Optimal capital allocation with copulas

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Abstract

In this paper, we investigate optimal capital allocation problems for a portfolio consisting of different lines of risks linked by a Farlie-Gumbel-Morgenstern copula, modelling the dependence between them. Based on the Tail Mean-Variance principle, we examine the bivariate case and then the multivariate case. Explicit formulae for optimal capital allocations are obtained for exponential loss distributions. Finally, the results are illustrated by various numerical examples.

Keywords: Optimal Capital allocation; Tail Mean-Variance; Copula

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1. Introduction

A fundamental question in the fields such as quantitative risk management, insurance and finance is how much money to ask from a policyholder in a heterogeneous portfolio? A closely related question is how to allocate a given amount of capital between the different classes in a portfolio. The first question is known as the premium calculation problem and the second is the capital allocation problem. The two problems are well-known and have been studied, see e.g. Laeven and Goovaerts [13], Zaks et al. [20], Frostig et al. [10], Belles-Sampera [4], Bauer and Zanjani [2,3] and references therein.

In recent years, there has been growing interest in studying the optimal allocation problems because they play a central role in Solvency II. For example, Dhaene et al. [9] assumed a portfolio consisting of $n$ risks $X_1, X_2, \ldots, X_n$ and a company wished to allocate the total capital $d = d_1 + d_2 + \cdots + d_n$ to the corresponding risks. They formulated capital allocation as an optimization problem and provided a reasonable criterion, which is to set the amount $d_i$ to $X_i$ so that the potential loss measured by some appropriate
distance measure is as small as possible. And further, they proposed to model the capital allocation problem by the following optimization problem:

$$\min_{d_1, \ldots, d_n} \sum_{i=1}^{n} v_i E \left[ \zeta_i D \left( \frac{X_i - d_i}{v_i} \right) \right], \text{ such that } \sum_{i=1}^{n} d_i = d, \tag{1.1}$$

where the $v_i$, $i = 1, 2, \cdots, n$, are non-negative real numbers such that $\sum_{i=1}^{n} v_i = 1$, $D$ is a non-negative function and the $\zeta_i$, $i = 1, 2, \cdots, n$, are non-negative random variables such that $E[\zeta_i] = 1$. The non-negative real number $v_i$ is a measure of exposure or business volume of the $i$th unit, such as revenue, insurance premium, etc. $D$ is a distance measurement function that measures the loss of allocation. The non-negative random variables $\zeta_i$, $i = 1, 2, \cdots, n$, are used as weight factors to the different possible outcomes of $D(X_i - d_i)$. The framework is general and includes many optimisation criterion as special cases, such as Quantile, Haircut and Covariance. A special case of (1.1) is the following optimization problem:

$$\min_{d_1, \ldots, d_n} \sum_{i=1}^{n} v_i E \left[ (X_i - d_i)^2 \right], \text{ such that } \sum_{i=1}^{n} d_i = d,$$

which is showed in Dhaene et al. [9]. The solution to the equation above is

$$d_i = \frac{d}{n} + E(X_i) - \frac{1}{n} \sum_{j=1}^{n} E(X_j), \quad i = 1, 2, \cdots, n.$$

Xu and Hu [17] generalized this idea and defined the following loss function:

$$L(p) = \sum_{i=1}^{n} D(X_i - p_i).$$

Then, they proposed to model the capital allocation problem by the following optimization problem:

$$\min_{\vec{d} \in A} \Pr(L(\vec{d}) \geq t), \quad \forall t \geq 0. \tag{1.2}$$


Note that allocation criterion based on minimizing the loss function does not take into account the factor of variability of loss function. We also note that both allocation rule (1.1) and (1.2) do not take into account tail risk. Xu and Mao [18] proposed a Tail Mean-Variance (TMV) framework to overcome this limitation. In more detail, they provided the following allocation rule:

$$\min_{d_1, \ldots, d_n} \left\{ E \left[ \sum_{i=1}^{n} (X_i - d_i)^2 | S > \text{VaR}_\kappa(S) \right] + \beta \text{Var} \left( \sum_{i=1}^{n} (X_i - d_i)^2 | S > \text{VaR}_\kappa(S) \right) \right\},$$

such that $\sum_{i=1}^{n} d_i = d, \tag{1.3}$

where $\beta > 0$, $S = \sum_{i=1}^{n} X_i$ is the aggregate risk, $\text{VaR}_\kappa(S) = \inf(x \in R, F(S) \geq \kappa)$ is the value at risk at level $\kappa$, $0 < \kappa < 1$ of $S$ and $F(S) = \Pr(S \leq s)$ is the distribution of $S$. Xu and Mao [18] studied the optimal capital allocations when the risks have multivariate elliptical distributions.
In most papers described above, the dependence between different lines of business of a portfolio is due to construction of a multivariate distribution. Copulas are currently seen as flexible and effective tools to describe dependence between the random variables. In this paper, we propose introducing dependence with a copula. We use the Farlie-Gumbel-Morgenstern (FGM) copula, like Bargès et al. [1] and Cossette et al. [7], to describe the dependence. Based on the simplicity and tractability of FGM copula, we derive the closed-form expressions for the optimal capital allocations based on the Tail Mean-Variance model when the FGM copula represent dependence between the risk marginals.

The work of this paper can be seen as a complement to the research on the the optimal capital allocations based on the Tail Mean-Variance principle and extend the former results by introducing dependence with a copula. When the optimal capital allocation problems for a portfolio consisting of different lines of risks linked by a Farlie-Gumbel-Morgenstern copula are considered, we show that, the explicit capital allocation formulae can be obtained under the Tail Mean-Variance principle.

The rest of the paper is organized as follows. In Section 2, we describe basic definitions and properties of the FGM copula and the exponential distributions. In Section 3, we derive explicit capital allocation formulae for the bivariate case, where both losses are exponentially distributed with the dependence linked by the FGM copula. In Section 4, we extend the results to the multivariate case. Some numerical examples to calculate the optimal capital allocation are presented to illustrate the solution procedure in Section 5.

2. Definitions

Here, we briefly recall the characteristics and definition of the FGM copula. The FGM copula is a perturbation of the independence copula and it is not Archimedean. The FGM copula is a first order approximation of the Plackett copula (Nelsen [15], p.100) and of the Frank copula (p.133). Among the recent applications of the FGM copula, we mention Bargès et al. [1] and Cossette et al. [7] who deal with the application of the TVaR-based allocation rule using the FGM copula. The FGM copula is applied in the context of risk measurement by Gebizlioglu and Yagci [11] and in the context of sums of dependent r.v. by Geluk and Tang [12]. The FGM copula is also used in a ruin context by Cossette et al. [8], Xie and Zou [16]. The FGM copula is attractive due to its tractability and its simplicity.

We assume the lines of business of an portfolio are linked with a FGM copula. Since the exponential distribution is a classic distribution for the risk random variables and its practical mathematic properties allow explicit results, we assume marginal risks are distributed as exponentials.

2.1 The bivariate case

Firstly, we consider two business lines whose losses \(X_1\) and \(X_2\) follow the exponential distribution: \(X_i \sim \text{Exp}(\lambda_i), i = 1, 2\). Their probability density functions (pdf) \(f_{X_i}\) and cumulative distribution functions (cdf) \(F_{X_i}\) are given by

\[
 f_{X_i} = \lambda_i e^{-\lambda_i x_i}, \quad F_{X_i} = 1 - e^{-\lambda_i x_i}, \quad \text{for } i = 1, 2.
\]

We assume that the couple \((X_1, X_2)\) has a bivariate distribution defined with FGM copula. The FGM copula is defined as follows:

\[
 C^{FGM}_{\theta}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2),
\]
for \((u_1, u_2) \in [0, 1] \times [0, 1]\). The dependence parameter \(\theta\) takes value in \([-1, 1]\), where \(\theta < 0 (> 0)\) corresponds to a negative (positive) dependence relation. The density of the bivariate FGM copula is
\[
C^\text{FGM}_\theta(u_1, u_2) = \frac{\partial^2 C^\text{FGM}_\theta(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + \theta(2\bar{u}_1 - 1)(2\bar{u}_2 - 1),
\]
where \(\bar{u}_i = 1 - u_i\), \(i = 1, 2\). Then, the joint c.d.f. \(F_{X_1, X_2}(x_1, x_2)\) of \((X_1, X_2)\) with marginals \(F_{X_1}\) and \(F_{X_2}\) and defined with the FGM copula is given by
\[
F_{X_1, X_2}(x_1, x_2) = C^\text{FGM}_\theta(F_{X_1}(x_1), F_{X_2}(x_2)) = F_{X_1}(x_1)F_{X_2}(x_2) + \theta F_{X_1}(x_1)F_{X_2}(x_2)(1 - F_{X_1}(x_1))(1 - F_{X_2}(x_2)).
\]

