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NEW OSTROWSKI TYPE INEQUALITIES FOR m-CONVEX FUNCTIONS AND APPLICATIONS

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Abstract

In this paper we establish new inequalities of Ostrowski type, for functions whose derivatives in absolute value are m-convex. We also give some applications to special means of positive real numbers. Finally, we obtain some error estimates for the midpoint formula.

Keywords: m-convex function, Starshaped function, Convex function, Ostrowski inequality, Hermite-Hadamard inequality, Hölder inequality, Power Mean inequality, Special means, The midpoint formula, Lipschitzian mapping.

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1. Introduction

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L([a, b])$ where $a, b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality holds (see [2]):

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \le \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right].$$

This inequality is well known in the literature as the *Ostrowski inequality*. For some results which generalize, improve, and extend the above inequality, see [2, 5, 6, 8, 10], and references therein.

In [14], G. Toader defined m-convexity, an intermediate between usual convexity and the starshaped property, as the following:

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1.1. Definition. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be *m-convex*, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m(b)$ the set of m-convex functions on $[0, b]$ for which $f(0) \le 0$.

1.2. Definition. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be starshaped if for every $x \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx) \le tf(x).$$

For $m = 1$, we recapture the concept of convex functions defined on $[0, b]$, and for $m = 0$ the concept of starshaped functions on $[0, b]$.

The following theorem contains the Hermite-Hadamard integral inequality (see [7]).

1.3. Theorem. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an M-Lipschitzian mapping on I, and $a, b \in I$ with $a < b$. Then we have the inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le M\frac{(b-a)}{4}. \tag{1.1}$$

In [13], E. Set, M. E. Özdemir and M.Z. Sarikaya established the following theorem.

1.4. Theorem. *Let $f : I^\circ \subset [0, b^*] \to \mathbb{R}$, $b^* > 0$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is m-convex on $[a, b]$, $q > 1$ and $m \in (0, 1]$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le (b-a)\left(\frac{3^{1-\left(\frac{1}{q}\right)}}{8}\right)\left(\left|f'(a)\right| + m^{\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|\right), \tag{1.2}$$

where $\frac{b}{m} < b^*$. □

In [11], U. Kirmaci proved the following theorem.

1.5. Theorem. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.3}$$ □

S.S. Dragomir and G. Toader proved the following Hermite-Hadamard type inequality for m-convex functions, see [9, p.7].

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.4}$$

Some generalizations of this result can be found in [4].

In [3], M. K. Bakula, M. E. Özdemir and J. Pečarić proved the following theorems.

1.6. Theorem. *Let I be an open real interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$, where $0 \le a < b < \infty$. If $|f'|^q$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q \in [1,\infty)$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left(\mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}}\right), \tag{1.5}$$

where

$$\mu_1 = \min\left\{\frac{|f'(a)|^q + m\left|f'(\frac{a+b}{2m})\right|^q}{2}, \frac{\left|f'(\frac{a+b}{2})\right|^q + m\left|f'(\frac{a}{m})\right|^q}{2}\right\},$$
$$\mu_2 = \min\left\{\frac{|f'(b)|^q + m\left|f'(\frac{a+b}{2m})\right|^q}{2}, \frac{\left|f'(\frac{a+b}{2})\right|^q + m\left|f'(\frac{b}{m})\right|^q}{2}\right\}. \qquad \square$$

1.7. Theorem. *Let I be an open real interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$, where $0 \le a < b < \infty$. If $|f'|^q$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q \in [1,\infty)$, then*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad \le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m\left|f'(\frac{b}{m})\right|^q}{2}\right)^{\frac{1}{q}}, \left(\frac{m\left|f'(\frac{a}{m})\right|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}. \qquad \square \end{aligned} \tag{1.6}$$

The main purpose of this paper is to establish new Ostrowski type inequalities for functions whose derivatives in absolute value are m-convex. Using these results we give some applications to special means of positive real numbers, and obtain some error estimates for the midpoint formula.

2. The results

In [1], in order to prove some inequalities related to the Ostrowski inequality, M. Alomari and M. Darus used essentially the following lemma, in which however the constant $(b-a)$ has been changed to $(a-b)$ in the formulation of equality (2.1).

