Signed degree sequences in signed multipartite graphs

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Abstract
A signed $k$-partite graph (signed multipartite graph) is a $k$-partite graph in which each edge is assigned a positive or a negative sign. If $G(V_1, V_2, \ldots, V_k)$ is a signed $k$-partite graph with $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$, $1 \leq i \leq k$, the signed degree of $v_{ij}$ is $sdeg(v_{ij}) = d_{ij} = d^+_i - d^-_i$, where $1 \leq i \leq k$, $1 \leq j \leq n_i$, and $d^+_i(d^-_i)$ is the number of positive (negative) edges incident with $v_{ij}$. The sequences $\sigma_i = [d_{i1}, d_{i2}, \ldots, d_{in_i}]$, $1 \leq i \leq k$, are called the signed degree sequences of $G(V_1, V_2, \ldots, V_k)$. The set of distinct signed degrees of the vertices in a signed $k$-partite graph $G(V_1, V_2, \ldots, V_k)$ is called its signed degree set. In this paper, we characterize signed degree sequences of signed $k$-partite graphs. Also, we give the existence of signed $k$-partite graphs with given signed degree sets.

Keywords: Signed graphs, signed multipartite graph, signed degree, signed set.

2000 AMS Classification: 05C22.

Received 17/09/2011 : Accepted 24/06/2014 Doi: 10.15672/HJMS.2015449661

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [3]. Let $G$ be a signed graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The signed degree of $v_i$ is $sdeg(v_i) = d_i = d^+_i - d^-_i$, where $1 \leq i \leq n$ and $d^+_i(d^-_i)$ is the number of positive (negative) edges incident with $v_i$. A signed degree sequence $\sigma = [d_1, d_2, \ldots, d_n]$ of a signed graph $G$ is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is $s$-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \ldots, d_n]$ is a standard sequence if $\sigma$ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$ and

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Theorem 3. Let \( \alpha = [d_1, d_2, \ldots, d_p] \) and \( \beta = [e_1, e_2, \ldots, e_q] \) be standard sequences. Then, \( \alpha \) and \( \beta \) are the signed degree sequences of a signed bipartite graph if and only if there exist integers \( r \) and \( s \) with \( d_1 = r - s \) and \( 0 \leq s \leq \frac{q - d_1}{2} \) such that \( \alpha' \) and \( \beta' \) are the signed degree sequences of a signed bipartite graph, where \( \alpha' \) is obtained from \( \alpha \) by deleting \( d_1 \) and \( \beta' \) is obtained from \( \beta \) by reducing \( r \) greatest entries of \( \beta \) by 1 each and adding \( s \) least entries of \( \beta \) by 1 each.
Further, so that each edge is assigned a positive or a negative sign. Let $sdeg(v_{ij}) = d_{ij} = d_{ij}^+ - d_{ij}^-$, where $1 \leq i \leq k$, $1 \leq j \leq n_i$ and $d_{ij}^+$ ($d_{ij}^-$) is the number of positive (negative) edges incident with $v_{ij}$. The sequences $\alpha_i = [d_{11}, d_{22}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, are called the signed degree sequences of $G(V_1, V_2, \cdots, V_k)$. Also the sequences $\alpha_i = [d_{11}, d_{22}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, of integers are $s$-graphical if $\alpha_i$'s are the signed degree sequences of some signed $k$-partite graph. Denote a positive edge $xy$ by $xy^+$ and a negative edge $xy$ by $xy^-$. Several results on signed degree sequences in signed multipartite graphs can be found in [9]. We start with the following observation.

**Theorem 4.** Let $G(V_1, V_2, \cdots, V_k)$ be a signed $k$-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \leq i \leq k$ and having $q$ edges. Then

$$p = \sum_{i=1}^k \sum_{j=1}^{n_i} s \deg(v_{ij}) \equiv 2q \pmod{4},$$

and the number of positive edges and negative edges of $G(V_1, V_2, \cdots, V_k)$ are respectively $\frac{2q}{2} = \frac{2p}{2}$ and $\frac{2q}{2} = \frac{2p}{2}$.

**Proof.** Let $v_{ij}$ ($1 \leq i \leq k$, $1 \leq j \leq n_i$) be incident with $d_{ij}^+$ positive edges and $d_{ij}^-$ negative edges so that

$$sdeg(v_{ij}) = d_{ij}^+ - d_{ij}^- \text{ while } \deg(v_{ij}) = d_{ij}^+ + d_{ij}^-.$$

Obviously, $\sum_{i=1}^k \sum_{j=1}^{n_i} \deg(v_{ij}) = 2q$.

