in B, c \in C\}.
\]

If \( A \) is a singleton set, then \( \delta(A, B, C) = \delta(a, B, C) \). In case \( B \) and \( C \) are also singleton sets, then

\[
\delta(A, B, C) = D(A, B, C) = d(a, b, c)
\]

for every \( A = \{a\}, B = \{b\}, C = \{c\} \). From the definition of \( \delta \) we can say that,

\[
\delta(A, B, C) = \delta(A, C, B) = \delta(C, A, B) = \delta(B, C, A) = \delta(C, B, A) = \delta(B, A, C) \geq 0.
\]

Also,

\[
\delta(A, B, C) \leq \delta(A, B, E) = \delta(A, E, C) = \delta(E, B, C);
\]

for all \( A, B, C, E \in B(X) \). Let us note that \( \delta(A, B, C) = 0 \) if at least two of \( A, B \) and \( C \) are equal singleton sets.

We need the following definitions and lemmas for our main theorems:

1.1. **Definition.** A sequence \( \{A_n\}_{n=1}^\infty \) of subsets of \( X \) is said to be convergent to a subset \( A \) of \( X \) if:

i. Given \( a \in A \), there is a sequence \( \{a_n\} \) of \( X \) such that \( a_n \in A_n \) for \( n = 1, 2, 3, \ldots \) and \( \lim_{n \to \infty} d(a_n, a, c) = 0 \).

ii. Given \( \epsilon > 0 \), there exists a positive integer \( n_0 \) such that \( A_n \subseteq A_\epsilon \) for every \( n > n_0 \), where \( A_\epsilon \) is the union of all open spheres with centers in \( A \) and radius \( \epsilon \).

1.2. **Definition.** [1] Let \( G : X \to X \) and \( F : X \to B(X) \). Then the pair \( \{G, F\} \) is said to be weakly commuting if \( GFx \in B(X) \) and

\[
\delta(GFx, GFx, C) \leq \max\{\delta(GFx, Fx, C), \delta(GFx, GFx, C)\}
\]

for every \( x \in X \) and \( C \in B(X) \).
1.3. Definition. [1] Let $G : X \to X$ and $F : X \to B(X)$. Then the pair \{$G, F\}$ is said to be \textit{R-weakly commuting} if

$$
\delta(FGx, GFx, C) \leq R \cdot \max\{\delta(Gx, Fx, C), \delta(GFx, GFx, C)\}
$$

for every $x \in X$, $C \in B(X)$ and $R > 0$.

1.4. Remark. If $F$ is a single valued function, then Definitions 1.2 and 1.3 reduce to the following:

$$
\delta(FGx, GFx, C) = d(FGx, GFx, C) \leq d(Gx, Fx, C) = \delta(Gx, Fx, C))
$$

and

$$
\delta(FGx, GFx, C) = d(FGx, GFx, C) \leq R \cdot d(Gx, Fx, C) = R \cdot \delta(Gx, Fx, C),
$$

respectively.

In recent years, common fixed points of Greguš [10] type have been proved by Diviccaro, Fisher and Sessa [5], Fisher and Sessa [6], Mukherjee and Verma [14], Murthy, Cho and Fisher [16], M. R. Singh, L. S. Singh and P. P. Murthy [19] under weaker conditions.

In this paper, we have extended and generalized a theorem of M. R. Singh, L. S. Singh and P. P. Murthy [19] in a 2-metric space.

2. Main results

Let $S$ and $T$ be mappings of 2-metric space $(X, d)$ into itself and $A, B : X \to B(X)$ are two set valued mappings satisfying the following conditions:

\begin{align*}
(2.1) & \quad \bigcup A(X) \subset T(X) \text{ and } \bigcup B(X) \subset S(X); \\
(2.2) & \quad \text{For every } x, y \in X, \ C \in B(X) \text{ and } p > 0, \\
& \quad \delta^p(Ax, By, C) \leq \varphi(a \cdot \delta^p(Sx, Ty, C) + (1 - a) \max\{\delta^p(Ax, Sx, C), \delta^p(By, Ty, C), \\
& \quad b \cdot \delta^p(Sx, By, C) + c \cdot \delta^p(Ty, Ax, C)\})
\end{align*}

where $a \in (0, 1)$ and $\varphi : [0, \infty) \to [0, \infty)$ is

(i) non-increasing;
(ii) upper-semi continuous,
(iii) satisfies $\varphi(t) < t$ for every $t > 0$.

