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Signed degree sequences in signed multipartite graphs

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Abstract

A signed k-partite graph (signed multipartite graph) is a k-partite graph in which each edge is assigned a positive or a negative sign. If $G(V_1, V_2, \cdots, V_k)$ is a signed k-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \le i \le k$, the signed degree of v_{ij} is $sdeg(v_{ij}) = d_{ij} = d^+_{ij} - d^-_{ij}$, where $1 \le i \le k$, $1 \le j \le n_i$ and $d^+_{ij}(d^-_{ij})$ is the number of positive (negative) edges incident with v_{ij}. The sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \le i \le k$, are called the signed degree sequences of $G(V_1, V_2, \cdots, V_k)$. The set of distinct signed degrees of the vertices in a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ is called its signed degree set. In this paper, we characterize signed degree sequences of signed k-partite graphs. Also, we give the existence of signed k-partite graphs with given signed degree sets.

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1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [3]. Let G be a signed graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The signed degree of v_i is $sdeg(v_i) = d_i = d^+_i - d^-_i$, where $1 \le i \le n$ and $d^+_i(d^-_i)$ is the number of positive(negative) edges incident with v_i. A signed degree sequence $\sigma = [d_1, d_2, \cdots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is a standard sequence if σ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$ and

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$|d_1| \geq |d_n|$.

The following result, due to Charttrand et al. [1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, and this is similar to Hakimi's result for degree sequences in graphs [2].

Theorem 1. A standard integral sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is s-graphical if and only if

$$\sigma' = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+s+1} - 1, d_{d_1+s+2}, \cdots, d_{n-s}, d_{n-s+1} + 1, \cdots, d_n + 1]$$

is s-graphical for some s, $0 \leq s \leq \frac{n-1-d_1}{2}$.

The next result [12] provides a good candidate for parameter s in Theorem 1.

Theorem 2. A standard integral sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is s-graphical if and only if

$$\sigma'_m = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \cdots, d_{n-m}, d_{n-m+1} + 1, \cdots, d_n + 1]$$

is s-graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$.

The set of distinct signed degrees of the vertices in a signed graph G is called its signed degree set. In [6], it is proved that every set of positive (negative) integers is the signed degree set of some connected signed graph and the smallest possible order for such a signed graph is also determined. Hoffman and Jordan [4] have shown that the degree sequences of signed graphs can be characterized by a system of linear inequalities. The set of all n-tuples satisfying this system of linear inequalities is a polytope P_n. In [5], Jordan et al. have proved that P_n is the convex hull of the set of degree sequences of signed graphs of order n. We can find more results on signed degrees in [4,5].

A signed bipartite graph is a bipartite graph in which each edge is assigned a positive or a negative sign. Let $G(U, V)$ be a signed bipartite graph with $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$. Then signed degree of u_i is $sdeg(u_i) = d_i = d_i^+ - d_i^-$, where $1 \leq i \leq p$ and $d_i^+(d_i^-)$ is the number of positive (negative) edges incident with u_i and signed degree of v_j is $sdeg(v_j) = e_j = e_j^+ - e_j^-$, where $1 \leq j \leq q$ and $e_j^+(e_j^-)$ is the number of positive (negative) edges incident with v_j. The sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are called the signed degree sequences of the signed bipartite graph $G(U, V)$. Two sequences $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ are standard sequences if α is non-zero and non-increasing, $|d_1| \geq |d_p|$, $\sum_{i=1}^{p} d_i = \sum_{j=1}^{q} e_j$, $d_1 > 0$, each $|d_i| \leq q$, each $|e_j| \leq p$ and $|e_j| \leq |d_1|$.

The following result due to Pirzada et al. [8], gives necessary and sufficient conditions for two sequences of integers to be the signed degree sequences of some signed bipartite graph. .

Theorem 3. Let $\alpha = [d_1, d_2, \cdots, d_p]$ and $\beta = [e_1, e_2, \cdots, e_q]$ be standard sequences. Then, α and β are the signed degree sequences of a signed bipartite graph if and only if there exist integers r and s with $d_1 = r - s$ and $0 \leq s \leq \frac{q-d_1}{2}$ such that α' and β' are the signed degree sequences of a signed bipartite graph, where α' is obtained from α by deleting d_1 and β' is obtained from β by reducing r greatest entries of β by 1 each and adding s least entries of β by 1 each.

The set of distinct signed degrees of the vertices in a signed bipartite graph $G(U,V)$ is called its signed degree set. The work for signed degree sets in signed bipartite graphs can be found in [7]. Also the work on signed degrees in signed tripartite graphs can be found in [10, 11].