Let $G(V_1, V_2, \cdots, V_k)$ have $q$ positive edges and $h$ negative edges. Then $q = g + h$, $\sum_{i=1}^k \sum_{j=1}^{n_i} d_{ij}^+ = 2g$ and $\sum_{i=1}^k \sum_{j=1}^{n_i} d_{ij}^- = 2h$.

Further,

$$\sum_{i=1}^k \sum_{j=1}^{n_i} \deg(v_{ij}) = \sum_{i=1}^k \sum_{j=1}^{n_i} (d_{ij}^+ - d_{ij}^-)$$

$$= \sum_{i=1}^k \sum_{j=1}^{n_i} d_{ij}^+ - \sum_{i=1}^k \sum_{j=1}^{n_i} d_{ij}^-$$

$$= 2g - 2h.$$

Hence,

$$p = \sum_{i=1}^k \sum_{j=1}^{n_i} s \deg(v_{ij}) \equiv 2g - 2h$$

$$= 2(q - h) - 2h$$

$$= 2q - 4h,$$

so that $p \equiv 2q \pmod{4}$. Again, from $g + h = q$ and $2g - 2h = p$, we have $g = \frac{2q + p}{4}$ and $h = \frac{2q - p}{4}$. □
Corollary 5. A necessary condition for the $k$ sequences $\alpha_i = [d_{i1}, d_{i2}, \ldots, d_{in}]$, $1 \leq i \leq k$, of integers to be $s$-graphical is that $\sum_{i=1}^{k} \sum_{j=1}^{n} d_{ij}$ is even.

A zero sequence is a finite sequence each term of which is 0. Clearly, every $k$ finite zero sequences are the signed degree sequences of a signed $k$-partite graph. If $\beta = [a_1, a_2, \ldots, a_n]$ is a sequence of integers, then the negative of $\beta$ is the sequence $\beta = [-a_1, -a_2, \ldots, -a_n]$.

The next result follows by interchanging positive edges with negative edges.

Theorem 6. The sequences $\alpha_i = [d_{i1}, d_{i2}, \ldots, d_{in}]$, $1 \leq i \leq k$, are the signed degree sequences of some signed $k$-partite graph if and only if $-\alpha_i = [-d_{i1}, -d_{i2}, \ldots, -d_{in}]$ are the signed degree sequences of some signed $k$-partite graph.

Assume without loss of generality, that a non-zero sequence $\beta = [a_1, a_2, \ldots, a_n]$ is non-increasing and $|a_1| \geq |a_n|$, we can always replace $\beta$ by $-\beta$ if necessary. The next result follows by interchanging positive edges with negative edges.

Theorem 7. Let $\alpha_i = [d_{i1}, d_{i2}, \ldots, d_{in}]$, $1 \leq i \leq k$, be standard sequences and let $r = \frac{1}{2} \left( d_{i1} + \sum_{j=2}^{k} n_j \right)$. Let $\alpha'_i$ be obtained from $\alpha_i$ by deleting $d_{i1}$ and $\alpha'_2, \alpha'_3, \ldots, \alpha'_k$ be obtained from $\alpha_2, \alpha_3, \ldots, \alpha_k$ by reducing $r$ greatest entries of $\alpha_2, \alpha_3, \ldots, \alpha_k$ by 1 each and adding remaining entries of $\alpha_2, \alpha_3, \ldots, \alpha_k$ by 1 each. Then $\alpha_i$ are the signed degree sequences of some complete signed $k$-partite graph if and only if $\alpha'_i$ are also signed degree sequences of some complete signed $k$-partite graph, $1 \leq i \leq k$.