Let $x_0$ be an arbitrary point of $X$. Since $\bigcup A(X) \subset T(X)$, then there exists a point $x_1 \in X$ such that $Tx_1 \in Ax_0 = y_0$. Again, since $\bigcup B(X) \subset S(X)$, for the point $x_1 \in X$ we can find a point $x_2 \in X$ such that $Sx_1 \in Bx_0 = y_1$, and so on. Inductively, we can define a sequence $\{x_n\}$ in $X$ such that

\begin{align*}
(2.3) & \quad \begin{cases} \\
Tx_{n+1} \in Ax_n = y_n, \quad \text{when } n \text{ is even} \\
Sx_{n+1} \in Bx_n = y_n, \quad \text{when } n \text{ is odd}
\end{cases}
\end{align*}

Now we are ready to prove the following lemma for our theorem:

2.1. Lemma. Let $(X, d)$ be a 2-metric space. Let $S, T$ be self maps of $X$ and $A, B : X \to B(X)$ satisfying the conditions $(2.1)$ and $(2.2)$. Then for every $n \in \mathbb{N}$ we have

$$
\lim_{n \to \infty} \delta(y_n, y_{n+1}, y_{n+2}) = 0.
$$
Hence we conclude that

Proof. Since

\[ \delta(y_{2n+1}, y_{2n+2}, y_{2n}) = \delta(Ax_{2n+1}, Bx_{2n+1}, y_{2n}) \]

we have

\[
\delta(y_{2n+1}, y_{2n+2}, y_{2n}) \leq \left[ \varphi(a \cdot \delta^p(Sx_{2n+1}, Tx_{2n+1}, y_{2n})) + (1 - a) \cdot \max\{\delta^p(Sx_{2n+1}, Ax_{2n+2}, y_{2n}), \delta^p(Tx_{2n+1}, Bx_{2n+1}, y_{2n})\}, \right.
\]

\[
\left. b \cdot D^p(Sx_{2n+1}, Bx_{2n+1}, y_{2n}) + c \cdot D^p(Tx_{2n+1}, Ax_{2n+2}, y_{2n}) \right]^{\frac{1}{p}},
\]

(2.4)

Again we consider,

\[
\delta(y_{2n+3}, y_{2n+2}, y_{2n+1}) = \delta(Bx_{2n+3}, Ax_{2n+2}, y_{2n+1}) \leq \left[ \varphi(a \cdot \delta^p(Sx_{2n+2}, Tx_{2n+3}, y_{2n+1})) + (1 - a) \cdot \max\{\delta^p(Sx_{2n+2}, Ax_{2n+3}, y_{2n+1}), \delta^p(Tx_{2n+3}, Bx_{2n+1}, y_{2n+1})\}, \right.
\]

\[
\left. b \cdot D^p(Sx_{2n+2}, Bx_{2n+1}, y_{2n}) + c \cdot D^p(Tx_{2n+3}, Ax_{2n+2}, y_{2n}) \right]^{\frac{1}{p}},
\]

(2.5)

(since \(0 < a < 1\)). By the definition of \(\varphi\), this implies

\[ \delta(y_{n+1}, y_{n+2}, y_{n}) \to 0. \]

Hence we conclude that

(2.6)

\[
\lim_{n \to \infty} \delta(y_n, y_{n+1}, y_{n+2}) = 0.
\]

\(\square\)

2.2. Lemma. [1] If \(\{A_n\}\) and \(\{B_n\}\) are sequences in \(B(X)\) converging to \(A\) and \(B\) in \(B(X)\) respectively, then the sequence \(\{\delta(A_n, B_n, C)\}\) converges to \(\{\delta(A, B, C)\}\).