2. Signed degree sequences in signed k-partite graphs

A signed k-partite graph (signed multipartite graph) is a k-partite graph in which each edge is assigned a positive or a negative sign. Let $G(V_1, V_2, \cdots, V_k)$ be a signed k-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \le i \le k$. The signed degree of v_{ij} is $sdeg(v_{ij}) = d_{ij} = d^+_{ij} - d^-_{ij}$, where $1 \le i \le k$, $1 \le j \le n_i$ and d^+_{ij} (d^-_{ij}) is the number of positive (negative) edges incident with v_{ij}. The sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \le i \le k$, are called the signed degree sequences of $G(V_1, V_2, \cdots, V_k)$. Also the sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \le i \le k$, of integers are s-graphical if $\alpha_i' s$ are the signed degree sequences of some signed k-partite graph. Denote a positive edge xy by xy^+ and a negative edge xy by xy^-. Several results on signed degree sequences in signed multipartite graphs can be found in [9]. We start with the following observation.

Theorem 4. Let $G(V_1, V_2, \cdots, V_k)$ be a signed k-partite graph with $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \le i \le k$ and having q edges. Then

$$p = \sum_{i=1}^{k} \sum_{j=1}^{n_i} s\deg(v_{ij}) \equiv 2q (mod\ 4),$$

and the number of positive edges and negative edges of $G(V_1, V_2, \cdots, V_k)$ are respectively $\frac{2q+p}{4}$ and $\frac{2q-p}{4}$.

Proof. Let v_{ij} ($1 \le i \le k$, $1 \le j \le n_i$) be incident with d^+_{ij} positive edges and d^-_{ij} negative edges so that

$$sdeg(v_{ij}) = d^+_{ij} - d^-_{ij} \text{ while } deg(v_{ij}) = d^+_{ij} + d^-_{ij}.$$

Obviously, $\sum_{i=1}^{k} \sum_{j=1}^{n_i} \deg(v_{ij}) = 2q$.

Let $G(V_1, V_2, \cdots, V_k)$ have g positive edges and h negative edges. Then $q = g + h$,

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} d^+_{ij} = 2g \text{ and } \sum_{i=1}^{k} \sum_{j=1}^{n_i} d^-_{ij} = 2h.$$

Further,

$$\begin{aligned} \sum_{i=1}^{k} \sum_{j=1}^{n_i} s\deg(v_{ij}) &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (d^+_{ij} - d^-_{ij}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} d^+_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} d^-_{ij} \\ &= 2g - 2h. \end{aligned}$$

Hence,

$$\begin{aligned} p = \sum_{i=1}^{k} \sum_{j=1}^{n_i} s\deg(v_{ij}) &\equiv 2g - 2h \\ &= 2(q-h) - 2h \\ &= 2q - 4h, \end{aligned}$$

so that $p \equiv 2q(mod\ 4)$. Again, from $g + h = q$ and $2g - 2h = p$, we have $g = \frac{2q+p}{4}$ and $h = \frac{2q-p}{4}$. $\square$

Corollary 5. A necessary condition for the k sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, of integers to be s-graphical is that $\sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}$ is even.

A zero sequence is a finite sequence each term of which is 0. Clearly, every k finite zero sequences are the signed degree sequences of a signed k-partite graph. If $\beta = [a_1, a_2, \cdots, a_n]$ is a sequence of integers, then the negative of β is the sequence $\beta = [-a_1, -a_2, \cdots, -a_n]$.

The next result follows by interchanging positive edges with negative edges.

Theorem 6. The sequences $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, are the signed degree sequences of some signed k-partite graph if and only if $-\alpha_i = [-d_{i1}, -d_{i2}, \cdots, -d_{in_i}]$ are the signed degree sequences of some signed k-partite graph.

Assume without loss of generality, that a non-zero sequence $\beta = [a_1, a_2, \cdots, a_n]$ is non-increasing and $|a_1| \geq |a_n|$, for we can always replace β by $-\beta$ if necessary. The k sequences of integers $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, are said to be standard sequences if α_1 is non-zero and non-increasing, $\sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{ij}$ is even, $d_{11} > 0$, each $|d_{ij}| \leq \sum_{r=1, r \neq i}^{k} n_r$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, $|d_{11}| \geq \left|d_{1n_1}\right|$ and $|d_{11}| \geq |d_{ij}|$ for each $2 \leq i \leq k$,$1 \leq j \leq n_i$.

A complete signed k-partite graph is a complete k-partite graph in which each edge is assigned a positive or a negative sign. The following result provides a useful recursive test whether the k sequences of integers form the signed degree sequences of some complete signed k-partite graph.

Theorem 7. Let $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be standard sequences and let $r = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{k} n_j\right)$. Let α'_1 be obtained from α_1 by deleting d_{11} and $\alpha'_2, \alpha'_3, \cdots, \alpha'_k$ be obtained from $\alpha_2, \alpha_3, \cdots, \alpha_k$ by reducing r greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding remaining entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each. Then α_i are the signed degree sequences of some complete signed k-partite graph if and only if α'_i are also signed degree sequences of some complete signed k-partite graph, $1 \leq i \leq k$.