Proof. Let $G'(V'_2, V'_3, \ldots, V'_k)$ be a complete signed $k$-partite graph with signed degree sequences $\alpha'_i$, $1 \leq i \leq k$. Let $V'_2 = \{v_{12}, v_{13}, \ldots, v_{1n_1}\}$ and $V'_i = \{v_{i2}, v_{i3}, \ldots, v_{in_i}\}$, $2 \leq i \leq k$. Then a complete signed $k$-partite graph with signed degree sequences $\alpha_i$, $1 \leq i \leq k$, can be obtained by adding a vertex $v_{i1}$ to $V'_i$ so that there are $r$ positive edges from $v_{i1}$ to those $r$ vertices of $V'_2, V'_3, \ldots, V'_i$, whose signed degrees were reduced by 1 in going from $\alpha_i$ to $\alpha'_i$, and there are negative edges from $v_{i1}$ to the remaining vertices of $V'_2, V'_3, \ldots, V'_k$, whose signed degrees were increased by 1 in going from $\alpha_i$ to $\alpha'_i$. Note that the signed degree of $v_{i1}$ is $r - \left( \sum_{j=2}^{k} n_j - r \right) = 2r - \sum_{j=2}^{k} n_j = d_{i1}$.

Conversely, let $\alpha_i$, $1 \leq i \leq k$, be the signed degree sequences of a complete signed $k$-partite graph. Let the vertex sets of the complete signed $k$-partite graph be $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ such that $sdeg(v_{ij}) = d_{ij}$, $1 \leq i \leq k$, $1 \leq j \leq n_i$.

Among all the complete signed $k$-partite graphs with $\alpha_i$, $1 \leq i \leq k$, as the signed degree sequences, let $G(V_1, V_2, \ldots, V_k)$ be one with the property that the sum $S$ of the signed degrees of the vertices $V_2, V_3, \ldots, V_k$ joined to $v_{i1}$ by positive edges is maximum. Let $d_{11}^+$ and $d_{11}^-$ be respectively the number of positive edges and the number of negative edges incident with $v_{i1}$. Then $sdeg(v_{i1}) = d_{11} = d_{11}^+ - d_{11}^-$, $deg(v_{i1}) = d_{11}^+ + d_{11}^- = \sum_{j=2}^{k} n_j$, and hence $d_{11}^- = \frac{1}{2} \left( d_{11} + \sum_{j=2}^{k} n_j \right) = r$. Let $U$ be the set of $r$
vertices of \(V_2, V_3, \ldots, V_k\) with highest signed degrees and let \(W = \bigcup_{j=2}^{k} V_j - U\). We claim that \(v_{11}\) must be joined by positive edges to the vertices of \(U\). If this is not true, then there exist vertices \(v_{gh} \in U\) and \(v_{ij} \in W\) such that the edge \(v_{11}v_{gh}\) is negative and the edge \(v_{11}v_{ij}\) is positive. Since \(sdeg(v_{gh}) \geq sdeg(v_{ij})\), there exist vertices \(v_{mn}\) and \(v_{pq}\) such that the edge \(v_{gh}v_{mn}\) is positive and the edge \(v_{ij}v_{pq}\) is negative. If the edge \(v_{gh}v_{pq}\) is positive, then change the signs of the edges \(v_{11}v_{gh}, v_{gh}v_{pq}, v_{pq}v_{ij}, v_{ij}v_{11}\) so that the edges \(v_{11}v_{gh}\) and \(v_{pq}v_{ij}\) are positive and the edges \(v_{11}v_{ij}\) and \(v_{gh}v_{pq}\) are negative. But if the edge \(v_{gh}v_{pq}\) is negative, then \(sdeg(v_{gh}) < sdeg(v_{ij})\), which is a contradiction. The case when \(v_{mn} = v_{pq}\) follows by the same argument as in above.

Hence we obtain a complete signed \(k\)-partite graph with signed degree sequences \(\alpha_i, 1 \leq i \leq k\), in which the sum of the signed degrees of the vertices of \(V_2, V_3, \ldots, V_k\) joined to \(v_{11}\) by positive edges exceeds \(S\), a contradiction.

Thus we may assume that \(v_{11}\) is joined by positive edges to the vertices of \(U\) and by negative edges to the vertices of \(W\). So \(G(V_1, V_2, \ldots, V_k) - v_{11}\) is a complete signed \(k\)-partite graph with \(\alpha'_i, 1 \leq i \leq k\), as the signed degree sequences. \(\square\)