2.3. Theorem. Let \(S\) and \(T\) be mappings of a 2-metric space \((X, d)\) into itself, and \(A, B : X \to B(X)\) two set-valued mappings satisfying the conditions (2.1), (2.2), (2.3), and the following:

(2.6)

\(S(X)\) or \(T(X)\) is a complete subspace of \(X\);

(2.7)

The pairs \(\{A, S\}\) and \(\{B, T\}\) are \(R\)-weakly commuting, then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).
Proof. From Lemma 2.1, the sequence \(\{y_n\}\) is a Cauchy sequence. Assume \(T(X)\) is a complete subspace of \(X\). Since the sequence \(\{x_n\}\) defined by (2.3) is a subsequence, then \(\{T x_{2n+1}\}\) is Cauchy and converges to a point \(z\) in \(T(X)\). Since \(T(X)\) is a complete subspace of \(X\), for some \(u \in X\), \(T x_{2n+1} \to z = T(u)\). By using (2.2), we have

\[
\delta(S x_{2n+2}, T x_{2n+1}, C) \leq \delta(y_{2n+1}, y_{2n}, C).
\]

Letting \(n \to \infty\),

\[
\lim_{n \to \infty} \delta(S x_{2n+2}, T x_{2n+1}, C) \leq \lim_{n \to \infty} \delta(y_{2n+1}, y_{2n}, C) = 0.
\]

The above implies

\[
\lim_{n \to \infty} \delta(S x_{2n+2}, T x_{2n+1}, C) = 0.
\]

Therefore, we get

\[
\lim_{n \to \infty} S x_{2n+2} = \lim_{n \to \infty} T x_{2n+1} = z.
\]

We can also show that

\[
\lim_{n \to \infty} \delta(A x_{2n+2}, z, C) = 0.
\]

Now, we shall show that \(u\) is a coincidence point of \(B\) and \(T\).

For \(n = 0, 1, 2, \ldots\) and using (2.2) we have

\[
\delta^p(A x_{2n}, B u, C) \leq \varphi(a \cdot \delta^p(S x_{2n}, T u, C) + (1 - a) \max\{\delta^p(S x_{2n}, A x_{2n}, C),
\]

\[
\delta^p(T u, B u, C), b \cdot D^p(S x_{2n}, T u, C) + c \cdot D^p(T u, A x_{2n}, C)\}\}.
\]

Now letting \(n \to \infty\), the above inequality implies that

\[
\lim_{n \to \infty} \delta^p(A x_{2n}, B u, C) \leq \varphi((1 - a) \max\{\delta^p(S x_{2n}, A x_{2n}, C),
\]

\[
\delta^p(T u, B u, C), b \cdot D^p(z, B u, C)\}\}
\]

and so

\[
\lim_{n \to \infty} \delta^p(A x_{2n}, B u, C) \leq \varphi(\delta^p(T u, B u, C)) < \delta^p(z, B u, C),
\]

which is a contradiction. Thus \(\{z\} = B u = \{T u\}\).

Since \(\bigcup B(X) \subseteq S(X)\), for some \(v \in X\) we have \(\{S v\} = B u = \{T u\}\).

If \(A v \neq B u\), then we have from (2.2),

\[
\delta^p(\bigcup A v, B u, C) \leq \varphi(a \cdot \delta^p(S v, T u, C) + (1 - a) \max\{\delta^p(\bigcup A v, S v, C), \delta^p(B u, T u, C),
\]

\[
b \cdot D^p(S v, B u, C) + c \cdot D^p(T u, A v, C)\}\},
\]

which implies

\[
\delta^p(\bigcup A v, B u, C) \leq \varphi(a \cdot \delta^p(S v, T u, C) + (1 - a) \max\{\delta^p(\bigcup A v, S v, C), \delta^p(B u, T u, C),
\]

\[
b \cdot \delta^p(S v, B u, C) + c \cdot \delta^p(T u, A v, C)\}.
\]

or equivalently

\[
\delta^p(\bigcup A v, B u, C) \leq \varphi((1 - a) \max\{\delta^p(\bigcup A v, S v, C), c \cdot \delta^p(T u, A v, C)\}).
\]