Proof. Let $G'(V'_1, V'_2, \cdots, V'_k)$ be a complete signed k-partite graph with signed degree sequences α'_i, $1 \leq i \leq k$. Let $V'_1 = \{v_{12}, v_{13}, \cdots, v_{1n_1}\}$ and $V'_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $2 \leq i \leq k$. Then a complete signed k-partite graph with signed degree sequences α_i , $1 \leq i \leq k$, can be obtained by adding a vertex v_{11} to V'_1 so that there are r positive edges from v_{11} to those r vertices of $V'_2, V'_3, \cdots, V'_k$, whose signed degrees were reduced by 1 in going from α_i to α'_i, and there are negative edges from v_{11} to the remaining vertices of $V'_2, V'_3, \cdots, V'_k$, whose signed degrees were increased by 1 in going from α_i to α'_i. Note that the signed degree of v_{11} is $r - \left(\sum_{j=2}^{k} n_j - r\right) = 2r - \sum_{j=2}^{k} n_j = d_{11}$.

Conversely, let α_i , $1 \leq i \leq k$, be the signed degree sequences of a complete signed k-partite graph. Let the vertex sets of the complete signed k-partite graph be $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ such that $sdeg(v_{ij}) = d_{ij}$, $1 \leq i \leq k$, $1 \leq j \leq n_i$.

Among all the complete signed k-partite graphs with α_i, $1 \leq i \leq k$, as the signed degree sequences, let $G(V_1, V_2, \cdots, V_k)$ be one with the property that the sum S of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges is maximum. Let d^+_{11} and d^-_{11} be respectively the number of positive edges and the number of negative edges incident with v_{11}. Then $sdeg(v_{11}) = d_{11} = d^+_{11} - d^-_{11}$, $deg(v_{11}) = d^+_{11} + d^-_{11} = \sum_{j=2}^{k} n_j$, and hence $d^+_{11} = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{k} n_j\right) = r$. Let U be the set of r

vertices of $V_2, V_2, \cdots, V_k$ with highest signed degrees and let $W = \cup_{j=2}^{k} V_j - U$. We claim that v_{11} must be joined by positive edges to the vertices of U. If this is not true, then there exist vertices $v_{gh} \in U$ and $v_{ij} \in W$ such that the edge $v_{11}v_{gh}$ is negative and the edge $v_{11}v_{ij}$ is positive. Since $sdeg\,(v_{gh}) \geq sdeg\,(v_{ij})$, there exist vertices v_{mn} and v_{pq} such that the edge $v_{gh}v_{mn}$ is positive and the edge $v_{ij}v_{pq}$ is negative. If the edge $v_{gh}v_{pq}$ is positive, then change the signs of the edges $v_{11}v_{gh}$, $v_{gh}v_{pq}$, $v_{pq}v_{ij}$, $v_{ij}v_{11}$ so that the edges $v_{11}v_{gh}$ and $v_{pq}v_{ij}$ are positive and the edges $v_{11}v_{ij}$and $v_{gh}v_{pq}$ are negative. But if the edge $v_{gh}v_{pq}$ is negative, then $sdeg\,(v_{gh}) < sdeg\,(v_{ij})$, which is a contradiction. The case when $v_{mn} = v_{pq}$ follows by the same argument as in above.

Hence we obtain a complete signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, in which the sum of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges exceeds S, a contradiction.

Thus we may assume that v_{11} is joined by positive edges to the vertices of U and by negative edges to the vertices of W. So $G(V_1, V_2, \cdots, V_k) - v_{11}$ is a complete signed k-partite graph with α_i', $1 \leq i \leq k$, as the signed degree sequences. □

Theorem 7 provides an algorithm of checking whether the standard sequences α_i, $1 \leq i \leq k$, are the signed degree sequences, and for constructing a corresponding complete signed k-partite graph. Suppose $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be the standard signed degree sequences of a complete signed k-partite graph with parts $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Deleting d_{11} and reducing $r = \frac{1}{2}\left(d_{11} + \sum_{j=2}^{k} n_j\right)$ greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding remaining entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each to form $\alpha_2', \alpha_3', \cdots, \alpha_k'$. Then edges are defined by $v_{11}\ v_{ij}^{+}$ if $d_{ij}'s$ are reduced by 1 and $v_{11}v_{ij}^{-}$ if $d_{ij}'s$ are increased by 1. For $-\alpha_i$, $1 \leq i \leq k$, edges are defined by $v_{11}v_{ij}^{-}$ if $d_{ij}'s$ are reduced by 1 and $v_{11}v_{ij}^{+}$ if $d_{ij}'s$ are increased by 1. If the conditions of standard sequences do not hold, then we delete d_{i1} for that i for which the conditions of standard sequences get satisfied. If this method is applied recursively, then a complete signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, is constructed.