Theorem 7 provides an algorithm of checking whether the standard sequences \(\alpha_i, 1 \leq i \leq k\), are the signed degree sequences, and for constructing a corresponding complete signed \(k\)-partite graph. Suppose \(\alpha_i = [d_{i1}, d_{i2}, \ldots, d_{in_i}], 1 \leq i \leq k\), be the standard signed degree sequences of a complete signed \(k\)-partite graph with parts \(V_i = \{v_{1i}, v_{2i}, \ldots, v_{ni}\}\). Deleting \(d_{11}\) and reducing \(r = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{n} n_j\right)\) greatest entries of \(\alpha_2, \alpha_3, \ldots, \alpha_k\) by 1 each and adding remaining entries of \(\alpha_2, \alpha_3, \ldots, \alpha_k\) by 1 each to form \(\alpha'_2, \alpha'_3, \ldots, \alpha'_k\). Then edges are defined by \(v_{11} v_{ij}'\) if \(d_{ij}'s\) are reduced by 1 and \(v_{11} v_{ij}\) if \(d_{ij}'s\) are increased by 1. For \(-\alpha_i, 1 \leq i \leq k\), edges are defined by \(v_{11} v_{ij}'\) if \(d_{ij}'s\) are reduced by 1 and \(v_{11} v_{ij}\) if \(d_{ij}'s\) are increased by 1. If the conditions of standard sequences do not hold, then we delete \(d_{11}\) for that \(i\) for which the conditions of standard sequences get satisfied. If this method is applied recursively, then a complete signed \(k\)-partite graph with signed degree sequences \(\alpha_i, 1 \leq i \leq k\), is constructed.

The next result gives necessary and sufficient conditions for the \(k\) sequences of integers to be the signed degree sequences of some signed \(k\)-partite graph.

**Theorem 8.** Let \(\alpha_i = [d_{i1}, d_{i2}, \ldots, d_{in_i}], 1 \leq i \leq k\), be standard sequences. Then \(\alpha_i, 1 \leq i \leq k\), are the signed degree sequences of a signed \(k\)-partite graph if and only if there exist integers \(r\) and \(s\) with \(d_{11} = r - s\) and \(0 \leq s \leq \frac{1}{2}\left(\sum_{j=2}^{k} n_j - d_{11}\right)\) such that \(\alpha'_i\) are the signed degree sequences of a signed \(k\)-partite graph, where \(\alpha'_i\) is obtained from \(\alpha_i\) by deleting \(d_{11}\) and \(\alpha'_2, \alpha'_3, \ldots, \alpha'_k\) are obtained from \(\alpha_2, \alpha_3, \ldots, \alpha_k\) by reducing \(r\) greatest entries of \(\alpha_2, \alpha_3, \ldots, \alpha_k\) by 1 each and adding \(s\) least entries of \(\alpha_2, \alpha_3, \ldots, \alpha_k\) by 1 each.

**Proof.** Let \(r\) and \(s\) be integers with \(d_{11} = r - s\) and \(0 \leq s \leq \frac{1}{2}\left(\sum_{j=2}^{k} n_j - d_{11}\right)\) such that \(\alpha'_i, 1 \leq i \leq k\), are the signed degree sequences of a signed \(k\)-partite graph \(G'\)

\[V'_1, V'_2, \ldots, V'_k\).

Let \(V'_1 = \{v_{11}, v_{21}, \ldots, v_{n_1}\}\) and \(V'_2 = \{v_{12}, v_{22}, \ldots, v_{n_2}\}\), \(2 \leq i \leq k\). Let \(U\) be the set of \(r\) vertices of \(V'_2, V'_3, \ldots, V'_k\) with highest signed degrees, \(W\) be the set of \(s\) vertices of \(V'_2, V'_3, \ldots, V'_k\) with least signed degrees and let \(Z = \bigcup_{j=2}^{k} V'_j - U - W\). Then a signed \(k\)-partite graph with signed degree sequences \(\alpha_i, 1 \leq i \leq k\), can be obtained by adding a vertex \(v_{11}\) to \(V'_1\) so that there are \(r\) positive edges from \(v_{11}\) to the vertices of \(U\) and \(s\) negative edges from \(v_{11}\) to the vertices of \(W\). Note that the signed degree of \(v_{11}\) is \(r - s = d_{11}\).

Conversely, let \(\alpha_i, 1 \leq i \leq k\), be the signed degree sequences of a signed \(k\)-partite
graph. Let the vertex sets of the signed $k$-partite graph be $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ such that $sdeg(v_{ij}) = d_{ij}, 1 \leq i \leq k, 1 \leq j \leq n_i$.