The joint pdf of \((X_1, X_2)\) is
\[
f_{X_1, X_2}(x_1, x_2) = C^\text{FGM}_\theta(F_{X_1}(x_1), F_{X_2}(x_2))f_{X_1}(x_1)f_{X_2}(x_2)
= (1 + \theta)\lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} - \theta \lambda_2 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}
- \theta \lambda_1 e^{-\lambda_1 x_1} 2\lambda_2 e^{-\lambda_2 x_2} + \theta \lambda_1 e^{-\lambda_1 x_1} 2\lambda_2 e^{-\lambda_2 x_2}.
\]

\[\text{(2.1)}\]

2.2 The multivariate case

Now, we consider \(n\) business lines whose losses \(X_1, X_2, \ldots, X_n\) follow the exponential distribution: \(X_i \sim \text{Exp}(\lambda_i), i = 1, 2, \ldots, n\). We also assume that the \(n\) different risks jointed by a multivariate FGM \(n\)-copula, which has \(2^n - n - 1\) parameters, is defined as follows:
\[
C^\text{FGM}_\theta(u_1, \ldots, u_n) = u_1 \cdots u_n \left(1 + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n} \theta_{p_1 \cdots p_q} \bar{u}_{p_1} \cdots \bar{u}_{p_q}\right),
\]
where \(\theta \in [-1, 1]\), \(\bar{u}_i = 1 - u_i\), \(i = 1, 2, \ldots, n\). Its density can be written as
\[
c^\text{FGM}_\theta(u_1, \ldots, u_n) = 1 + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n} \theta_{p_1 \cdots p_q}(2\bar{u}_{p_1} - 1) \cdots (2\bar{u}_{p_q} - 1).
\]

Then, the joint pdf of \((X_1, \ldots, X_n)\) is
\[
f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = C^\text{FGM}_\theta(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))f_{X_1}(x_1) \cdots f_{X_n}(x_n)
= f_{X_1}(x_1) \cdots f_{X_n}(x_n)(1 + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n} \theta_{p_1 \cdots p_q}(1 - F_{X_{p_1}}(x_{p_1})) \cdots (1 - F_{X_{p_q}}(x_{p_q}))
= \omega(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n) + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n} \theta_{p_1 \cdots p_q} \times
\]
\[
\left(\sum_{l=0}^{q-1} \sum_{(i_1, \ldots, i_q) \in \tau_{l,q}} (-1)^l \omega(x_{p_1}, \ldots, x_{p_q}; x_{i_{q+1}}, \ldots, x_{i_n}; 2^{q-1} \lambda_{i_1}, \ldots, 2^{q-1} \lambda_{i_q}, \lambda_{i_{q+1}}, \ldots, \lambda_{i_n})\right),
\]
\[\text{(2.2)}\]
3. Calculation of the optimal capital allocation for two risks

In this section, we obtain the close form expression of the optimal capital allocation based on the Tail Mean-Variance (TMV) model for two exponentially distributed risks linked by a FGM copula. Let $X_1$ and $X_2$ are two exponentially distributed with $\lambda_1$ and $\lambda_2$. In order to simplify the calculation, we assume that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 2\lambda_2$, and $\lambda_2 \neq 2\lambda_1$. Applying the similar technique proposed below, one can obtain the adjusted results without these restrictions.

**Theorem 3.1.** Let $X_1$ and $X_2$ are two exponentially distributed with $\lambda_1$ and $\lambda_2$. The dependence structure for $(X_1, X_2)$ is defined by the bivariate FGM copula with parameter $\theta \in [-1, 1]$. Then the optimal allocation solution $d^* = (d_1^*, d_2^*)$ to the TMV model

$$
\min_{d_1, d_2} \left\{ \mathbb{E} \left[ \sum_{i=1}^{2} (X_i - d_i)^2 | S > \text{VaR}_k(S) \right] \right\}
$$

is presented as follows:

$$
d_1 = \frac{1}{2(1 - 2\beta(2E^{X_1}X_2 + (E^{X_1} - E^{X_2})^2 - (E^{X_1^2} + E^{X_2^2})) \times \left(2\beta(E^{X_1}X_2 - E^{X_1^2}X_2 + E^{X_1^2} - E^{X_2^2}) + (E^{X_1} - E^{X_2})(1 - 2\beta(E^{X_1^2} + E^{X_2^2})) \right) + d(1 - 4\beta(E^{X_1}X_2 - E^{X_1^2} + E^{X_2}(E^{X_2} - E^{X_1})))}, \tag{3.1}
$$

$$
d_2 = \frac{1}{2(1 - 2\beta(2E^{X_1}X_2 + (E^{X_1} - E^{X_2})^2 - (E^{X_1^2} + E^{X_2^2})) \times \left(-2\beta(E^{X_1}X_2 - E^{X_1^2}X_2 + E^{X_1^2} - E^{X_2^2}) - (E^{X_1} - E^{X_2})(1 - 2\beta(E^{X_1^2} + E^{X_2^2})) \right) + d(1 - 4\beta(E^{X_1}X_2 - E^{X_1^2} + E^{X_2}(E^{X_2} - E^{X_1})))}, \tag{3.2}
$$

where

$$
E^{X_i} = \mathbb{E} \left[ X_i^k | S > \text{VaR}_k(S) \right] = \frac{1}{1 - \kappa} [(1 + \theta)\xi_k(\text{VaR}_k(S); \lambda_1; \lambda_j) - \theta\xi_k(\text{VaR}_k(S); 2\lambda_1; \lambda_j)]
$$

$$
E^{X_iX_j} = \mathbb{E} \left[ X_i^k X_j^k | S > \text{VaR}_k(S) \right] = \frac{1}{1 - \kappa} [(1 + \theta)\chi(\text{VaR}_k(S); \lambda_1; \lambda_j) - \theta\chi(\text{VaR}_k(S); 2\lambda_1; 2\lambda_j)], \quad i, j = 1, 2, \quad \kappa = 1, 2, 3
$$

$$
\xi_1(x; \alpha_i; \alpha_j) = \frac{\alpha_j e^{-\alpha_i x} (x + \frac{1}{\alpha_i})}{\alpha_j - \alpha_i} - \frac{\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x}}{(\alpha_j - \alpha_i)^2},
$$

$$
\xi_2(x; \alpha_i; \alpha_j) = \frac{e^{-\alpha_i x} (\frac{2}{\alpha_i} + \frac{6x}{\alpha_i} + x^2)}{(\alpha_j - \alpha_i)^2} + \frac{2(\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x})}{(\alpha_j - \alpha_i)^2},
$$

$$
\xi_3(x; \alpha_i; \alpha_j) = \frac{e^{-\alpha_i x} (\frac{6e}{\alpha_i} + \frac{3e}{\alpha_i} + x^3)}{(\alpha_j - \alpha_i)^2} - \frac{3e^{-\alpha_i x} (\frac{2}{\alpha_i} + \frac{2x}{\alpha_i} + x^2)}{(\alpha_j - \alpha_i)^2} + \frac{6(\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x})}{(\alpha_j - \alpha_i)^3},
$$

such that $d_1 + d_2 = d$. 

\[ \chi(x; \alpha_i; \alpha_j) = \frac{\alpha_i e^{-\alpha_i x} \left( \frac{2}{\alpha_i} + \frac{2x}{\alpha_i} + x^2 \right)}{(\alpha_j - \alpha_i)^2} - \frac{4\alpha_j e^{-\alpha_j x} \left( x + \frac{1}{\alpha_j} \right) + 2\alpha_i e^{-\alpha_j x} \left( x + \frac{1}{\alpha_i} \right)}{(\alpha_j - \alpha_i)^3} + \frac{6\alpha_i e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x}}{(\alpha_j - \alpha_i)^4}, \]

\[ \varphi(x; \alpha_i; \alpha_j) = \frac{6\alpha_i e^{-\alpha_i x} \left( x + \frac{1}{\alpha_i} \right) + 2\alpha_i e^{-\alpha_j x} \left( x + \frac{1}{\alpha_j} \right)}{(\alpha_j - \alpha_i)^2} + \frac{2\alpha_i e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x}}{(\alpha_j - \alpha_i)^3}. \]