The next result gives necessary and sufficient conditions for the k sequences of integers to be the signed degree sequences of some signed k-partite graph.

Theorem 8. Let $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \leq i \leq k$, be standard sequences. Then α_i , $1 \leq i \leq k$, are the signed degree sequences of a signed k-partite graph if and only if there exist integers r and s with $d_{11} = r - s$ and $0 \leq s \leq \frac{1}{2}\left(\sum_{j=2}^{k} n_j - d_{11}\right)$ such that α_i' are the signed degree sequences of a signed k-partite graph, where α_1' is obtained from α_1 by deleting d_{11} and $\alpha_2', \alpha_3', \cdots, \alpha_k'$ are obtained from $\alpha_2, \alpha_3, \cdots, \alpha_k$ by reducing r greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding s least entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each.

Proof. Let r and s be integers with $d_{11} = r - s$ and $\leq s \leq \frac{1}{2}\left(\sum_{j=2}^{k} n_j - d_{11}\right)$ such that α_i', $1 \leq i \leq k$, are the signed degree sequences of a signed k-partite graph $G'(V_1', V_2', \prime, V_k')$.

Let $V_1' = \{v_{12}, v_{13}, \cdots, v_{1n_1}\}$ and $V_i' = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $2 \leq i \leq k$. Let U be the set of r vertices of $V_2', V_3', \cdots, V_k'$ with highest signed degrees, W be the set of s vertices of $V_2', V_3', \cdots, V_k'$ with least signed degrees and let $Z = \cup_{j=2}^{k} V_j' - U - W$. Then a signed k-partite graph with signed degree sequences α_i, $1 \leq i \leq k$, can be obtained by adding a vertex v_{11} to V_1' so that there are r positive edges from v_{11} to the vertices of U and s negative edges from v_{11} to the vertices of W. Note that the signed degree of v_{11} is $r - s = d_{11}$.

Conversely, let α_i, $1 \leq i \leq k$, be the signed degree sequences of a signed k-partite graph. Let the vertex sets of the signed k-partite graph be $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ such that $sdeg(v_{ij}) = d_{ij}$, $1 \le i \le k$, $1 \le j \le n_i$.

Among all the signed k-partite graphs with α_i, $1 \le i \le k$, as the signed degree sequences, let $G(V_1, V_2, \cdots, V_k)$ be one with the property that the sum S of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges is maximum. Let $d_{11}^+ = r$ and $d_{11}^- = s$ be respectively the number of positive edges and the number of negative edges incident with v_{11}. Then $sdeg(v_{11}) = d_{11} = d_{11}^+ - d_{11}^- = r - s$ and $deg(v_{11}) = d_{11}^+ + d_{11}^- = r + s \le \sum_{j=2}^k n_j$, and hence $0 \le s \le \frac{1}{2}\left(\sum_{j=2}^k n_j - d_{11}\right)$. Let U be the set of r vertices of $V_2, V_3, \cdots, V_k$ with highest signed degrees and let $W = \cup_{j=2}^k V_j - U$.

We claim that v_{11} must be joined by positive edges to the vertices of U. If this is not true, then there exist vertices $v_{gh} \in U$ and $v_{mn} \in W$ such that the edge $v_{11}v_{mn}$ is positive and either (i) $v_{11}v_{gh}$ is a negative edge or (ii) v_{11} and v_{gh} are not adjacent in $G(V_1, V_2, \cdots, V_k)$. As $sdeg(v_{gh}) \ge sdeg(v_{mn})$, that is $d_{gh} \ge d_{mn}$, therefore we consider only (i) and then (ii) is similar to (i).

We note that if there exists a vertex v_{pq} ($\ne v_{11}$) such that $v_{pq}v_{gh}$ is a positive edge and $v_{pq}v_{mn}$ is a negative edge, then change the signs of these edges so that $v_{11}v_{gh}$ and $v_{pq}v_{mn}$ are positive, and $v_{11}v_{mn}$ and $v_{pq}v_{gh}$ are negative. Hence we obtain a signed k-partite graph with signed degree sequences α_i, $1 \le i \le k$, in which the sum of the signed degrees of the vertices of $V_2, V_3, \cdots, V_k$ joined to v_{11} by positive edges exceeds S, a contradiction. So assume that no such vertex v_{pq} exists.

Now, suppose that v_{gh} is not incident to any positive edge. Since $sdeg(v_{gh}) \ge sdeg(v_{mn})$, that is $d_{gh} \ge d_{mn}$, then there exist at least two vertices v_{pq} and v_{lt} (both distinct from v_{11}) such that $v_{pq}v_{mn}$ and $v_{lt}v_{mn}$ are negative edges and both v_{pq} and v_{lt} are not adjacent to v_{gh}. Then by changing the edges so that $v_{11}v_{gh}$ is a positive edge, and $v_{11}v_{mn}$,$v_{gh}v_{pq}$, $v_{gh}v_{lt}$ are negative edges, we again get a contradiction. Hence v_{gh} is incident to at least one positive edge.