Among all the signed $k$-partite graphs with $\alpha_i$, $1 \leq i \leq k$, as the signed degree sequences, let $G(V_1, V_2, \ldots, V_k)$ be one with the property that the sum $S$ of the signed degrees of the vertices of $V_2, V_3, \ldots, V_k$ joined to $v_{11}$ by positive edges is maximum. Let $d_{11}^+ = r$ and $d_{11}^- = s$ be respectively the number of positive edges and the number of negative edges incident with $v_{11}$. Then $sdeg(v_{11}) = d_{11}^- = d_{11}^+ - d_{11}^+ = r - s$ and $deg(v_{11}) = d_{11}^+ + d_{11}^- = r + s \leq \sum_{j=2}^{k} n_j$, and hence $0 \leq s \leq \frac{1}{2} \left( \sum_{j=2}^{k} n_j - d_{11}^- \right)$. Let $U$ be the set of $r$ vertices of $V_2, V_3, \ldots, V_k$ with highest signed degrees and let $W = \bigcup_{j=2}^{k} V_j - U$.

We claim that $v_{11}$ must be joined by positive edges to the vertices of $U$. If this is not true, then there exist vertices $v_{gh} \in U$ and $v_{mn} \in W$ such that the edge $v_{11}v_{mn}$ is positive and either (i) $v_{11}v_{gh}$ is a negative edge or (ii) $v_{11}$ and $v_{gh}$ are not adjacent in $G(V_1, V_2, \ldots, V_k)$. As $sdeg(v_{gh}) \geq sdeg(v_{mn})$, that is $d_{gh} \geq d_{mn}$, therefore we consider only (i) and then (ii) is similar to (i).

We note that if there exists a vertex $v_{pq} (\neq v_{11})$ such that $v_{pq}v_{gh}$ is a positive edge and $v_{pq}v_{mn}$ is a negative edge, then change the signs of these edges so that $v_{11}v_{gh}$ and $v_{pq}v_{mn}$ are positive, and $v_{11}v_{mn}$ and $v_{pq}v_{gh}$ are negative. Hence we obtain a signed $k$-partite graph with signed degree sequences $\alpha_i, 1 \leq i \leq k$, in which the sum of the signed degrees of the vertices of $V_2, V_3, \ldots, V_k$ joined to $v_{11}$ by positive edges exceeds $S$, a contradiction. So assume that no such vertex $v_{pq}$ exists.

Now, suppose that $v_{gh}$ is not incident to any positive edge. Since $sdeg(v_{gh}) \geq sdeg(v_{mn})$, that is $d_{gh} \geq d_{mn}$, then there exist at least two vertices $v_{pq}$ and $v_{lt}$ (both distinct from $v_{11}$) such that $v_{pq}v_{mn}$ and $v_{lt}v_{mn}$ are negative edges and both $v_{pq}$ and $v_{lt}$ are not adjacent to $v_{gh}$. Then by changing the edges so that $v_{11}v_{pq}$ is a positive edge, and $v_{11}v_{mn}, v_{gh}v_{pq}, v_{gh}v_{lt}$ are negative edges, we again get a contradiction. Hence $v_{gh}$ is incident to at least one positive edge.

We claim that there exists at least one vertex $v_{xy}$ such that $v_{yx}v_{xy}$ is a positive edge and $v_{yx}$ is not adjacent to $v_{mn}$. Suppose on contrary that whenever $v_{xy}$ is joined to a vertex by a positive edge, then $v_{mn}$ is also joined to this vertex by a positive edge. Since $sdeg(v_{xy}) \geq sdeg(v_{mn})$, that is $d_{xy} \geq d_{mn}$, then again we have the same situation as above, from which we get a contradiction. Thus there exists a vertex $v_{pq}$ such that $v_{pq}v_{xy}$ is a positive edge and $v_{pq}$ is not adjacent to $v_{mn}$. Similarly, it can be shown that there exists a vertex $v_{pq}$ such that $v_{pq}v_{mn}$ is a negative edge and $v_{pq}$ is not adjacent to $v_{gh}$. By changing the edges so that $v_{11}v_{gh}v_{mn}v_{xy}$ are positive edges, and $v_{11}v_{mn}, v_{gh}v_{pq}$ are negative edges, we again get a contradiction. Hence $v_{11}$ is joined by positive edges to the vertex of $U$.

In a similar way, it can be shown that $v_{11}$ is joined by negative edge to the $s$ vertices of $V_2, V_3, \ldots, V_k$ with least signed degrees.