2.1. Lemma. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $f' \in L([a,b])$, then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (a-b)\int_0^1 p(t) f'(ta + (1-t)b)\,dt \tag{2.1}$$

for each $t \in [0,1]$, where

$$p(t) = \begin{cases} t & \text{if } t \in \left[0, \frac{b-x}{b-a}\right], \\ t-1 & \text{if } t \in \left(\frac{b-x}{b-a}, 1\right], \end{cases}$$

for all $x \in [a,b]$. $\square$

2.2. Theorem. *Let I be an open real interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$, where $0 \le a < b < \infty$. If $|f'|$ is*

m-convex on $[a,b]$ for some fixed $m \in (0,1]$, then the following inequality holds:

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq (b-a)\min\left\{\left[\tfrac{1}{6}-\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2+\tfrac{2}{3}\left(\tfrac{b-x}{b-a}\right)^3\right]|f'(a)|\right. \\
&\qquad +m\left[\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2-\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3+\tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right]\left|f'\left(\tfrac{b}{m}\right)\right|, \\
&\qquad \left[\tfrac{1}{6}-\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2+\tfrac{2}{3}\left(\tfrac{b-x}{b-a}\right)^3\right]|f'(b)| \\
&\qquad \left.+m\left[\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2-\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3+\tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right]\left|f'\left(\tfrac{a}{m}\right)\right|\right\}
\end{aligned} \tag{2.2}$$

for each $x \in [a,b]$.

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad +(b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt.
\end{aligned}$$

Since $|f'|$ is m-convex on $[a,b]$ we know that for any $t \in [0,1]$,

$$\begin{aligned}
\left|f'(ta+(1-t)b)\right| &= \left|f'(ta+m(1-t)\frac{b}{m})\right| \\
&\leq t\left|f'(a)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left[t\left|f'(a)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt \\
&\qquad +(b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left[t\left|f'(a)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt \\
&=(b-a)\left\{\left[\tfrac{1}{6}-\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2+\tfrac{2}{3}\left(\tfrac{b-x}{b-a}\right)^3\right]|f'(a)|\right. \\
&\qquad \left.+m\left[\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2-\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3+\tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right]\left|f'\left(\tfrac{b}{m}\right)\right|\right\},
\end{aligned}$$

where we use the facts that

$$\begin{aligned}
&\int_0^{\frac{b-x}{b-a}} t\left[t\left|f'(a)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt \\
&\qquad =\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3\left|f'(a)\right|+m\left[\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2-\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3\right]\left|f'\left(\tfrac{b}{m}\right)\right|,
\end{aligned}$$

and

$$\int_{\frac{b-x}{b-a}}^{1}(1-t)\left[t\left|f'(a)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt$$
$$=\left[\tfrac{1}{6}-\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2+\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3\right]\left|f'(a)\right|+m\tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\left|f'\left(\tfrac{b}{m}\right)\right|.$$

Analogously we obtain

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le (b-a)\Big\{\left[\tfrac{1}{6}-\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2+\tfrac{2}{3}\left(\tfrac{b-x}{b-a}\right)^3\right]\left|f'(b)\right|$$
$$+m\left[\tfrac{1}{2}\left(\tfrac{b-x}{b-a}\right)^2-\tfrac{1}{3}\left(\tfrac{b-x}{b-a}\right)^3+\tfrac{1}{3}\left(\tfrac{x-a}{b-a}\right)^3\right]\left|f'\left(\frac{a}{m}\right)\right|\Big\},$$

and the proof is completed. □

2.3. Remark. Suppose that all the assumptions of Theorem 2.2 are satisfied. If we choose $x=\frac{a+b}{2}$, then we have

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le\frac{b-a}{8}\min\left\{\left|f'(a)\right|+m\left|f'\Big(\frac{b}{m}\Big)\right|,\left|f'(b)\right|+m\left|f'\Big(\frac{a}{m}\Big)\right|\right\},$$

which is the inequality (1.6) with $q=1$.

2.4. Remark. Suppose that all the assumptions of Theorem 2.2 are satisfied. Then

(A) If we choose $m=1$ and $x=\frac{a+b}{2}$, we obtain

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le\frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),$$

which is the inequality (1.3).

(B) If in addition we choose $\left|f'(x)\right|\le M$, $M>0$ in (A), then:

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le M\frac{(b-a)}{4},$$

which is the inequality (1.1).

2.5. Theorem. *Let I be an open real interval such that $[0,\infty)\subset I$. Let $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([a,b])$, where $0\le a<b<\infty$. If $|f'|^{\frac{p}{p-1}}$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\le\frac{1}{(p+1)^{\frac{1}{p}}}$$
$$\times\left\{\frac{(b-x)^2}{b-a}\left[\min\left\{\frac{|f'(b)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right\}\right]^{\frac{1}{q}}\right.\tag{2.3}$$
$$\left.+\frac{(x-a)^2}{b-a}\left[\min\left\{\frac{|f'(a)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right\}\right]^{\frac{1}{q}}\right\}$$

for each $x\in[a,b]$.