**Proof.** In order to find the optimal allocation solution \( \bar{d} = (d_1^*, d_2^*) \), we utilize the method of Lagrange multiplier. Assume

\[ l(d_1, d_2) = E \left( (X_1 - d_1)^2 + (X_2 - d_2)^2 \right| S > VaR_\alpha(S) \]

\[ + \beta \text{Var} \left( (X_1 - d_1)^2 + (X_2 - d_2)^2 \right| S > VaR_\alpha(S) \right), \]

and

\[ L(d_1, d_2, \gamma) = l(d_1, d_2) + \gamma (d - d_1 - d_2). \]

Since

\[ \text{Var} \left( (X_1 - d_1)^2 + (X_2 - d_2)^2 \right| S > VaR_\alpha(S) \right) = E \left( (X_1 - d_1)^2 + (X_2 - d_2)^2 \right| S > VaR_\alpha(S) \right) + (E_{X_1}^2 + E_{X_2}^2 - 2d_2 E_{X_2}) + 4d_1^2 (E_{X_1}^2 - E_{X_1} E_{X_2}'), \]

we get

\[ \frac{\partial L(d_1, d_2, \gamma)}{\partial d_1} = -2E_{X_1} + 2d_1 + 8\beta d_1 (E_{X_1}^2 - (E_{X_1})^2) \]

\[ + 4\beta [2d_2 (E_{X_1} E_{X_2} - E_{X_1}^2) + (E_{X_1}^2 - E_{X_1} E_{X_2}) - (E_{X_1}^2 - E_{X_1} E_{X_2})] - \gamma = 0. \] (3.3)

Similarly,

\[ \frac{\partial L(d_1, d_2, \gamma)}{\partial d_2} = -2E_{X_2} + 2d_2 + 8\beta d_2 (E_{X_2}^2 - (E_{X_2})^2) \]

\[ + 4\beta [2d_1 (E_{X_1} E_{X_2} - E_{X_1}^2) + (E_{X_2}^2 - E_{X_2} E_{X_2}) - (E_{X_2}^2 - E_{X_2} E_{X_2})] - \gamma = 0. \] (3.4)

and

\[ \frac{\partial L(d_1, d_2, \gamma)}{\partial \gamma} = d_1 + d_2 - d = 0. \] (3.5)

From Eqs. (3.3)-(3.5), the results of (3.1) and (3.2) follow immediately. Now, we need to find the formulae of \( E^{X_1} \), \( E^{X_2} \) and \( E^{X_1 X_2} \). Firstly, the explicit expression of \( E^{X_1} \) is calculated as follow.

\[ E^{X_1} = E \left( X_1 | S > VaR_\alpha(S) \right) = \int_{VaR_\alpha(S)}^\infty E \left( X_1 \right| S = s \right) f_S(S > VaR_\alpha(S)) \; ds \]

\[ = \frac{\int_{VaR_\alpha(S)}^\infty E \left( X_1 \right| S = s \right) f_S(s) \; ds \; ds}{Pr(S > VaR_\alpha(S))} = \int_{VaR_\alpha(S)}^\infty E \left( X_1 \right| S = s \right) f_S(s) \; x \; dx \; ds \]

\[ = \frac{\int_{VaR_\alpha(S)}^\infty x \; f_{X_1}(x, s) \; ds \; dx}{1 - \kappa}. \] (3.6)

Note that \( S = X_1 + X_2 \). From Eq. (2.1), we get

\[ \int_0^s x^3 f_{x_1}(x, s) \; dx = \int_0^s x^3 f_{x_1, x_2}(x, s - x) \; dx \]

\[ = (1 + \theta) \lambda_1 \lambda_2 \left( \frac{6(e^{-\lambda x} - e^{-\lambda x})}{(\lambda_2 - \lambda_1)^2} + \frac{6s e^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^3} - \frac{3s^2 e^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^2} + \frac{s^3 e^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^3} \right) \]
\[-\theta \lambda_1 \lambda_2 \left( \frac{6(e^{-\lambda_2 s} - e^{-2\lambda_1 s})}{(\lambda_2 - 2\lambda_1)^4} + \frac{6s e^{-2\lambda_1 s}}{(\lambda_2 - 2\lambda_1)^3} - \frac{3s^2 e^{-2\lambda_1 s}}{(\lambda_2 - 2\lambda_1)^2} + \frac{s^3 e^{-2\lambda_1 s}}{\lambda_2 - 2\lambda_1} \right) + \theta \lambda_1 \lambda_2 \left( \frac{6(e^{-2\lambda_2 s} - e^{-\lambda_1 s})}{(2\lambda_2 - \lambda_1)^4} + \frac{6s e^{-\lambda_1 s}}{(2\lambda_2 - \lambda_1)^3} - \frac{3s^2 e^{-\lambda_1 s}}{(2\lambda_2 - \lambda_1)^2} + \frac{s^3 e^{-\lambda_1 s}}{2\lambda_2 - \lambda_1} \right) + \theta \lambda_1 \lambda_2 \left( \frac{6(e^{-2\lambda_2 s} - e^{-\lambda_1 s})}{(2\lambda_2 - \lambda_1)^4} + \frac{6s e^{-\lambda_1 s}}{(2\lambda_2 - \lambda_1)^3} - \frac{3s^2 e^{-\lambda_1 s}}{(2\lambda_2 - \lambda_1)^2} + \frac{s^3 e^{-\lambda_1 s}}{2\lambda_2 - \lambda_1} \right) \right]. \quad (3.7)

Define

\[
\lambda_1 \lambda_2 \int_{VaR_n(S)}^{\infty} \left( \frac{6(e^{-\lambda_2 s} - e^{-\lambda_1 s})}{(\lambda_2 - \lambda_1)^4} + \frac{6se^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^3} - \frac{3s^2 e^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^2} + \frac{s^3 e^{-\lambda_1 s}}{\lambda_2 - \lambda_1} \right) ds
\]

\[
= e^{-\lambda_1 VaR_n(S)} \lambda_1 \left( \frac{6}{\chi_1} + \frac{6VaR_n(S)}{\lambda_1^2} + \frac{3(VaR_n(S))^2}{\lambda_1^3} + (VaR_n(S))^3 \right) - \frac{3e^{-\lambda_1 VaR_n(S)} \lambda_1 \left( \frac{2}{\chi_1} + \frac{2VaR_n(S)}{\lambda_1} + (VaR_n(S))^2 \right)}{(\lambda_2 - \lambda_1)^2}
\]

\[
+ \frac{6\lambda_2 e^{-\lambda_1 VaR_n(S)} (VaR_n(S) + \frac{1}{\chi_1}) - 6(\lambda_2 e^{-\lambda_1 VaR_n(S)} - \lambda_1 e^{-\lambda_2 VaR_n(S)})}{(\lambda_2 - \lambda_1)^4}
\]

\[
= \xi_3(VaR_n(S); \lambda_1; \lambda_2).
\]

Plugging Eqs. (3.7) and (3.8) into Eq. (3.6), we can derive the formula of

\[
E^{X_1^2} = \frac{1}{1 - \kappa} \left( 1 + \theta \xi_3(VaR_n(S); \lambda_1; \lambda_2) - \theta \xi_3(VaR_n(S); 2\lambda_1; \lambda_2) \right)
\]

\[-\theta \xi_3(VaR_n(S); \lambda_1; \lambda_2) - \theta \xi_3(VaR_n(S); 2\lambda_1; \lambda_2) + \theta \xi_3(VaR_n(S); 2\lambda_1; 2\lambda_2)) \right).
\]

The formula of \(E^{X_2^2}\) is symmetrically given by

\[
E^{X_2^2} = \frac{1}{1 - \kappa} \left( 1 + \theta \xi_3(VaR_n(S); \lambda_2; \lambda_1) - \theta \xi_3(VaR_n(S); 2\lambda_2; \lambda_1) \right)
\]

\[-\theta \xi_3(VaR_n(S); \lambda_2; \lambda_1) - \theta \xi_3(VaR_n(S); 2\lambda_2; \lambda_1) + \theta \xi_3(VaR_n(S); 2\lambda_2; 2\lambda_2)) \right).
\]

Since the formulae of \(E^{X_1^2}\) and \(E^{X_2^2}\) can be calculated similarly, we omit the proof. Now, we give the formula of \(E^{X_1^2X_2}\).

\[
E^{X_1^2X_2} = \mathbb{E} \left[ X_1^2X_2 | S > VaR_n(S) \right] = \int_{VaR_n(S)}^{\infty} \mathbb{E} \left[ X_1^2X_2 | S = s \right] f_S | S > VaR_n(S) (s) ds
\]

\[
= \int_{VaR_n(S)}^{\infty} \mathbb{E} \left[ X_1^2(s - X_1) | S = s \right] f_S | S > VaR_n(S) (s) ds
\]

\[
= \int_{VaR_n(S)}^{\infty} \mathbb{E} \left[ X_1^2(s - X_1) | S = s \right] f_S (s) ds
\]

\[
= \int_{VaR_n(S)}^{\infty} \mathbb{E} \left[ X_1^2(s - X_1) | S = s \right] f_S (s) ds
\]

\[
= \frac{\int_{VaR_n(S)}^{\infty} \int_0^s x^2(s - x)f_{x_1,S}(x,s)dxds}{\Pr(S > VaR_n(S))}.
\]

