Squared radial Ornstein-Uhlenbeck processes and inverse Laplace transforms of products of confluent hypergeometric functions

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Abstract
In this paper, we consider the squared radial Ornstein-Uhlenbeck process and associated Kolmogorov backward equation (the Laguerre heat equation). For this process, we obtain the Green function of the Laplace transform of the transition density function in terms of the confluent hypergeometric functions and present new representations for the inverse Laplace transform of the products of confluent hypergeometric functions.

Keywords: Laplace transform, Fourier transform, Squared radial Ornstein-Uhlenbeck process, Laguerre equation, Confluent hypergeometric function.

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1. Introduction
The so-called Ornstein-Uhlenbeck (OU) process \( V_t \) is defined by the following 2-parameter Langevin equation

\[
dV_t = -\lambda V_t dt + \gamma dB_t, \quad \lambda, \gamma > 0,
\]

where \( B_t \) is the \( n \)-dimensional Brownian motion. The OU and associated processes are intensively applied in the study of stochastic processes [2,17,25], astrophysics [10], neurophysiology [16], financial mathematics [9,20,26], and the first passage time problems [18].

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As a derivation of another stochastic process, by setting \( W_t = \|V_t\|^2 \) in the OU process, we can derive the squared radial Ornstein-Uhlenbeck process (SROU) of \( n \)-dimension as follows

\[
(1.2) \quad dW_t = 2\gamma\sqrt{W_t}dB_t + (\gamma^2 n - 2\lambda W_t)dt.
\]

For this type of stochastic process, see [15, 19, 21, 29] for some properties and different applications.

In this paper, we intend to consider the transition distribution as \( P(w,t|w_0) = \Pr\{W(t) \leq w|W(0) = w_0\} \) and study the associated partial differential equation (as Kolmogorov backward equation). This partial differential equation which can be presented as a heat equation with the Laguerre type operator, is derived from SROU process for the transition density function

\[
(1.3) \quad w_0 \frac{\partial^2 p}{\partial w_0^2} + (\alpha + 1 - w_0) \frac{\partial p}{\partial w_0} = \frac{\partial p}{\partial t}, \quad \alpha = \frac{n}{2} - 1,
\]

with an initial condition with respect to the Dirac delta function \( \delta(.) \)

\[
(1.4) \quad p(w, t|w_0) = \delta(w - w_0), \quad t = 0.
\]

The motivation of this paper is to find the Laplace transform of transition density \( p(w, t|w_0) \) and relate it to the confluent hypergeometric functions. In this sense, we present some new representations for the inverse Laplace transforms of the products of confluent hypergeometric functions.

For this purpose, in Section 2 we recall some properties of the confluent hypergeometric functions, and in Section 3 we consider the exponential operators for solving the Kolmogorov backward equation and obtaining its formal solution. Section 4 contains a brief summary of the Green function of Kolmogorov backward equation (the Laguerre heat equation). In Section 5, using the recurrence relations of confluent hypergeometric functions, we show some relations for the inverse Laplace transforms of the products of confluent hypergeometric functions.

### 2. The Confluent hypergeometric functions

The confluent hypergeometric functions (Kummer functions) are given as the independent solutions of Kummer’s equation

\[
(2.1) \quad z \frac{d^2w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0, \quad a, b \in \mathbb{R},
\]

in the following form

\[
(2.2) \quad w = c_1 M(a, b, z) + c_2 U(a, b, z).
\]

The function \( M(a, b, z) \) was introduced by Kummer in the year 1837 [24, p. 322, eq. (13.2.1)]

\[
(2.3) \quad M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \quad (a)_n = a(a + 1)(a + 2) \cdots (a + n - 1), \quad (a)_0 = 1,
\]

and in the year 1947, the function \( U(a, b, z) \) was introduced by Francesco Tricomi

\[
(2.4) \quad U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left[ \frac{M(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-a} M(1 + a - b, 2 - b, z) \right].
\]