We claim that there exists at least one vertex v_{yz} such that $v_{yz}v_{gh}$ is a positive edge and v_{yz} is not adjacent to v_{mn}. Suppose on contrary that whenever v_{gh} is joined to a vertex by a positive edge, then v_{mn} is also joined to this vertex by a positive edge. Since sdeg(v_{gh}) $\ge sdeg(v_{mn})$, that is $d_{gh} \ge d_{mn}$, then again we have the same situation as above, from which we get a contradiction. Thus there exists a vertex v_{yz} such that $v_{yz}v_{gh}$ is a positive edge and v_{yz} is not adjacent to v_{mn}. Similarly, it can be shown that there exists a vertex v_{pq} such that $v_{pq}v_{mn}$ is a negative edge and v_{pq} is not adjacent to v_{gh}. By changing the edges so that $v_{11}v_{gh}$,$v_{mn}v_{yz}$ are positive edges, and $v_{11}v_{mn}$, $v_{gh}v_{pq}$ are negative edges, we again get a contradiction. Hence v_{11} is joined by positive edges to the vertex of U.

In a similar way, it can be shown that v_{11} is joined by negative edge to the s vertices of V_2,V_3,$\cdots$, V_k with least signed degrees.

Hence $G(V_1, V_2, \cdots, V_k) - v_{11}$ is a signed k-partite graph with α_i', $1 \le i \le k$, as the signed degree sequences. □

Theorem 8 also provides an algorithm for determining whether or not the standard sequences α_i, $1 \le i \le k$, are the signed degree sequences, and for constructing a corresponding signed k-partite graph. Suppose $\alpha_i = [d_{i1}, d_{i2}, \cdots, d_{in_i}]$, $1 \le i \le k$, be the standard signed degrees sequences of a signed k-partite graph with parts $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Let $d_{11} = r - s$ and $0 \le s \le \frac{1}{2}\left(\sum_{j=2}^k n_j - d_{11}\right)$. Deleting d_{11} and reducing r greatest entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each and adding s least entries of $\alpha_2, \alpha_3, \cdots, \alpha_k$ by 1 each to form $\alpha_2', \alpha_3', \cdots, \alpha_k'$. Then edges are defined by $v_{11}v_{ij}^+$ if $d_{ij}'s$ are reduced by 1 ; $v_{11}v_{1j}^-$ if d_{ij}' s are increased by 1, and v_{11} and $v_{ij}$are not adjacent if d_{ij}' s are unchanged. For α_i, edges are defined by $v_{11}v_{ij}^-$ if d_{ij}' s are reduced by 1; $v_{11}v_{ij}^+$ if d_{ij}' s are increased by

1, and v_{11} and v_{ij} are not adjacent if d'_{ij} s are unchanged. If the conditions of standard sequences do not hold, then we delete d_{i1} for that i for which the conditions of standard sequences get satisfied. If this method is applied recursively, then a signed k-partite graph with signed degree sequences α_i, $1 \le i \le k$, is constructed.

3. Signed degree sets in signed k-partite graphs

Let $G(V_1, V_2, \cdots, V_k)$ be a signed k-partite graph with $X \subseteq V_i, Y \subseteq V_j$ $(i \ne j)$. If each vertex of X is joined to every vertex of Y by a positive (negative) edge, then it is denoted by $X \oplus Y(X \ominus Y)$.

The set S of distinct signed degrees of the vertices in a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ is called its signed degree set. Also, a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ is said to be connected if each vertex $v_i \in V_i$; is connected to every vertex $v_j \in V_j$.

The following result shows that every set of positive integers is a signed degree set of some connected signed k-partite graph.

Theorem 9. Let $d_1, d_2, \cdots, d_t$ be positive integers. Then there exists a connected signed k-partite graph with signed degree set

$$S = \{d_1, \sum_{i=1}^{2} d_i, \cdots, \sum_{i=1}^{t} d_i\}.$$

Proof. We consider the following two cases. (i) k even, (ii) k odd.
Case (i). Let $k = 2m$, where $m \ge 1$. Construct a signed k-partite graph $G(V_1, V_2, \cdots, V_{2m})$ as follows.
Let

$$\begin{aligned}
&V_1 = P_1 \cup Q_1 \cup R_1 \cup S_1 \cup X_1 \cup X_1' \cup X_1'' \cup X_2 \cup X_2' \cup X_2'' \cup \cdots \cup X_{t-1} \cup X_{t-1}' \cup X_{t-1}'',\\
&V_2 = P_2 \cup Q_2 \cup R_2 \cup S_2 \cup Y_1 \cup Y_1' \cup Y_2 \cup Y_2' \cup \cdots \cup Y_{t-1} \cup Y_{t-1}',\\
&V_3 = P_3 \cup Q_3,\\
&\vdots\\
&V_{2m-1} = P_{2m-1} \cup Q_{2m-1},\\
&V_{2m} = P_{2m} \cup Q_{2m},
\end{aligned}$$