Hence $G(V_1, V_2, \ldots, V_k) - v_{11}$ is a signed $k$-partite graph with $\alpha_i', 1 \leq i \leq k$, as the signed degree sequences. $\square$

Theorem 8 also provides an algorithm for determining whether or not the standard sequences $\alpha_i, 1 \leq i \leq k$, are the signed degree sequences, and for constructing a corresponding signed $k$-partite graph. Suppose $\alpha_i = [d_{i1}, d_{i2}, \ldots, d_{in_i}], 1 \leq i \leq k$, be the standard signed degree sequences of a signed $k$-partite graph with parts $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$. Let $d_{11} = r - s$ and $0 \leq s \leq \frac{1}{2} \left( \sum_{j=2}^{k} n_j - d_{11}^- \right)$. Deleting $d_{11}$ and reducing $r$ greatest entries of $\alpha_2, \alpha_3, \ldots, \alpha_k$ by 1 each and adding $s$ least entries of $\alpha_2, \alpha_3, \ldots, \alpha_k$ by 1 each to form $\alpha_2', \alpha_3', \ldots, \alpha_k'$. Then edges are defined by $v_{11}v_{ij}$ if $d_{ij}' > s$ are reduced by 1; $v_{11}v_{ij}$ if $d_{ij}' > s$ are increased by 1, and $v_{11}$ and $v_{ij}$ are not adjacent if $d_{ij}'$ are unchanged. For $\alpha_i$, edges are defined by $v_{11}v_{ij}$ if $d_{ij}' > s$ are reduced by 1; $v_{11}v_{ij}$ if $d_{ij}' > s$ are increased by
1, and \( v_{i1} \) and \( v_{ij} \) are not adjacent if \( d'_{ij} \) s are unchanged. If the conditions of standard sequences do not hold, then we delete \( d_{i1} \) for that \( i \) for which the conditions of standard sequences get satisfied. If this method is applied recursively, then a signed \( k \)-partite graph with signed degree sequences \( \alpha_i \), \( 1 \leq i \leq k \), is constructed.

3. Signed degree sets in signed \( k \)-partite graphs

Let \( G(V_1, V_2, \cdots, V_k) \) be a signed \( k \)-partite graph with \( X \subseteq V_i, Y \subseteq V_j \) \((i \neq j)\). If each vertex of \( G \) is joined to every vertex of \( Y \) by a positive (negative) edge, then it is denoted by \( X \ominus Y \). The set \( S \) of distinct signed degrees of the vertices in a signed \( k \)-partite graph \( G(V_1, V_2, \cdots, V_k) \) is called its signed degree set. Also, a signed \( k \)-partite graph \( G(V_1, V_2, \cdots, V_k) \) is said to be connected if each vertex \( v_i \in V_i \) is connected to every vertex \( v_j \in V_j \).

The following result shows that every set of positive integers is a signed degree set of some connected signed \( k \)-partite graph.

**Theorem 9.** Let \( d_1, d_2, \cdots, d_t \) be positive integers. Then there exists a connected signed \( k \)-partite graph with signed degree set

\[
S = \{ d_1, \sum_{i=1}^{2} d_i, \cdots, \sum_{i=1}^{t} d_i \}.
\]

**Proof.** We consider the following two cases. (i) \( k \) even, (ii) \( k \) odd.

**Case (i).** Let \( k = 2m \), where \( m \geq 1 \). Construct a signed \( k \)-partite graph \( G(V_1, V_2, \cdots, V_{2m}) \) as follows.

Let

\[
V_1 = P_1 \cup Q_1 \cup R_1 \cup S_1 \cup X_1' \cup X_1'' \cup X_1''' \cup X_2' \cup X_2'' \cup \cdots \cup X_{m-1}, V_{2m} = P_{2m} \cup Q_{2m},
\]

where

(a) \( P_1, Q_1, R_1, S_1, X_1', X_1'', X_1''', X_1'''', \cdots, X_{m-1} \) are pairwise disjoint,
(b) \( P_1, Q_2, R_2, S_2, Y_1', Y_1'', Y_1''', Y_{m-1}', Y_{m-1}'' \) are pairwise disjoint,
(c) For all \( i \), \( P_1 \cap Q_i = \phi, 3 \leq i \leq 2m \) and \( |P_1| = |Q_i| = d_1, 1 \leq i \leq 2m \); \( |R_i| = |S_i| = d_1, 1 \leq i \leq t - 1 \); \( |X_i'| = |Y_i'| = d_1, 1 \leq i \leq t - 1 \); \( |X_i''| = d_2 + d_3 + \cdots + d_{i+1}, 1 \leq i \leq t - 1 \).