Since \(0 \leq b + c \leq \frac{1}{4}\), \(0 < a < 1\), \(b, c \geq 0\), we have

\[
\delta^p(\bigcup A v, B u, C) < \varphi((1 - a) \cdot \delta^p(\bigcup A v, S v, C)),
\]

and

\[
\delta^p(\bigcup A v, B u, C) < \delta^p(\bigcup A v, S v, C))
\]

which implies \(\{S v\} = \bigcup A v\). Therefore, \(A v = \{S v\} = \{z\} = \{T u\} = B u\).
Since $Av = \{Sv\} = \{z\}$ and $\{A, S\}$ are $R$-weakly commuting maps, then
\[
\delta(ASv, SAv, C) < R \cdot \max\{d(Av, Sv, C), \delta(SAv, SAv, C)\},
\]
which implies that
\[
ASv = SAv \implies Az = \{Sz\}.
\]
Again, using (2.2),
\[
\delta^p(Az, z, C) \leq \delta^p(Az, Bu, C)
\]
\[
\leq \varphi(a \cdot \delta^p(Sz, Tu, C) + (1 - a) \max\{\delta^p(Az, Sz, C), \delta^p(Bu, Tu, C),
\]
\[
b \cdot D^p(Sz, Bu, C) + c \cdot D^p(Tu, Az, C)\}),
\]
or equivalently
\[
\delta^p(Az, z, C) \leq \varphi(a \cdot \delta^p(Sz, Tu, C) + (1 - a) \max\{\delta^p(Az, Sz, C), \delta^p(Bu, Tu, C),
\]
\[
b \cdot D^p(Sz, Bu, C) + c \cdot D^p(Tu, Az, C)\})
\]
or equivalently
\[
\delta^p(Az, z, C) \leq \varphi(a \cdot \delta^p(Az, z, C) + (1 - a) \max\{0, 0, b \cdot \delta^p(Az, z, C)
\]
\[
+ c \cdot \delta^p(z, Az, C)\})
\]
\[
\leq \varphi(\delta^p(Az, z, C))
\]
\[
\leq \delta^p(Az, z, C),
\]
which is a contradiction. Thus $Az = \{Sz\} = \{z\}$, and $z$ is a common fixed point of $A$ and $S$.

Similarly, we can show that $\{z\}$ is a common fixed point of $B$ and $T$ by assuming
$\{B, T\}$ is a pair of $R$-weakly commuting maps. Hence, $Az = Bz = \{z\} = \{Sz\} = \{Tz\}$.

Now we shall prove that $\{z\}$ is a unique fixed point of $A, B, S, T$. Let $z^*$ be a second fixed point of $A, B, S$ and $T$. Then from (2.2) we have,
\[
d^p(z, z^*, C) \leq \delta^p(Az, Bz^*, C)
\]
\[
\leq \varphi(a \cdot \delta^p(Sz, Tz^*, C) + (1 - a) \max\{\delta^p(Az, Sz, C), \delta^p(Bz^*, Tz^*, C), b \cdot D^p(Sz, Bz^*, C) + c \cdot D^p(Tz^*, Az, C)\})
\]
\[
\leq \varphi(a \cdot \delta^p(Sz, Tz^*, C) + (1 - a) \cdot \max\{\delta^p(Az, Sz, C), \delta^p(Bz^*, Tz^*, C), b \cdot \delta^p(Sz, Bz^*, C) + c \cdot \delta^p(Tz^*, Az, C)\})
\]
\[
\leq \varphi(a \cdot \delta^p(z, z^*, C) + (1 - a) \cdot \delta^p(z, z^*, C))
\]
\[
\leq \delta^p(z, z^*, C)
\]
\[
< d^p(z, z^*, C),
\]
which is a contradiction. Hence we get $z = z^*$.

That means that $z$ is a unique common fixed point of $A, B, S$ and $T$ in $X$, which completes the proof.

\[\square\]

References


New Common Fixed Point Theorems