where
(a) $P_1, Q_1, R_1, S_1, X_1, X_1', X_1'', X_2, X_2', X_2'', \cdots, X_{t-1}, X_{t-1}', X_{t-1}''$ are pairwise disjoint,
(b) $P_2, Q_2, R_2, S_2, Y_1, Y_1', Y_2, Y_2', \cdots, Y_{t-1}, Y_{t-1}'$ are pairwise disjoint,
(c) For all i, $P_i \cap Q_i = \phi$, $3 \le i \le 2m$ and $|P_i| = |Q_i| = d_1$, $1 \le i \le 2m$; $|R_i| = |S_i| = d_1$, $1 \le i \le 2$; $|X_i| = |X_i'| = |Y_i| = |Y_i'| = d_1$, $1 \le i \le t-1$; $|X_i''| = d_2 + d_3 + \cdots + d_{i+1}$, $1 \le i \le t-1$.

For all i, let $P_i \oplus Q_{i+1}$, $1 \le i \le 2m-1$; $Q_i \oplus P_{i+1}$, $1 \le i \le 2m-1$; $Q_1 \oplus R_2, R_1 \oplus Q_2, R_1 \oplus S_2, S_1 \oplus R_2, X_1 \oplus S_2, X_1' \oplus R_2, X_i \oplus Y_i'$, $1 \le i \le t-1$; $X_i' \oplus Y_i$, $1 \le i \le t-1$; $X_i'' \oplus Y_i'$, $1 \le i \le t-1$; $X_i \oplus Y_{i-1}'$, $2 \le i \le t-1$; $X_i' \oplus Y_{i-1}$, $2 \le i \le t-1$;
for all even i, $P_i \ominus P_{i+1}$, $2 \le i \le 2m-2$; $Q_i \ominus Q_{i+1}$, $2 \le i \le 2m-2$;
and for all i, $Q_1 \ominus Q_2, R_1 \ominus R_2, X_1 \ominus R_2, X_1' \ominus S_2, X_i \ominus Y_{i-1}$, $2 \le i \le t-1$; $X_i' \ominus Y_{i-1}'$, $2 \le i \le t-1$.
Then the signed degrees of the vertices of $G(V_1, V_2, \cdots, V_{2m})$ are as follows.