At this point, we mention some properties of the confluent hypergeometric functions that are used in the next sections.
i): The Wronskian of the confluent hypergeometric functions is shown by [24, p. 324, eq. (13.2.24)]

\[
W \left\{ M(a, b, z), U(a, b, z) \right\} = -\frac{\Gamma(b)z^{-b}e^z}{\Gamma(a)}.
\]

ii): Two recurrence relations for the confluent hypergeometric function are given by [24, p. 325], [1]

\[
(a + 1)zM(a + 2, b + 2, z) + (b + 1)(b - z)M(a + 1, b + 1, z) - b(b + 1)M(a, b, z) = 0,
\]

\[
(a + 1)zU(a + 2, b + 2, z) + (z - b)U(a + 1, b + 1, z) - U(a, b, z) = 0.
\]

iii): The following relations are derivative and integral formulas for the functions \(M\) and \(U\), [24, pp. 325, 326, 332]

\[
d\frac{dz}{dz}M(a, b, z) = \frac{a}{b}M(a + 1, b + 1, z),
\]

\[
d\frac{dz}{dz}U(a, b, z) = -\frac{a}{b}U(a + 1, b + 1, z),
\]

\[
\int M(a, b, z) = \frac{b - 1}{a - 1}M(a - 1, b - 1, z),
\]

\[
\int U(a, b, z) = -\frac{1}{a - 1}U(a - 1, b - 1, z).
\]

3. The Laguerre heat equation

In this section, we intend to find a formal solution for the Kolmogorov backward equation (1.3). This equation, can be interpreted as the Laguerre heat equation which is derived from the squared radial Ornstein-Uhlenbeck process \(W_t, t \geq 0\) with initial value \(W(0) = w_0\). For this purpose, we recall the definition of exponential operators

\[
\exp \left( \lambda[q(x)\frac{dx}{dx} + v(x)] \right)f(x) = f(x(\lambda))g(\lambda),
\]

where \(x(\lambda)\) and \(g(\lambda)\) satisfy the following system with first-order differential equations as follows

\[
\frac{dx}{dx}x(\lambda) = q(x(\lambda)), \quad x(0) = x,
\]

\[
\frac{dg}{dg}g(\lambda) = v(x(\lambda))g(\lambda), \quad g(0) = 1.
\]

This type of differential operator, has many applications in the different fields of applied mathematics, such as fractional calculus and mathematical physics [3–8], [11–14, 23].

3.1. Theorem. The solution of Kolmogorov backward equation

\[
w_0 \frac{\partial^2 p}{\partial w_0^2} + (\alpha + 1 - w_0) \frac{\partial p}{\partial w_0} = \frac{\partial p}{\partial t}, \quad t > 0, \quad w_0 \in \mathbb{R},
\]

with initial condition

\[p(w, t|w_0) = \delta(w - w_0), \quad t = 0,
\]

is given by

\[
p(w, t|w_0) = \frac{e^{\alpha w}}{\pi} \int_0^{\infty} \frac{e^{-k^2(w_0(1-e^{-t}))}}{e^{-2t+k^2(e^{-t}-1)^2}} \left( e^{-2t} + k^2(e^{-t}-1)^2 \right)^{\frac{\alpha + 1}{2}}
\]

\[\times \cos \left( k(w_0 - \frac{we^{-t}}{e^{-2t+k^2(e^{-t}-1)^2}}) + (\alpha - 1) \arctan(\frac{k(e^{-t}-1)}{e^{-t}-1}) \right) dk.
\]
Proof. The proof will be divided into three steps. First, we recall the definition of Fourier transform of transition density

\[ F(p(w, t|w_0), k) = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikw_0} p(w, t|w_0) dw_0, \]

and apply the Fourier transform on the Laguerre heat equation (1.3), to get

\[ (ik - ik\alpha + 1) F(k) + (ik^2 + k) \frac{\partial}{\partial k} F(k) = F_t. \]

In the second step, we solve the first order differential equation (3.6) and incorporate its initial condition to obtain

\[ F(k) = \frac{1}{\sqrt{2\pi}} e^{(ik - ik\alpha + 1 + ik^2 + k) \frac{\partial}{\partial k}} \left[ e^{-iwk} \right]. \]