Note that

\[
\int_0^s x^2(s - x)f_{x_1,S}(x,s)dx = s \int_0^s x^2f_{x_1,x_2}(x,s-x)dx - \int_0^s x^3f_{x_1,x_2}(x,s-x)dx.
\]

By the similar calculations, we can derive

\[
E^{X_1^2X_2} = \frac{1}{1 - \kappa} \left( 1 + \theta \chi(VaR_n(S); \lambda_1; \lambda_2) - \theta \chi(VaR_n(S); 2\lambda_1; \lambda_2) \right)
\]

\[-\theta \chi(VaR_n(S); \lambda_1; 2\lambda_2) + \theta \chi(VaR_n(S); 2\lambda_1; 2\lambda_2)) \right).
\]

The formula of \(E^{X_1^2X_1}\) is also symmetrically given by

\[
E^{X_2^2X_1} = \frac{1}{1 - \kappa} \left( 1 + \theta \chi(VaR_n(S); \lambda_2; \lambda_1) - \theta \chi(VaR_n(S); 2\lambda_2; \lambda_1) \right)
\]

\[-\theta \chi(VaR_n(S); \lambda_2; 2\lambda_1) + \theta \chi(VaR_n(S); 2\lambda_2; 2\lambda_2)) \right).
\]
Since the formula of $E^{X_1, X_2}$ can be calculated similarly, we also omit the proof. This completes the proof of this theorem.

**Remark 3.1.** In Theorem 3.1, we assume that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 2\lambda_2$, and $\lambda_2 \neq 2\lambda_1$. In fact, one can obtain the adjusted results without these restrictions. Since the first part on the right-hand side of equation (3.6) can be rewritten as

$$
(1 + \theta)\lambda_1 \lambda_2 e^{-\lambda_1 s} \left( \frac{6(e^{(\lambda_1 - \lambda_2)s} - 1)}{(\lambda_2 - \lambda_1)^4} + \frac{6s^2}{(\lambda_2 - \lambda_1)^3} - \frac{3s^3}{(\lambda_2 - \lambda_1)^2} + \frac{s^3}{\lambda_2 - \lambda_1} \right),
$$

where $e^{(\lambda_1 - \lambda_2)s} - 1$ could be expanded in Taylor series if $\lambda_2 \to \lambda_1$,

$$
e^{(\lambda_1 - \lambda_2)s} - 1 = (\lambda_1 - \lambda_2)s + \frac{(\lambda_1 - \lambda_2)^2 s^2}{2} + \frac{(\lambda_1 - \lambda_2)^3 s^3}{6} + \sum_{n=4}^{\infty} \frac{(\lambda_1 - \lambda_2)^n s^n}{n!},$$

then the limit of the first part on the right-hand side of equation (3.6) is

$$
\lim_{\lambda_2 \to \lambda_1} (1 + \theta)\lambda_1 \lambda_2 e^{-\lambda_1 s} \left( \frac{6(e^{(\lambda_1 - \lambda_2)s} - 1)}{(\lambda_2 - \lambda_1)^4} + \frac{6s^2}{(\lambda_2 - \lambda_1)^3} - \frac{3s^3}{(\lambda_2 - \lambda_1)^2} + \frac{s^3}{\lambda_2 - \lambda_1} \right) = \lim_{\lambda_2 \to \lambda_1} 6(1 + \theta)\lambda_1 \lambda_2 e^{-\lambda_1 s} \sum_{n=4}^{\infty} \frac{(\lambda_1 - \lambda_2)^n s^n}{n!} = \frac{(1 + \theta)\lambda_1^2}{4} e^{-\lambda_1 s} s^4. \tag{3.9}
$$

For the case $\lambda_1 = \lambda_2$, the first component of $f_{X_1, X_2}(x_1, x_2)$ can be rewritten as $(1 + \theta)\lambda_1^2 e^{-\lambda_1 (x_1 + x_2)}$, the first part on the right-hand side of equation (3.6) can be simplified as

$$
\int_0^s x^3 (1 + \theta)\lambda_1^2 e^{-\lambda_1 (x + (s-x))} \, dx = \frac{(1 + \theta)\lambda_1^2}{4} e^{-\lambda_1 s} s^4. \tag{3.10}
$$

By the results of (3.9) and (3.10), for the case $\lambda_1 = \lambda_2$, we can apply the similar technique to obtain the adjusted results. For the cases $\lambda_1 = 2\lambda_2$ and $\lambda_2 = 2\lambda_1$, one obtains the similar results.

4. Calculation of the optimal capital allocation for $n$ risks

Explicit formulae for the optimal capital allocation for risks linked by a FGM copula cannot only be derived in the bivariate case but even for an undefined number of risks. In order to derive the explicit formulae for the optimal capital allocation, we first give the following lemma.

**Lemma 4.1.** Let $\bar{X} = (X_1, X_2, \cdots, X_n)$. Then the optimal allocation solution $\bar{d} = (d_1, d_2, \cdots, d_n)$ to the TMV model,

$$
\min_{d_1, \cdots, d_n} \left\{ E \left[ \sum_{i=1}^{n} (X_i - d_i)^2 I(S > \text{VaR}_n(S)) \right] + \beta \text{Var} \left( \sum_{i=1}^{n} (X_i - d_i)^2 I(S > \text{VaR}_n(S)) \right) \right\},
$$

such that $\sum_{i=1}^{n} d_i = d$,

is presented as follows:

$$
\bar{d} = A^{-1} \bar{z}, \tag{4.1}
$$

where $A^{-1} = (a_{ij})_{n \times n}$ is the inverse matrix of $A = 8\beta \Sigma + 2I_n$, $\Sigma$ is the conditional covariance matrix of $(\bar{X}|S > \text{VaR}_n(S))$, and $\bar{z} = (a_{\bar{X}} + \delta, \Lambda + \delta_2, \cdots, \Lambda + \delta_n)$ with $\delta_i = 4\beta \sum_{j=1}^{n} \text{Cov}(X_j^2, X_i|S > \text{VaR}_n(S)) + 2E[X_i|S > \text{VaR}_n(S)], i = 1, 2, \cdots, n$, where $\Lambda = (d - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \delta_j)/(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij})$. 

Proposition 4.2. Let \( \bar{X} = (X_1, X_2, \ldots, X_n) \), \( X_1, X_2, \ldots, X_n \) are \( n \) exponentially distributed with parameters \( \lambda_1, \lambda_2, \ldots, \lambda_n \). The dependence structure for \( (X_1, X_2, \ldots, X_n) \) is defined by a multivariate FGM copula. Then quantities \( E^{X_i^k} \), \( E^{X_i^2 X_j} \) and \( E^{X_j X_i} \) for \( i \neq j \), are presented as follows:

\[
E^{X_i^k} = \frac{(-1)^{n-k} q}{1 - \kappa} \left( \bar{X}^+ \right)_k (\text{Var}_N(S); \lambda_1; \ldots; \lambda_n) + \sum_{p=2}^{n} \sum_{l=0}^{q} \sum_{\tau \in \iota_{n,q-2}} \theta_{p_1 \ldots p_q} \left( \begin{array}{c}
\sum_{l=0}^{q} \sum_{\tau \in \iota_{n,q-2}} \theta_{p_1 \ldots p_q} \\
\sum_{l=0}^{q} \sum_{\tau \in \iota_{n,q-2}} \theta_{p_1 \ldots p_q}
\end{array} \right)
\]

In the following proposition, we provide the explicit formulae of \( E^{X_i^k} \), \( E^{X_i^2 X_j} \) and \( E^{X_j X_i} \) for \( n \) exponentially distributed risks linked by a multivariate FGM copula. Let \( \bar{X} = (X_1, X_2, \ldots, X_n) \), \( X_1, X_2, \ldots, X_n \) are \( n \) exponentially distributed with \( \lambda_1, \lambda_2, \ldots, \lambda_n \). As in the bivariate case, we assume that \( \lambda_i \neq \lambda_j \) and \( \lambda_i \neq 2\lambda_j \) for \( i \neq j \).