For all \( i \), let \( P_1 \cap Q_{i+1}, 1 \leq i \leq 2m - 1 \); \( Q_1 \cap P_{i+1}, 1 \leq i \leq 2m - 1 \); \( Q_1 \cap R_2, R_1 \cap Q_2, R_1 \cap S_2, S_1 \cap R_2, X_1 \cap S_2, X_1' \cap R_2, X_1' \cap Y_1', 1 \leq i \leq t - 1 ; X_1'' \cap Y_1, 1 \leq i \leq t - 1 \); \( X_1''' \cap Y_1', 1 \leq i \leq t - 1 \); \( X_1'''', X_1'''' \cap Y_1', 1 \leq i \leq t - 1 \); \( X_1'''' \cap Y_1', 1 \leq i \leq t - 1 \); \( X_1'''' \cap Y_{m-1}', 2 \leq i \leq t - 1 \); \( X_1''' \cap Y_{m-1}, 2 \leq i \leq t - 1 \); \( X_1'' \cap Y_{m-1}, 2 \leq i \leq t - 1 \); \( X_1' \cap Y_{m-1}', 2 \leq i \leq t - 1 \).

Then the signed degrees of the vertices of \( G(V_1, V_2, \cdots, V_{2m}) \) are as follows.
Every set of negative integers is a signed degree set of some connected signed $k$-partite graph.
Theorem 11. Every set of the integers is a signed degree set of some connected signed $k$-partite graph.

Proof. Let $S$ be a set of integers. Then we have the following five cases.

Case (i). $S$ is a set of positive (negative) integers. Then the result follows by Theorem 9 (Corollary 10).

Case (ii). $S = \{0\}$. Then a signed $k$-partite graph $G(V_1, V_2, \cdots, V_k)$ with $V_i = \{v_i, v_i'\}$ for all $i$, $1 \leq i \leq k$, in which $v_i v_{i+1}$, $v_i' v_{i+1}$ for all $i$, $1 \leq i \leq k-1$, are positive edges and $v_i v_{i+1}$, $v_i' v_{i+1}$ for all $i$, $1 \leq i \leq k-1$, are negative edges has signed degree set $S$.

Case (iii). $S$ is a set of non-negative (non-positive) integers. Let $S = S' \cup \{0\}$, where $S'$ be a set of positive (negative) integers. Then by Theorem 9 (Corollary 10), there is a connected signed $k$-partite graph $G'(V_1', V_2', \cdots, V_k')$ with signed degree set $S'$. Construct a new signed $k$-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let $V_1 = V_1' \cup \{x_1\} \cup \{y_1\}$, $V_2 = V_2' \cup \{x_2\} \cup \{y_2\}$, $V_3 = V_3'$, \ldots, $V_k = V_k'$, with $V_i \cap \{x_1\} = \phi, V_i \cap \{y_1\} = \phi, V_i \cap \{x_2\} = \phi, V_i \cap \{y_2\} = \phi, \{x_2\} \cap \{y_2\} = \phi, \{x_1\} \cap \{y_1\} = \phi$.

Let $v_1 x_1 v_2 y_2, y_1 v_2$ be positive edges, $v_1 y_2$, $x_1 y_2$ be negative edges, where $v_1 \in V_1'$, $v_2 \in V_2'$ and let there be all the edges of $G'(V'_1, V'_2, \cdots, V'_k)$. Then $G(V_1, V_2, \cdots, V_k)$ has signed degree set $S$. We note that addition of the positive edges $v_1 x_1, v_2 y_2, y_1 v_2$ and negative edges $v_1 y_2, x_1 y_2, y_1 v_2$ do not effect the signed degrees of the vertices of $G'(V'_1, V'_2, \cdots, V'_k)$, and the vertices $x_1, y_1, x_2, y_2$ have signed degree zero each.