Now, according to the functions \( q(x) = ix^2 + x \) and \( v(x) = ix - i\alpha x + 1 \) in equation (3.2), the Fourier transform of the transition density is

\[ F(k) = \mathcal{F}(p(w, t|w_0), k) = \frac{1}{\sqrt{2\pi}} e^{ix \left( ik - ike^{-t} + e^{-t} \right)} \left( \alpha - 1 \right) e^{-iw \left( \frac{k}{\alpha (1 - e^{-t})} \right)}. \]

Finally, by applying the of inverse Fourier transform on the above relation

\[ p(w, t|w_0) = \mathcal{F}^{-1} (F(k), w_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikw_0} F(k) dk, \]

we get the relation (3.4). The graph of this solution has been plotted in Figure 1 for \( w = \alpha = 0 \). \( \square \)
4. The Green function of Laguerre differential equation

We consider the Laplace transform of transition density function \( p(w,t|w_0) \) in the SROU process

\[
\bar{p}(w,s|w_0) = \int_0^{+\infty} e^{-st} p(w,t|w_0) dt, \quad \Re(s) > 0,
\]

and apply this transform on the Kolmogorov backward equation (1.3), to get the following Laguerre differential equation

\[
w_0 \frac{d^2 \bar{p}(w,s|w_0)}{dw_0^2} + (\alpha + 1 - w_0) \frac{d \bar{p}(w,s|w_0)}{dw_0} - s \bar{p}(w,s|w_0) = -\delta(w - w_0).
\]

According to [24], two linear independent solutions of equation (4.2) are the confluent hypergeometric functions \( M(s,\alpha + 1, w_0) \) and \( U(s,\alpha + 1, w_0) \). Now, we use the idea of Veestraeten’s papers [27, 28] and state the following theorem for obtaining the Green function of the Laguerre differential equation.

4.1. Theorem. The Green function of the Laguerre differential equation (4.2) is given by

\[
\bar{p}(w,s|w_0) = \begin{cases}
G(w,w_0) & -\infty \leq w_0 \leq w = \Gamma(s) \Gamma(\alpha + 1) e^{-w w_0^\alpha} U(s,\alpha + 1, w) M(s,\alpha + 1, w_0), \\
G(w,w_0) & w \leq w_0 \leq +\infty = \Gamma(s) \Gamma(\alpha + 1) e^{-w w_0^\alpha} M(s,\alpha + 1, w) U(s,\alpha + 1, w_0),
\end{cases}
\]

Proof. According to the properties of the Green function, we claim that the Green function of the equation (4.2) is

\[
G(w,w_0) = G(w,w_0)|_{-\infty \leq w_0 \leq w} + G(w,w_0)|_{w \leq w_0 \leq +\infty},
\]

where

\[
\begin{cases}
G(w,w_0)|_{-\infty \leq w_0 \leq w} = A M(s,\alpha + 1, w_0), \\
G(w,w_0)|_{w \leq w_0 \leq +\infty} = B U(s,\alpha + 1, w_0),
\end{cases}
\]

and \( A \) and \( B \) are two unknown coefficients that should be determined by the following boundary conditions [22]

\[
\begin{cases}
\frac{dG(w,w_0)}{dw_0}|_{w \leq w_0 \leq +\infty} - \frac{dG(w,w_0)}{dw_0}|_{-\infty \leq w_0 \leq w} = -\frac{1}{w}, & w = w_0, \\
\frac{dG(w,w_0)}{dw_0}|_{w \leq w_0 \leq +\infty} = -\frac{1}{w}, & w = w_0.
\end{cases}
\]

After applying the boundary conditions and incorporating the Wronskian of confluent hypergeometric functions given by (2.5), the coefficients \( A \) and \( B \) can be easily obtained as

\[
A = \frac{\Gamma(s)}{\Gamma(\alpha + 1)} e^{-w w_0^\alpha} U(s,\alpha + 1, w),
\]

\[
B = \frac{\Gamma(s)}{\Gamma(\alpha + 1)} e^{-w w_0^\alpha} M(s,\alpha + 1, w).
\]