Proposition 4.2. Let \( \bar{X} = (X_1, X_2, \ldots, X_n) \), \( X_1, X_2, \ldots, X_n \) are \( n \) exponentially distributed with parameters \( \lambda_1, \lambda_2, \ldots, \lambda_n \). The dependence structure for \( (X_1, X_2, \ldots, X_n) \) is defined by a multivariate FGM copula. Then quantities \( E^{X_i^k} \), \( E^{X_i^2 X_j} \) and \( E^{X_j X_i} \), \( i \neq j \), are presented as follows:

\[
E^{X_i^2 X_j} = \frac{(-1)^{n-2} q}{1 - \kappa} \left( \bar{X}^+ \right)_2 (\text{Var}_N(S); \lambda_1; \ldots; \lambda_n) + \sum_{p=2}^{n} \sum_{l=0}^{q} \sum_{\tau \in \iota_{n,q-2}} \theta_{p_1 \ldots p_q} \left( \begin{array}{c}
\sum_{l=0}^{q} \sum_{\tau \in \iota_{n,q-2}} \theta_{p_1 \ldots p_q} \\
\sum_{l=0}^{q} \sum_{\tau \in \iota_{n,q-2}} \theta_{p_1 \ldots p_q}
\end{array} \right)
\]

Proof. The idea of proof is also based on the method of Lagrange multiplier. One can refer to Xu and Mao [17] for the full proof.
\[ (-1)^{j+1} \int_{\text{Var}(S)}^{\infty} f_{X_i}(s) \int_{s}^{\infty} f_{X_j}(x) f_{X_i}(x, j, s - x_i - x_j) dx_i dx_j ds \]
+ \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n \cap i \notin \{p_1, \ldots, p_q\}} \sum_{l=0}^{q} \theta_{p_1 \cdots p_q} (1 - 2F_{x_i}(x_i)) \left[ \sum_{l=0}^{q} \sum_{(e_1, \ldots, e_q) - j \in \tau_{q-1}} (-1)^j \omega_{i-j}(x_{p_1}; \cdots; x_{q+1}; \cdots; x_{n}; 2^q \lambda_{p_1}; \cdots; 2^q \lambda_{p_q}; \lambda_{q+1}; \cdots; \lambda_n) \right]

(4.5)

where \( \omega_{i-j}(x_1; x_2; \cdots; x_n; \alpha_1; \alpha_2; \cdots; \alpha_n) = \alpha_1 e^{-\alpha_1 x_1} \cdots \alpha_i e^{-\alpha_i x_i-1} \cdots \alpha_{q+1} e^{-\alpha_{q+1} x_{q+1}} \alpha_{q+1} e^{-\alpha_{q+1} x_{q+1}} \cdots \alpha_n e^{-\alpha_n x_n} \). Assume that

\[
\int_0^{s-x} \int_0^{s-x} \cdots \int_0^{s-x} \omega_{i-j}(x_1; x_2; \cdots; x_n; \alpha_1; \alpha_2; \cdots; \alpha_n) dx_1 \cdots dx_{i-1} \times dx_{i+1} \cdots dx_{j-1} \cdots dx_{n-1}
\]

\[
\sum_{t=1, t \neq i, j}^{n} \prod_{q=1, q \neq i, q \neq j}^{n} \frac{\alpha_q}{\alpha_q - \alpha_i} \alpha_i e^{-\alpha_i (s-x-x_j)} = h_{i-j}(s-x-x_j; \alpha_1; \cdots; \alpha_n).
\]

(4.6)

Note that

\[
fx_{i}, x_{j}, s-x_i-x_j (x_i, x_j, s-x_i-x_j) = \int_0^{s-x_i-x_j} \int_0^{s-x_i-x_j} \cdots \int_0^{s-x_i-x_j} 1_{n-1} f x_{i}, x_{j}, s-x_i-x_j (x_i, x_j, s-x_i-x_j)
\]

\[
\sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n \cap i \notin \{p_1, \ldots, p_q\}} \sum_{l=0}^{q} \theta_{p_1 \cdots p_q} (1 - 2F_{x_i}(x_i)) (1 - 2F_{x_j}(x_j)) \left[ \sum_{l=0}^{q} \sum_{(e_1, \ldots, e_q) - j \in \tau_{q-2}} (-1)^j h_{i-j}(s-x_i-x_j; 2^q \lambda_{p_1}; \cdots; 2^q \lambda_{p_q}; \lambda_{q+1}; \cdots; \lambda_n) \right]
\]

(4.7)

Plugging (4.5) into (4.7), we have

\[
fx_{i}, x_{j}, s-x_i-x_j (x_i, x_j, s-x_i-x_j) = \int_0^{s-x_i-x_j} \int_0^{s-x_i-x_j} \cdots \int_0^{s-x_i-x_j} 1_{n-1} f x_{i}, x_{j}, s-x_i-x_j (x_i, x_j, s-x_i-x_j)
\]

\[
\sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n \cap j \notin \{p_1, \ldots, p_q\}} \sum_{l=0}^{q} \theta_{p_1 \cdots p_q} (1 - 2F_{x_i}(x_i)) (1 - 2F_{x_j}(x_j)) \left[ \sum_{l=0}^{q} \sum_{(e_1, \ldots, e_q) - j \in \tau_{q-1}} (-1)^j h_{i-j}(s-x_i-x_j; 2^q \lambda_{p_1}; \cdots; 2^q \lambda_{p_q}; \lambda_{q+1}; \cdots; \lambda_n) \right]
\]
\[ + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \ldots < p_q \leq \sum_{j=1}^{n} \sum_{r \in \{p_1, \ldots, p_q\}} \theta_{p_1 \ldots p_q} (1-2F_{X_j}(x_j)) \times \]

\[ \left[ \sum_{i=0}^{q} \sum_{(x_1, \ldots, x_q) \in \pi_{i,q-1}} (-1)^i h_{-i-j}(s-x_i-x_j; 2^{s_i} \lambda_{p_1}; \ldots; 2^{s_q} \lambda_{p_q}; \lambda_{i+1}; \ldots; \lambda_{i_n}) \right] \]

\[ + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \ldots < p_q \leq \sum_{j=1}^{n} \sum_{r \in \{p_1, \ldots, p_q\}} \theta_{p_1 \ldots p_q} \times \]

\[ \left[ \sum_{i=0}^{q} \sum_{(x_1, \ldots, x_q) \in \pi_{i,q-1}} (-1)^i h_{-i-j}(s-x_i-x_j; 2^{s_i} \lambda_{p_1}; \ldots; 2^{s_q} \lambda_{p_q}; \lambda_{i+1}; \ldots; \lambda_{i_n}) \right]. \quad (4.8) \]

Utilizing the dividend difference presented in Chiragiev and Landsman [6], note that

\[ h_{-i-j}(x; \alpha_1; \ldots; \alpha_{i-1}; \alpha_{i+1}; \ldots; \alpha_n) = (-1)^{n-3} \times \alpha_1 \times \cdots \times \alpha_{i-1} \times \alpha_{i+1} \times \cdots \alpha_n \times \Lambda(x; \alpha_1; \ldots; \alpha_{i-1}; \alpha_{i+1}; \ldots; \alpha_n), \]

where \( \Lambda(x; \alpha_1; \ldots; \alpha_{i-1}; \alpha_{i+1}; \ldots; \alpha_n) \) is the \((n-3)\)th order dividend difference of \( \Lambda(x; \alpha) = e^{-Ax} \). Then, we get

\[ \int_{0}^{s_{-x_i}} \int_{0}^{s_{-x_j}} x_i^2 f_{X_j}(x_j)f_{X_j}(x_i)h_{-i-j}(s-x_i-x_j; \lambda_1; \ldots; \lambda_{i-1}; \lambda_{i+1}; \ldots; \lambda_{j-1}; \lambda_{j+1}; \ldots; \lambda_n) \, dx_j \, dx_i \]

\[ = \int_{0}^{s_{-x_i}} \int_{0}^{s_{-x_j}} x_i^2 f_{X_j}(x_j)f_{X_j}(x_i)(-1)^{n-3} \eta_{-i-j} \Lambda(x; \lambda_1; \ldots; \lambda_{i-1}; \lambda_{i+1}; \ldots; \lambda_n) \, dx_j \, dx_i \]

\[ = (-1)^{n-2} \eta L_{X_i^2 X_j} (s; \lambda_1; \ldots; \lambda_{i-1}; \lambda_{i+1}; \ldots; \lambda_{j-1}; \lambda_{j+1}; \ldots; \lambda_n), \quad (4.9) \]

where \( \eta_{-i-j} = \lambda_1 \times \cdots \lambda_{i-1} \times \lambda_{i+1} \times \cdots \lambda_{j-1} \times \lambda_{j+1} \times \cdots \lambda_n \). \( L_{X_i^2 X_j} (s; \lambda) = \int_{0}^{s_{-x_i}} \int_{0}^{s_{-x_j}} x_i^2 f_{X_j}(x_j)f_{X_j}(x_i)h_{-i-j}(s-x_i-x_j; \lambda_1; \ldots; \lambda_{i-1}; \lambda_{i+1}; \ldots; \lambda_n) \, dx_j \, dx_i \).