$sdeg(p_1) = |Q_2| - 0 = d_1$ for all $p_1 \in P_1$;
for even i, $2 \leq i \leq 2m - 2$
$sdeg(p_i) = |Q_{i-1}| + |Q_{i+1}| - |P_{i+1}| = d_1 + d_1 - d_1 = d_1$, for all $p_i \in P_i$;
for odd i, $3 \leq i \leq 2m - 1$
$sdeg(p_i) = |Q_{i-1}| + |Q_{i+1}| - |P_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $p_i \in P_i$,
$sdeg(p_{2m}) = |Q_{2m-1}| - 0 = d_1$, for all $p_{2m} \in P_{2m}$;
$sdeg(q_1) = |P_2| + |R_2| - |Q_2| = d_1 + d_1 - d_1 = d_1$, for all $q_1 \in Q_1$;
$sdeg(q_2) = |P_1| + |R_1| + |P_3| - (|Q_1| + |Q_3|) = d_1 + d_1 + d_1 - (d_1 + d_1) = d_1$, for all $q_2 \in Q_2$;
for odd i, $3 \leq i \leq 2m - 1$
$sdeg(q_i) = |P_{i-1}| + |P_{i+1}| - |Q_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $q_i \in Q_i$;
for even i, $4 \leq i \leq 2m - 2$
$sdeg(q_i) = |_{i-1}| + |P_{i+1}| - |Q_{i+1}| = d_1 + d_1 - d_1 = d_1$, for all $q_i \in Q_i$, $sdeg(q_{2m}) = |P_{2m-1}| - 0 = d_1$, for all $q_{2m} \in Q_{2m}$, $sdeg(r_1) = |Q_2| + |S_2| - |R_2| = d_1 + d_1 - d_1 = d_1$, for all $r_1 \in R_1$,
$sdeg(s_1) = |R_2| - 0 = d_1$, for all $s_1 \in S_1$,
$sdeg(r_2) = |Q_1| + |S_1| + |X'_1| - (|R_1| + |X_1|) = d_1 + d_1 + d_1 - (d_1 + d_1) = d_1$, for all $r_2 \in R_2$,
$sdeg(s_2) = |R_1| + |X_1| - |X'_1| = d_1 + d_1 - d_1 = d_1$, for all $s_2 \in S_2$,
$sdeg(x_1) = |S_2| + |Y'_1| - |R_2| = d_1 + d_1 - d_1 = d_1$, for all $x_1 \in X_1$,
$sdeg(x'_1) = |R_2| + |Y_1| - |S_2| = d_1 + d_1 - d_1 = d_1$, for all $x'_1 \in X'_1$,
$sdeg(x''_1) = |Y'_1| - 0 = d_1$, for all $x''_1 \in X''_1$;
for $2 \leq i \leq t - 1$
$sdeg(x_i) = |Y'_{i-1}| + |Y'_i| - |Y_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $x_i \in X_i$;
for $2 \leq i \leq t - 1$
$sdeg(x'_i) = |Y_{i-1}| + |Y_i| - |Y'_{i-1}| = d_1 + d_1 - d_1 = d_1$, for all $x'_i \in X'_i$;
for $2 \leq i \leq t - 1$
$sdeg(x''_i) = |Y'_i| - 0 = d_1$, for all $x''_i \in X''_i$;
for $1 \leq i \leq t - 2$
$sdeg(y_i) = |X'_i| + |X'_{i+1}| - |X_{i+1}| = d_1 + d_1 - d_1 = d_1$, for all $y_i \in Y_i$
$sdeg(y_{t-1}) = |X'_{t-1}| - 0 = d_1$, for all $y_{t-1} \in Y_{t-1}$;
for $1 \leq i \leq t - 2$
$sdeg(y'_i) = |X_i| + |X''_i| + |X_{i+1}| - |X'_{i+1}| = d_1 + d_2 + d_3 + \cdots + d_{i+1} + d_1 - d_1 = \sum_{j=1}^{i+1} d_j$, for all $y'_i \in Y'_i$,
and $sdeg(y'_{t-1}) = |X_{t-1}| + |X''_{t-1}| = d_1 + d_2 + d_3 + \cdots + d_t = \sum_{j=1}^{t} d_j$, for all $y'_{t-1} \in Y'_{t-1}$.
Therefore signed degree set of $G(V_1, V_2, \prime, V_{2m})$ is $S = \{d_1, \sum_{i=1}^{2} d_i, \prime, \sum_{i=1}^{t} d_i\}$.
Case (ii). Let $k = 2m + 1$, where $m \geq 1$. This follows by using the construction as in case (i), and taking another partite set $V_{2m+1} = P_{2m+1} \cup Q_{2m+1}$ with $P_{2m+1} \cap Q_{2m+1} = \phi, |P_{2m+1}| = |Q_{2m+1}| = d_1, P_{2m} \oplus Q_{2m+1}, Q_{2m} \oplus P_{2m+1}, P_{2m+1} \oplus P_1, P_{2m+1} \oplus R_2, Q_{2m+1} \oplus S_1, Q_1 \oplus S_2$ and $P_{2m} \ominus P_{2m+1}, Q_{2m} \ominus Q_{2m+1}, P_{2m+1} \ominus Q_1, P_1 \ominus R_2$, $S_1 \ominus S_2$.
Clearly, by construction , the above signed k-partite graphs are connected. Hence the result follows. □

By interchanging positive edges with negative edges in Theorem 9, we obtain the following result.

Corollary 10. Every set of negative integers is a signed degree set of some connected signed k-partite graph.

Finally, we have the following result.

Theorem 11. Every set of integers is a signed degree set of some connected signed k-partite graph.

Proof. Let S be a set of integers. Then we have the following five cases.

Case (i). S is a set of positive (negative) integers. Then the result follows by Theorem 9 (Corollary 10).

Case (ii). $S = \{0\}$. Then a signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ with $V_i = \{v_i, v_i'\}$ for all i, $1 \leq i \leq k$, in which $v_i v_{i+1}', v_i' v_{i+1}$ for all i, $1 \leq i \leq k-1$, are positive edges and $v_i v_{i+1}, v_i' v_{i+1}'$ for all i, $1 \leq i \leq k-1$, are negative edges has signed degree set S.

Case (iii). S is a set of non-negative (non-positive) integers. Let $S = S' \cup \{0\}$, where S' be a set of positive(negative) integers. Then by Theorem 9(Corollary 10), there is a connected signed k-partite graph $G'(V_1', V_2', \cdots, V_k')$ with signed degree set S'. Construct a new signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let $V_1 = V_1' \cup \{x_1\} \cup \{y_1\}$, $V_2 = V_2' \cup \{x_2\} \cup \{y_2\}$, $V_3 = V_3'$, $\cdots$, $V_k = V_k'$, with $V_1' \cap \{x_1\} = \phi, V_1' \cap \{y_1\} = \phi, \{x_1\} \cap \{y_1\} = \phi, V_2' \cap \{x_2\} = \phi, V_2' \cap \{y_2\} = \phi, \{x_2\} \cap \{y_2\} = \phi$. Let $v_1' x_2, x_1 v_2', y_1 y_2$ be positive edges, $v_1' y_2, x_1 x_2, y_1 v_2'$ be negative edges, where $v_1' \in V_1', v_2' \in V_2'$ and let there be all the edges of $G'(V_1', V_2', \cdots, V_k')$. Then $G(V_1, V_2, \cdots, V_k)$ has signed degree set S. We note that addition of the positive edges $v_1' x_2, x_1 v_2', y_1 y_2$ and negative edges $v_1' y_2, x_1 x_2, y_1 v_2'$ do not effect the signed degrees of the vertices of $G'(V_1', V_2', \cdots, V_k')$, and the vertices x_1, y_1, x_2, y_2 have signed degrees zero each.