Case (iv). $S$ is a set of non-zero integers. Let $S = S' \cup S''$, where $S'$ and $S''$ are sets of positive and negative integers respectively. Then by Theorem 9 (Corollary 10), there are connected signed $k$-partite graphs $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets $S'$ and $S''$ respectively. Suppose $G'_1(V_{11}', V_{21}', \cdots, V_{k1}')$ and $G''_2(V_{12}'', V_{22}'', \cdots, V_{k2}'')$ are the copies of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$, with signed degree sets $S'$ and $S''$ respectively. Construct a new signed $k$-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let

\begin{align*}
V_1 &= V_1' \cup V_{11}' \cup V_{12}' \cup V_{12}'', \\
V_2 &= V_2' \cup V_{21}' \cup V_{22}' \cup V_{22}'', \\
V_3 &= V_3' \cup V_{31}' \cup V_{32}' \cup V_{32}'', \\
&\vdots \\
V_k &= V_k' \cup V_{k1}' \cup V_{k2}' \cup V_{k2}'',
\end{align*}

with $V_i \cap V_{i1}' = \phi, V_i \cap V_{i2}' = \phi, V_i \cap V_{i2}'' = \phi, V_i \cap V_{i1}'' = \phi, V_i \cap V_{i2}'' = \phi, V_i \cap V_{i1}'' = \phi$.

Let $v_1 v_2, v_3 v_4$ be positive edges, $v_3 v_4, v_3 v_4$ be negative edges, where $v_1 \in V_1', v_1 \in V_{i1}', v_2 \in V_2', v_2 \in V_{i2}'$ and let there be all the edges of $G'(V'_1, V'_2, \cdots, V'_k), G'_1(V_{11}', V_{21}', \cdots, V_{k1}')$, $G''(V''_1, V''_2, \cdots, V''_k)$ and $G''_2(V_{12}'', V_{22}'', \cdots, V_{k2}'')$. Then $G(V_1, V_2, \cdots, V_k)$ has signed degree set $S$.

We note that addition of the positive edges $v_1 v_2, v_3 v_4$ and negative edges $v_3 v_4, v_3 v_4$ do not effect the signed degrees of the vertices of $G'(V'_1, V'_2, \cdots, V'_k)$, $G'_1(V_{11}', V_{21}', \cdots, V_{k1}')$, $G''(V''_1, V''_2, \cdots, V''_k)$ and $G''_2(V_{12}'', V_{22}'', \cdots, V_{k2}'')$.

Case (v). $S$ is the set of all integers. Let $S = S' \cup S'' \cup \{0\}$, where $S'$ and $S''$ are sets of positive and negative integers respectively. Then by Theorem 9 (Corollary 10), there exist connected signed $k$-partite graphs $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets $S'$ and $S''$ respectively. Construct a new signed $k$-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.
Let
\[ V_1 = V'_1 \cup V''_1 \cup \{x\}, \]
\[ V_2 = V'_2 \cup V''_2 \cup \{y\}, \]
\[ V_3 = V'_3 \cup V''_3, \]
\[ \vdots \]
\[ V_k = V'_k \cup V''_k, \]
with \( V'_i \cap V''_i = \emptyset \), \( V'_i \cap \{x\} = \emptyset \), \( V''_i \cap \{x\} = \emptyset \), \( V'_i \cap \{y\} = \emptyset \), \( V''_i \cap \{y\} = \emptyset \). Let \( v'_1 v'_2, v''_1 y, xv'_2 \) be positive edges, \( v''_1 y, v'_1 v'_2, xv''_2 \) be negative edges, where \( v'_1 \in V'_1 \), \( v''_1 \in V''_1 \), \( v'_2 \in V'_2 \), \( v''_2 \in V''_2 \), and let there be all the edges of \( G'(V'_1, V'_2, \ldots, V'_k) \) and \( G''(V''_1, V''_2, \ldots, V''_k) \). Therefore \( G(V_1, V_2, \ldots, V_k) \) has signed degree set \( S \). We note that addition of the positive edges \( v'_1 v'_2, v''_1 y, xv'_2 \) and negative edges \( v''_1 y, v'_1 v'_2, xv''_2 \) do not effect the signed degrees of the vertices of \( G'(V'_1, V'_2, \ldots, V'_k) \) and \( G''(V''_1, V''_2, \ldots, V''_k) \), and the vertices \( x \) and \( y \) have signed degrees zero each.

Clearly, by construction, all the signed \( k \)-partite graphs are connected. This proves the result. \( \square \)

**References**