Finally, by substituting the coefficients \( A \) and \( B \) into (4.4) we derive the result (4.3). □
Now, we intend to define the new function \( p_1(w, t|w_1) \) which is obtained from integration of the transition density \( p(w, t|w_0) \) as follows

\[
p_1(w, t|w_1) = \int_0^{w_1} p(w, t|w_0) dw_0
\]

\[
= \frac{e^{\alpha t}}{\pi} \int_0^{+\infty} e^{-\frac{q^2 w (\alpha + 1 + t)}{2}} \left( e^{-2t} + k^2 (-1 + e^{-t})^2 \right)^{-1} \times
\]

\[
\left\{ \frac{1}{k} \sin \left( k w_1 - \frac{k w e^{-t}}{e^{-2t} + k^2 (-1 + e^{-t})^2} + (\alpha - 1) \arctan(k(e^t - 1)) \right) \right\} dk.
\]

(4.8)

Also, the Laplace transform of \( p_1(w, t|w_1) \) is

\[
\bar{p}_1(w, s|w_0) = \int_0^{+\infty} e^{-st} p_1(w, t|w_1) dt
\]

(4.9)

which by setting the relation (4.3) in the latter integral, we get the Laplace transform of \( p_1(w, t|w_1) \) as

(4.10) \( \bar{p}_1(w, s|w_1) = \frac{\Gamma(s - 1)}{\Gamma(\alpha)} e^{-w \alpha^s U(s, \alpha + 1, w)} \left[ M(s - 1, \alpha, w_1) - 1 \right] \).

5. The inverse Laplace transform of the products of Confluent hypergeometric functions

In this section, we use the recurrence relations of the confluent hypergeometric functions to show new relations for the inverse Laplace transforms of the products of confluent hypergeometric functions. For this purpose, we consider the inverse transforms of the functions \( \bar{p}(w, s|w_0) \) and \( \bar{p}_1(w, s|w_1) \) and for simplicity of the formulas (3.4) and (4.8), we set the change of variables \( x = -1 + e^{-t}, y = e^{-2t} + k^2 (-1 + e^{-t})^2 \) and \( q = (\alpha - 1) \arctan(k(e^t - 1)) \). Therefore, we have

(5.1) \( L^{-1} \left\{ \Gamma(s) U(s, \alpha + 1, w) M(s - 1, \alpha, w_1) \right\} \)

\[
= \frac{\Gamma(\alpha) e^{w+\alpha} w^{-\alpha}}{\pi} \int_0^{+\infty} e^{-\frac{q^2 w x}{y}} \frac{1}{y} \sin \left( k w_1 - \frac{k w (x + 1)}{y} + q \right) + \cos \left( -\frac{k}{y} w(x+1)+q \right) + \frac{1}{k} \sin \left( -\frac{k w (x+1)}{y} + q \right) \right\} dk,
\]

and

(5.2) \( L^{-1} \left\{ \Gamma(s) U(s, \alpha + 1, w) M(s, \alpha + 1, w_0) \right\} \)

\[
= \frac{\Gamma(\alpha +1) e^{w+\alpha} w^{-\alpha}}{\pi} \int_0^{+\infty} e^{-\frac{q^2 w x}{y}} \frac{1}{y} \frac{1}{\alpha} \cos \left( \frac{k (w_0 - w (x+1))}{y} + q \right) \right\} dk.
\]

At this point, we use the recurrence relation (2.6) for the confluent hypergeometric function \( M(a, b, z) \) and establish a table for the inverse Laplace transform of the products of confluent hypergeometric functions. For this purpose, we fix the confluent hypergeometric function \( U(s, \alpha + 1, w) \) and consider the relation

\[
z \Gamma(s) M(s + 1, \alpha + 2, z) U(s, \alpha + 1, w) = - (\alpha + 1) (\alpha - z) \Gamma(s) M(s, \alpha + 1, z) U(s, \alpha + 1, w)
\]