Case (iv). S is a set of non-zero integers. Let $S = S' \cup S''$, where S' and S'' are sets of positive and negative integers respectively. Then by Theorem 9 (Corollary 10), there are connected signed k-partite graphs $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets S' and S'' respectively. Suppose $G_1'(V_{11}', V_{21}', \cdots, V_{k1}')$ and $G_2''(V_{12}'', V_{22}'', \cdots, V_{k2}'')$ are the copies of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets S' and S'' respectively. Construct a new signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let

$$V_1 = V_1' \cup V_{11}' \cup V_1'' \cup V_{12}'',$$

$$V_2 = V_2' \cup V_{21}' \cup V_2'' \cup V_{22}'',$$

$$V_3 = V_3' \cup V_{31}' \cup V_3'' \cup V_{32}'',$$

$$\vdots$$

$$V_k = V_k' \cup V_{k1}' \cup V_k'' \cup V_{k2}'',$$

with $V_i' \cap V_{i1}' = \phi, V_i' \cap V_i'' = \phi, V_i' \cap V_{i2}'' = \phi, V_{i1}' \cap V_i'' = \phi, V_{i1}' \cap V_{i2}'' = \phi, V_i'' \cap V_{i2}'' = \phi$. Let $v_1' v_{22}'', v_{11}' v_2''$ be positive edges, $v_1' v_2', v_{11}' v_{22}''$ be negative edges, where $v_1' \in V_1', v_{11}' \in V_{11}', v_2'' \in V_2'', v_{22}'' \in V_{22}''$ and let there be all the edges of $G'(V_1', V_2', \cdots, V_k'), G_1'(V_{11}', V_{21}', \cdots, V_{k1}'), G''(V_1'', V_2'', \cdots, V_k'')$ and $G_2''(V_{12}'', V_{22}'', \cdots, V_{k2}'')$. Then $G(V_1, V_2, \cdots, V_k)$ has signed degree set S.

We note that addition of the positive edges $v_1' v_{22}'', v_{11}' v_2''$ and negative edges $v_1' v_2'', v_{11}' v_{22}''$ do not effect the signed degrees of the vertices of $G'(V_1', V_2', \cdots, V_k')$, $G_1'(V_{11}', V_{21}', \cdots, V_{k1}')$, $G''(V_1'', V_2'', \cdots, V_k'')$ and $G_2''(V_{12}'', V_{22}'', \cdots, V_{k2}'')$.

Case (v). S is the set of all integers. Let $S = S' \cup S'' \cup \{0\}$, where S' and S'' are sets of positive and negative integers respectively. Then by Theorem 9(Corollary 10), there exist connected signed k-partite graphs $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$ with signed degree sets S' and S'' respectively. Construct a new signed k-partite graph $G(V_1, V_2, \cdots, V_k)$ as follows.

Let

$$V_1 = V_1' \cup V_1'' \cup \{x\},$$
$$V_2 = V_2' \cup V_2'' \cup \{y\},$$
$$V_3 = V_3' \cup V_3'',$$
$$\vdots$$
$$V_k = V_k' \cup V_k'',$$

with $V_i' \cap V_i'' = \phi, V_1' \cap \{x\} = \phi, V_1'' \cap \{x\} = \phi, V_2' \cap \{y\} = \phi, V_2'' \cap \{y\} = \phi$. Let $v_1'v_2'', v_1''y, xv_2'$ be positive edges, $v_1'y, v_1''v_2', xv_2''$ be negative edges, where $v_1' \in V_1', v_1'' \in V_1'', v_2' \in V_2', v_2'' \in V_2''$, and let there be all the edges of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$. Therefore $G(V_1, V_2, \cdots, V_k)$ has signed degree set S. We note that addition of the positive edges $v_1'v_2'', v_1''y, xv_2'$ and negative edges $v_1'y, v_1''v_2', xv_2''$ do not effect the signed degrees of the vertices of $G'(V_1', V_2', \cdots, V_k')$ and $G''(V_1'', V_2'', \cdots, V_k'')$, and the vertices x and y have signed degrees zero each.

Clearly, by construction, all the signed k-partite graphs are connected. This proves the result. □

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