Given that the risks are exponentially distributed, then \( L_{X_i^2 X_j}, M_{X_i^2 X_j}, N_{X_i^2 X_j} \) can be rewritten as

\[ \lambda_i(\lambda_i-\lambda)]^2 (\lambda_i-\lambda]]^2 \lambda_i(\lambda_i-\lambda)]^2 - 2(2\lambda_i-4\lambda_j+3\lambda_i) e^{-\lambda_i} + 2(\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 \]

\[ \frac{2s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i}}{2s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i}} \]

\[ M_{X_i^2 X_j} (s) = -2\left( \frac{2e^{-\lambda_i}}{(2\lambda_i-4\lambda_j+3\lambda_i) e^{-\lambda_i} + 2(\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 \right) \]

\[ \frac{2s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i}}{2s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i} s^2 e^{-2\lambda_i}} \]

\[ N_{X_i^2 X_j} (s) = -2\left( \frac{2e^{-\lambda_i}}{(2\lambda_i-4\lambda_j+3\lambda_i) e^{-\lambda_i} + 2(\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 (\lambda_i-\lambda)]^2 \right) \]
\[
\frac{2\lambda e^{-\lambda s}}{(\lambda - 2\lambda)^2(2\lambda - \lambda)} - \frac{s^2 - e^{-\lambda s}}{(\lambda - 2\lambda)^2(\lambda - \lambda)} - \frac{2s(3\lambda - 2\lambda - 2\lambda)e^{-\lambda s}}{(\lambda - 2\lambda)^2(\lambda - \lambda)^2} - 2(6\lambda^2 + (2\lambda)^2 + 23\lambda + 3\lambda^2 - 4\lambda(2\lambda + 2\lambda)e^{-\lambda s}}{(\lambda - 2\lambda)^2(\lambda - \lambda)^3}.
\]

\[
O_{X^2 \chi_j}(s; \lambda) = -4\frac{2e^{-\lambda s}}{(\lambda - 2\lambda)^3(2\lambda - \lambda)} + 2\frac{(2\lambda - 4)(2\lambda + 3\lambda) e^{-2\lambda s}}{(2\lambda - 2\lambda)^4(2\lambda - \lambda)^2} - \frac{2s(3(2\lambda - 2\lambda - 2\lambda - 2\lambda)e^{-2\lambda s}}{(2\lambda - 2\lambda)^3(2\lambda - \lambda)^2} - 2(6(2\lambda)^2 + (2\lambda)^2 + 23\lambda + 3\lambda^2 - 4(2\lambda)(2\lambda + 2\lambda)e^{-2\lambda s}}{(2\lambda - 2\lambda)^4(2\lambda - \lambda)^3}.
\]

Utilizing the results of (4.8) and (4.10)-(4.13), one can rewrite \( \int_0^1 \int_0^{s-x} x^2 f_{x_{i,j}}, s \, dx_j dx_i \) as:

\[
\int_0^a \int_0^{s-x} x_j f_{x_{i,j}}, s \, dx_j dx_i = (-1)^{n-2}\eta(L_{X^2 \chi_j}(s; \lambda_1; \cdots; \lambda_n)) + \sum_{q=2}^n \sum_{1 \leq p_1 < \cdots < p_q \leq n} \sum_{\{\tau \}_{1}\in\{p_1, \cdots, p_q\}} (-1)^q 2^{q-1-l} x \]

\[
[L_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)]
\]

\[
-M_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)
\]

\[
+N_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)
\]

\[
\sum_{q=2}^n \sum_{1 \leq p_1 < \cdots < p_q \leq n} \sum_{\{\tau \}_{1}\in\{p_1, \cdots, p_q\}} (-1)^q 2^{q-1-l} x \]

\[
[L_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)]
\]

\[
-M_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)
\]

\[
+N_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)
\]

\[
\sum_{q=2}^n \sum_{1 \leq p_1 < \cdots < p_q \leq n} \sum_{\{\tau \}_{1}\in\{p_1, \cdots, p_q\}} (-1)^q 2^{q-1-l} x \]

\[
(L_{X^2 \chi_j}(s; 2^{\tau} \lambda_1; \cdots; 2^{\tau} \lambda_p; \lambda_{q+1}; \cdots; \lambda_n)).
\]

In order to calculate the explicit formula of \( E_{X^2 \chi_j} \), \( i \neq j \), as in (4.4), the \( L_{X^2 \chi_j}(s; \lambda) \), \( M_{X^2 \chi_j}(s; \lambda) \), \( N_{X^2 \chi_j}(s; \lambda) \) and \( O_{X^2 \chi_j}(s; \lambda) \) terms in the equation above must be integrated on \( s \) as follows:

\[
\bar{L}_{X^2 \chi_j}(V; \lambda) = \int_{\nu}^\infty L_{X^2 \chi_j}(s; \lambda) ds = \frac{2\lambda_1 - 4\lambda_j + 3\lambda e^{-\lambda_j V}}{(\lambda_1 - \lambda_j)^4(\lambda_1 - \lambda)^2} + \frac{2(1 + \nu V_j) e^{-\lambda_j V}}{\lambda_1^2(\lambda_1 - \lambda_j)^4(\lambda_1 - \lambda)^3}.
\]
\[
\begin{align*}
+ (2 + \lambda V(2 + \lambda V)) & e^{-\lambda V} + \frac{2(1 + V\lambda)(3\lambda_i - \lambda_j - 2\lambda)}{\lambda^2(\lambda_i - \lambda_j)^2(\lambda_i - \lambda)^2} e^{-\lambda V} \\
+ \frac{2(2\lambda^2 + \lambda_j^2 + 2\lambda\lambda_j + 3\lambda^2 - 4\lambda_i(\lambda_i + 2\lambda))}{\lambda_i(\lambda_i - \lambda_j)^4(\lambda_i - \lambda)^2} e^{-\lambda V} - \frac{2e^{-\lambda V}}{\lambda(\lambda_i - \lambda)^2(\lambda_i - \lambda)^2},
\end{align*}
\]

(4.14)

\[
\begin{align*}
\tilde{M}_{X_iX_j}(V; \lambda) &= 2\frac{(2\lambda_i - 4\lambda_j + 3\lambda)e^{-\lambda V}}{\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_i - \lambda)^2} + \frac{2(1 + V\lambda_i)(3\lambda_i - \lambda_j - 2\lambda)}{\lambda^2(2\lambda_i - \lambda_j)^4(\lambda_i - \lambda)^2} e^{-\lambda V} \\
+ \frac{2(1 + V2\lambda_i)(3(2\lambda_i) - \lambda_j - 2\lambda)}{\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_i - \lambda)^2} e^{-\lambda V} + \frac{2(6(2\lambda_i)^2 + \lambda_j^2 + 2\lambda\lambda_j + 3\lambda^2 - 4\lambda_i(\lambda_i + 2\lambda))}{\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_i - \lambda)^2} e^{-\lambda V} - \frac{2e^{-\lambda V}}{\lambda(\lambda_i - \lambda)^2(\lambda_i - \lambda)^2},
\end{align*}
\]

(4.15)

\[
\begin{align*}
\tilde{N}_{X_iX_j}(V; \lambda) &= 2\frac{(2\lambda_i - 4(2\lambda_j) + 3\lambda)e^{-\lambda V}}{2\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_j - \lambda)^2} + \frac{2(1 + V2\lambda_i)e^{-2\lambda_j V}}{2\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_j - \lambda)^2} e^{-2\lambda_j V} \\
+ \frac{2(1 + V2\lambda_i)(3(2\lambda_i) - 2\lambda_j - 2\lambda)}{2\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_j - \lambda)^2} e^{-2\lambda_j V} + \frac{2(6(2\lambda_i)^2 + \lambda_j^2 + 2\lambda\lambda_j + 3\lambda^2 - 4\lambda_i(2\lambda_i + 2\lambda))}{2\lambda_i(2\lambda_i - \lambda_j)^4(\lambda_j - \lambda)^2} e^{-2\lambda_j V} - \frac{2e^{-\lambda V}}{\lambda(2\lambda_i - \lambda_j)^4(\lambda_j - \lambda)^2},
\end{align*}
\]

(4.16)

Finally, we can derive the explicit formulae for \(E^{X_iX_j}, i \neq j; i, j = 1, 2, \ldots, n\).