(5.3) \( + \alpha (\alpha + 1) \Gamma(s) M(s - 1, \alpha, z) U(s, \alpha + 1, w). \)
We combine the inverse Laplace transforms (5.1) and (5.2) with the recurrence relation (5.3) to get

\[
L^{-1}\left\{ z\Gamma(s+1)U(s,\alpha+1, w)M(s+1, \alpha+2, z) \right\}
\]
\[
= \Gamma(\alpha+2)\frac{e^{w+1}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{\frac{k^2w}{y}} \frac{\alpha+1}{z} \left\{ (z-\alpha)\cos \left( k(z-w) \frac{(x+1)}{y} + q \right) \right\} dk.
\]

Similarly, we show our results for different orders of the product of confluent hypergeometric functions in Table 1, where

**Table 1.** The inverse Laplace transform of the products of confluent hypergeometric functions

<table>
<thead>
<tr>
<th>Laplace transform</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma(s)U(s, \alpha+1, w)M(s+1, \alpha, z) )</td>
<td>( \Gamma(\alpha+1)f )</td>
</tr>
<tr>
<td>( \Gamma(s)U(s, \alpha+1, w)M(s-1, \alpha, z) )</td>
<td>( \Gamma(\alpha)(f+h-j+r) )</td>
</tr>
<tr>
<td>( \Gamma(s+1)U(s, \alpha+1, w)M(s+1, \alpha+2, z) )</td>
<td>( \frac{\Gamma(\alpha+2)}{z} \left{ (z-\alpha+1)f+h-j+r \right} )</td>
</tr>
<tr>
<td>( \Gamma(s)U(s, \alpha+1, w)M(s-2, \alpha-1, z) )</td>
<td>( \frac{1}{(\alpha-1)(\alpha-2)} \left{ z(f-2j+2r) \right} )</td>
</tr>
<tr>
<td>( \Gamma(s)U(s, \alpha+1, w)M(s-3, \alpha-2, z) )</td>
<td>( \frac{1}{\alpha}\left{ h-f-n+(\alpha-1-z) \right} \times \Gamma(\alpha+1)(f+h-j+r) )</td>
</tr>
</tbody>
</table>

\[
f(t, \alpha, z, w) = \frac{e^{(w+1)}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{k^2w} \frac{\alpha+1}{z} \cos \left( k(z-w) \frac{(x+1)}{y} + q \right) dk,
\]

\[
h(t, \alpha, z, w) = \frac{e^{(w+1)}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{k^2w} \frac{\alpha+1}{z} \cos \left( \frac{k}{y}w(x+1) + q \right) dk,
\]

\[
m(t, \alpha, z, w) = \frac{e^{(w+1)}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{k^2w} \frac{\alpha+1}{z} \sin \left( k(z-w) \frac{(x+1)}{y} + q \right) dk,
\]

\[
n(t, \alpha, z, w) = \frac{e^{(w+1)}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{k^2w} \frac{\alpha+1}{z} \sin \left( \frac{k}{y}w(x+1) + q \right) dk,
\]

\[
r(t, \alpha, w) = \frac{e^{(w+1)}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{k^2w} \frac{\alpha+1}{z} \frac{1}{k} \sin \left( \frac{k}{y}w(x+1) + q \right) dk,
\]

\[
j(t, \alpha, z, w) = \frac{e^{(w+1)}w^{-\alpha}}{\pi} \int_0^{+\infty} e^{k^2w} \frac{\alpha+1}{z} \frac{1}{k} \sin \left( k(z-w) \frac{(x+1)}{y} + q \right) dk.
\]
5.1. Remark. If we fix the confluent hypergeometric function \( M(s, \alpha + 1, z) \) and use the recurrence relation (2.7), then we can get new relations for the inverse Laplace transform of other products of confluent hypergeometric functions such as \( \Gamma(s)U(s-2, \alpha-1, z)M(s, \alpha + 1, w) \).

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