Since the formulas of \(E^{X_i}, \tilde{X}_j\) can be calculated similarly, we omit the proof. We give the expressions of \(\tilde{L}_{X_iX_j}(V; \lambda), \tilde{M}_{X_iX_j}(V; \lambda), \tilde{N}_{X_iX_j}(V; \lambda)\) and \(\tilde{O}_{X_iX_j}(V; \lambda)\) directly.

\[
\begin{align*}
\tilde{L}_{X_iX_j}(V; \lambda) &= \left(e^{-\lambda V} - \frac{3\lambda_i - \lambda_j + 2\lambda}{\lambda(3\lambda_i - \lambda_j)^3(\lambda_i - \lambda)} e^{-\lambda V} - \frac{(1 + V\lambda_i)e^{-\lambda V}}{\lambda^2(3\lambda_i - \lambda_j)^3(\lambda_i - \lambda)}
\end{align*}
\]

(4.18)

\[
\begin{align*}
\tilde{M}_{X_iX_j}(V; \lambda) &= -2\frac{\lambda(2\lambda_i - \lambda_j)^2(\lambda_i - \lambda)}{\lambda(3\lambda_i - \lambda_j)^3(\lambda_i - \lambda)} e^{-\lambda V} - \frac{(1 + V\lambda_i)e^{-\lambda V}}{\lambda(3\lambda_i - \lambda_j)^3(\lambda_i - \lambda)}
\end{align*}
\]

(4.19)

\[
\begin{align*}
\tilde{O}_{X_iX_j}(V; \lambda) &= 4\frac{(3\lambda_i - \lambda_j + \lambda_l)}{\lambda(3\lambda_i - \lambda_j)^3(\lambda_i - \lambda)} e^{-\lambda V} + \frac{(1 + V\lambda_i)e^{-\lambda V}}{\lambda(3\lambda_i - \lambda_j)^3(\lambda_i - \lambda)}
\end{align*}
\]

(4.20)
Now, we give the formulæ of $E^{X_i}$, $i = 1, 2, \ldots, n$.

$$E^{X_i} = \mathbb{E} \left[ X_i^3 | S > \text{VaR}_n(S) \right] = \frac{\int_{\text{VaR}_n(S)}^{\infty} x^3 f_{X_i}(x, s) dx_i ds}{1 - \kappa}.$$  \hfill (4.22)

Now, we rewrite the $f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$ defined in Eq.(2.2) as

\[
f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = f_{X_i}(x_i) \left\{ \omega_{-i}(x_1; x_2; \ldots; x_n; \lambda_1; \lambda_2; \ldots; \lambda_n) \right\}
\]

\[
+ \sum_{q-2 \leq p_1 < \cdots < p_q \leq n} \sum_{n \in \{ p_1, \ldots, p_q \}} \theta_{p_1 \cdots p_q} (1 - 2F_{X_i}(x_i)) \left[ \omega_{-i}(x_1; x_2; \ldots; x_{p_1}; x_{p_2}; \ldots; x_{p_q}; x_{p_1+1}; \cdots; x_n; 2^{\alpha} \lambda_{p_1}; \cdots; 2^{\alpha} \lambda_{p_q}; \lambda_{p_1+1}; \cdots; \lambda_n) \right]
\]

\[
+ \sum_{q-2 \leq p_1 < \cdots < p_q \leq n} \sum_{n \in \{ p_1, \ldots, p_q \}} \theta_{p_1 \cdots p_q} \left[ \omega_{-i}(x_1; x_2; \ldots; x_{p_1}; x_{p_2}; \ldots; x_{p_q}; x_{p_1+1}; \cdots; x_n; 2^{\alpha} \lambda_{p_1}; \cdots; 2^{\alpha} \lambda_{p_q}; \lambda_{p_1+1}; \cdots; \lambda_n) \right],
\]  \hfill (4.23)

where $\omega_{-i}(x_1; x_2; \ldots; x_n; \alpha_1; \alpha_2; \cdots; \alpha_n) = \alpha_1 e^{-\alpha_1 x_1} \times \cdots \times \alpha_{i-1} e^{-\alpha_{i-1} x_{i-1}} \times \alpha_{i+1} \times e^{-\alpha_{i+1} x_{i+1}} \times \cdots \times \alpha_n e^{-\alpha_n x_n}$.

Assume that

\[
\int_0^{x-x_i} \int_0^{x-x_i-1} \cdots \int_0^{s_{i-1}-x_i-1} \omega_{-i}(x_1; \cdots; x_n; \alpha_1; \cdots; \alpha_n) dx_1 \cdots dx_{i-1} \times dx_{i+1} \cdots dx_n = \sum_{l=1}^{n} \left( \prod_{q=1, q \neq l; q \neq i}^{n} \frac{\alpha_q}{\alpha_q - \alpha_l} \right) \alpha_l e^{-\alpha_l (s-x_i)} = h_{-i}(s-x_i; \alpha_1; \cdots; \alpha_n).
\]  \hfill (4.24)

Utilizing the difference division presented in Chiragiev and Landsman [6], note that

\[
h_{-i}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n) = \left( -1 \right)^{n-2} \cdot \alpha_1 \times \cdots \times \alpha_{i-1} \times \alpha_{i+1} \times \cdots \times \alpha_n \times \Lambda(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n),
\]

where $\Lambda(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n)$ is the $(n-2)$th order divided difference of $\Lambda(x; \alpha)$. Then, we get

\[
\int_0^x X_i^3 f_{X_i}(x_i) h_{-i}(s-x_i; \lambda_1; \cdots; \lambda_{i-1}; \lambda_{i+1}; \cdots; \lambda_n) dx_i
\]

\[
= \left( -1 \right)^{n-1} \eta L_{X_i}^3(s; \lambda_1; \cdots; \lambda_{i-1}; \lambda_{i+1}; \cdots; \lambda_{j-1}; \lambda_{j+1}; \cdots; \lambda_n),
\]

where $\eta_{-i} = \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n$. $L_{X_i}^3(s; \lambda) = -\int_0^s x_i^3 e^{-\lambda x_i} \Lambda(s-x_i; \lambda) dx_i$. Let also $M_{X_i}^3(s; \lambda) = -\int_0^s 2x_i^3 e^{-2\lambda x_i} \Lambda(s-x_i; \lambda) dx_i$. Given that the risks are exponentially distributed, then $L_{X_i}^3$ and $M_{X_i}^3$ can be rewritten as

\[
L_{X_i}^3(s; \lambda) = \frac{3 \lambda^2 e^{-\lambda x_i}}{\lambda - \lambda_i} + \frac{3 \lambda e^{-\lambda x_i}}{\lambda - \lambda_i^2} + \frac{6 e^{-\lambda x_i}}{\lambda - \lambda_i^3} + \frac{6 e^{-\lambda x_i}}{(\lambda - \lambda_i)^2} - \frac{6 e^{-\lambda x_i}}{(\lambda - \lambda_i)^3},
\]  \hfill (4.25)

\[
M_{X_i}^3(s; \lambda) = 2 \left( \frac{3 \lambda^2 e^{-\lambda x_i}}{2 \lambda - \lambda_i} + \frac{3 \lambda e^{-\lambda x_i}}{(\lambda - \lambda_i)^2} + \frac{6 e^{-\lambda x_i}}{(2 \lambda - \lambda_i)^3} + \frac{6 e^{-\lambda x_i}}{(2 \lambda - \lambda_i)^2} - \frac{6 e^{-\lambda x_i}}{(2 \lambda - \lambda_i)^4} \right).
\]  \hfill (4.26)

Then, utilizing the results of (4.23), (4.25) and (4.26), we can express $\int_0^x X_i^3 f_{X_i}(x_i, s) dx_i$ as

\[
\int_0^x x_i^3 f_{X_i}(x_i, s) dx = \left( -1 \right)^{n-1} \eta \left( L_{X_i}^3(s; \lambda_1; \lambda_2; \cdots; \lambda_n) + \sum_{q=2}^{n} \sum_{p_1 < \cdots < p_q \leq n} \right) \theta_{p_1 \cdots p_q} \left( \omega_{-i}(x_1; x_2; \cdots; x_{p_1}; x_{p_2}; \cdots; x_{p_q}; x_{p_1+1}; \cdots; x_n; 2^{\alpha} \lambda_{p_1}; \cdots; 2^{\alpha} \lambda_{p_q}; \lambda_{p_1+1}; \cdots; \lambda_n) \right).
\]
\[ \theta_{p_1 \cdots p_q} \sum_{l=0}^{q-1} \sum_{(e_1, \ldots, e_q) \in \pi_{r_1 \cdots r_q}} \left\{ (-1)^{l+1} 2^{q-1-l} M^2 \left( s, \lambda_{p_1}, \ldots, \lambda_{q_1}, \ldots, \lambda_{n} \right) \right\} + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n} \left\{ (-1)^{q-1-l} M^2 \left( s, \lambda_{p_1}, \ldots, \lambda_{p_q}, \lambda_{q_1}, \ldots, \lambda_{n} \right) \right\} \]

In order to calculate the explicit formulae of \( E^X_i \), \( i = 1, 2, \ldots, n \), as in (4.22), the terms in the equation above must be integrated on \( s \) as follows.

\[ \tilde{L}_X^i (V; \lambda) = \int_V L_X^i (s; \lambda) ds = \frac{e^{-\lambda V(6 + V\lambda_1(6 + V\lambda_2(3 + V\lambda_3)))}}{\lambda^2(\lambda_1 - \lambda)} + \frac{6e^{-\lambda V}}{\lambda^3(\lambda_1 - \lambda)^2} \]

\[ \hat{M}_X^i (V; \lambda) = \int_V M^2 \left( s, \lambda_{p_1}, \ldots, \lambda_{q_1}, \ldots, \lambda_{n} \right) ds = \frac{e^{-2\lambda V}(6 + V\lambda_1(6 + V\lambda_2(3 + V\lambda_3)))}{\lambda^2(\lambda_1 - \lambda)^2} + \frac{12e^{-\lambda V}}{\lambda^3(2\lambda_1 - \lambda)^4} \]

Finally, we can derive the explicit formula (4.3) for \( E^X_i \), \( i = 1, 2, \ldots, n \).

Since the formulae of \( E^X_i \) and \( E^X_i \) can be calculated similarly, we also omit the proof. We give the expressions of \( \tilde{L}_X^2 (V; \lambda) \), \( \hat{M}_X^2 (V; \lambda) \), \( \tilde{L}_X (V; \lambda) \) and \( \hat{M}_X (V; \lambda) \) directly.

\[ \tilde{L}_X^2 (V; \lambda) = \frac{e^{-\lambda V}(2 + V\lambda_1(2 + V\lambda_2)))}{\lambda^2(\lambda_1 - \lambda)} + \frac{2e^{-\lambda V}(1 + V\lambda_1)e^{-\lambda V}}{\lambda^2(\lambda_1 - \lambda)^2} \]

\[ \hat{M}_X^2 (V; \lambda) = \frac{e^{-2\lambda V}(2 + V\lambda_1(2 + V\lambda_2)))}{\lambda^4(2\lambda_1 - \lambda)^2} + \frac{2e^{-\lambda V}(1 + V\lambda_1)e^{-\lambda V}}{\lambda^3(2\lambda_1 - \lambda)^3} \]

\[ \tilde{L}_X (V; \lambda) = \frac{e^{-\lambda V}(1 + V\lambda_1)e^{-\lambda V}}{\lambda^2(\lambda_1 - \lambda)} + \frac{e^{-\lambda V}(1 + V\lambda_1)e^{-\lambda V}}{\lambda^2(\lambda_1 - \lambda)^2} \]

\[ \hat{M}_X (V; \lambda) = \frac{e^{-2\lambda V}(1 + V\lambda_1)e^{-\lambda V}}{\lambda^2(2\lambda_1 - \lambda)^2} + \frac{2e^{-\lambda V}(1 + V\lambda_1)e^{-\lambda V}}{\lambda(2\lambda_1 - \lambda)^3} \]

This completes the proof of this proposition.

5. Numerical illustrations

5.1 Bivariate case

We illustrate our results with an example for the bivariate case. As an example, assume that risk variables \( X_1 \) and \( X_2 \) are exponentially distributed with parameters \( \lambda_1 = 2/5 \) and \( \lambda_2 = 3/4 \). We calculate the optimal capital allocations based on Tail Mean-Variance principle for different risk levels \( \kappa \), FGM copula parameters \( \theta \) and different \( \beta \).

The values of \( d_i (i = 1, 2) \) with a total capital \( d = 40 \) for \( \theta = -1, 0, 1, \beta = 0.1, 0.3, 0.5, 0.7, 0.9, \) and \( \kappa = 0.5, 0.75, 0.95, 0.99, 0.995 \) are listed in Table 1-3. We observe that for
fixed $\kappa$ and $\theta$, $d_1$ is a decreasing function with respect to $\beta$, but $d_2$ is an increasing function with respect to $\beta$. We also find that for fixed $\beta$ and $\theta$, $d_1$ is an increasing function with respect to $\kappa$, but $d_2$ is a decreasing function with respect to $\kappa$. Moreover, the larger the value of $\theta$, the greater changes in $d_1$ and $d_2$ for different $\kappa$ and $\beta$.

5.2 Multivariate case

For multivariate case, we illustrate our results with an example for three risk variables dependent through a trivariate FGM copula. Assume that risk variables $X_1$, $X_2$ and $X_3$ are exponentially distributed with parameters $\lambda_1 = 1/2$, $\lambda_2 = 1/3$ and $\lambda_3 = 1/5$. The trivariate FGM copula with 4 parameters can be written as

$$C(u_1, u_2, u_3) = u_1 u_2 u_3 (1 + \theta_{123} \bar{u}_1 \bar{u}_2 \bar{u}_3 + \theta_{12} \bar{u}_1 \bar{u}_2 + \theta_{13} \bar{u}_1 \bar{u}_3 + \theta_{23} \bar{u}_2 \bar{u}_3),$$

where $\bar{u}_i = 1 - u_i$, $i = 1, 2, 3$.

We calculate the optimal capital allocations (the amount $d_i$ to $X_i$, $i = 1, 2, 3$) based on Tail Mean-Variance principle for different risk levels $\kappa$, FGM copula parameters and different $\beta$.

The values of $d_i$ $(i = 1, 2, 3)$ with a total capital $d = 120$ for $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$, $\kappa = 0.5, 0.75, 0.95, 0.99, 0.995$, $\theta_{123} = 0.25, \theta_{12} = -0.2, 0.2, \theta_{13} = 0.5$, and $\theta_{23} = -0.6, 0.6$ are listed in Table 4-5. We observe that for fixed $\kappa$, $d_2$ and $d_3$ are decreasing functions with respect to $\beta$, but $d_1$ is an increasing function with respect to $\beta$ in both cases. We also find that for fixed $\beta$, $d_1$ is an increasing function with respect to $\kappa$, but $d_1$ and $d_2$ is decreasing function with respect to $\kappa$. Moreover, the changes in $d_1$, $d_2$ and $d_3$ for different $\kappa$ and $\beta$ in case $\theta_{123} = 0.25$, $\theta_{12} = 0.2$, $\theta_{13} = 0.5$ and $\theta_{23} = 0.6$ are greater than in case $\theta_{123} = 0.25$, $\theta_{12} = -0.2$, $\theta_{13} = 0.5$ and $\theta_{23} = -0.6$.

6. Concluding Remarks

In this paper, we introduce the use of copulas in optimal capital allocation based on Tail Mean-Variance principle. We obtain explicit expressions for the optimal capital allocation for exponential distributed risks linked by a FGM copula. The handy form of the FGM copula permits a direct calculation of the $E[\text{VaR}_k(S)]$, $E[X_i|S > \text{VaR}_k(S)]$, and $E[X_i|X_j|S > \text{VaR}_k(S)]$ when we assume only two risks. In the multivariate case, the dividend differences presented in Chiragiev and Landsman (2007) are used.

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References


Table 1. Optimal capital allocations for different $\beta$ and $\kappa$ with a total capital $d = 40$ and $\theta = -1$.

<table>
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<tr>
<th>$\beta$</th>
<th>$\kappa = 0.5$</th>
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<th>$\kappa = 0.95$</th>
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Table 2. Optimal capital allocations for different $\beta$ and $\kappa$ with a total capital $d = 40$ and $\theta = 0$.

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Table 3. Optimal capital allocations for different $\beta$ and $\kappa$ with a total capital $d = 40$ and $\theta = 1$.

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Table 4. Optimal capital allocations with $d = 120$ and $\theta_{123} = 0.25$, $\theta_{12} = -0.2$, $\theta_{13} = 0.5$, $\theta_{23} = -0.6$.

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Table 5. Optimal capital allocations with $d = 120$ and $\theta_{123} = 0.25$, $\theta_{12} = 0.2$, $\theta_{13} = 0.5$, $\theta_{23} = 0.6$.

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