Proof. From Lemma 2.1, and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right| dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right| dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}} \left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1 \left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq (b-a)\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{b-x}{b-a}\right)^{\frac{1}{q}} \\
&\quad\times\left(\min\left\{\frac{|f'(b)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right\}\right)^{\frac{1}{q}} \\
&\qquad + (b-a)\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}} \\
&\qquad\quad\times\left(\min\left\{\frac{|f'(a)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right\}\right)^{\frac{1}{q}} \\
&= \frac{1}{(p+1)^{\frac{1}{p}}}\frac{1}{b-a} \\
&\quad\times\left\{(b-x)^2\left[\min\left\{\frac{|f'(b)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right\}\right]^{\frac{1}{q}}\right. \\
&\quad\left.+(x-a)^2\left[\min\left\{\frac{|f'(a)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right\}\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where we use the facts that

$$\int_0^{\frac{b-x}{b-a}} t^p\,dt=\left(\frac{b-x}{b-a}\right)^{p+1}\frac{1}{p+1},\ \int_{\frac{b-x}{b-a}}^1 (1-t)^p\,dt=\left(\frac{x-a}{b-a}\right)^{p+1}\frac{1}{p+1},$$

and by (1.4) we get

$$\begin{aligned}
&\frac{b-a}{b-x}\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt \\
&\qquad\leq \min\left\{\frac{|f'(b)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right\}, \\
&\frac{b-a}{x-a}\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt \\
&\qquad\leq \min\left\{\frac{|f'(a)|^q+m\left|f'\left(\frac{x}{m}\right)\right|^q}{2},\ \frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right\}.
\end{aligned}$$

The proof is completed. □

2.6. Corollary. *Suppose that all the assumptions of Theorem 2.5 are satisfied. If we choose* $|f'(x)| \le M$, $M > 0$, *then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \le \left(\frac{1}{(p+1)^{\frac{1}{p}}} \right) \left(\frac{1+m}{2} \right)^{\frac{1}{q}} M \left[\frac{(b-x)^2 + (x-a)^2}{b-a} \right]. \qquad \square$$

2.7. Corollary. *Suppose that all the assumptions of Theorem 2.5 are satisfied. If we choose* $x = \frac{a+b}{2}$ *and* $\frac{1}{2} < \left(\frac{1}{p+1}\right)^{\frac{1}{p}} < 1$, *then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \le \frac{b-a}{4} \left(\mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}} \right),$$

where

$$\mu_1 = \min \left\{ \frac{|f'(b)|^q + m \left|f'(\frac{a+b}{2m})\right|^q}{2},\ \frac{\left|f'(\frac{a+b}{2})\right|^q + m \left|f'\left(\frac{b}{m}\right)\right|^q}{2} \right\},$$
$$\mu_2 = \min \left\{ \frac{|f'(a)|^q + m \left|f'(\frac{a+b}{2m})\right|^q}{2},\ \frac{\left|f'(\frac{a+b}{2})\right|^q + m \left|f'(\frac{a}{m})\right|^q}{2} \right\}. \qquad \square$$

2.8. Remark. Corollary 2.7 is similar to the inequality (1.5), but for the left-hand side of the Hermite-Hadamard inequality.

2.9. Theorem. *Let* I *be an open real interval such that* $[0, \infty) \subset I$. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I *such that* $f' \in L([a, b])$, *where* $0 \le a < b < \infty$. *If* $|f'|^q$ *is* m*-convex on* $[a, b]$ *for some fixed* $m \in (0, 1]$ *and* $q \in [1, \infty)$, $x \in [a, b]$, *then the following inequality holds:*

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| &\le (b-a) \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \Bigg\{ \left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{q}\right)} \Bigg[\frac{1}{3} \left(\frac{b-x}{b-a}\right)^3 |f'(a)|^q \\ &\quad + m \frac{(b-x)^2(b-3a+2x)}{6(b-a)^3} \left| f'\left(\frac{b}{m}\right) \right|^q \Bigg]^{\frac{1}{q}} \\ &\quad + \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{q}\right)} \Bigg[\left(\frac{1}{6} + \frac{(b-x)^2(3a-b-2x)}{6(b-a)^3} \right) |f'(a)|^q \\ &\quad + m \frac{1}{3} \left(\frac{x-a}{b-a}\right)^3 \left| f'\left(\frac{b}{m}\right) \right|^q \Bigg]^{\frac{1}{q}} \Bigg\} \end{aligned} \tag{2.4}$$

for each $x \in [a, b]$.

Proof. By Lemma 2.1, and using the well known power mean inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|dt \\
&\qquad + (b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|dt \\
&\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + (b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq (b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{b-x}{b-a}\right)^{2\left(1-\frac{1}{q}\right)}\left[\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3|f'(a)|^q\right.\right. \\
&\qquad\qquad \left.+m\frac{(b-x)^2(b-3a+2x)}{6(b-a)^3}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&\qquad + \left(\frac{x-a}{b-a}\right)^{2\left(1-\frac{1}{q}\right)}\left[\left(\frac{1}{6}+\frac{(b-x)^2(3a-b-2x)}{6(b-a)^3}\right)|f'(a)|^q\right. \\
&\qquad\qquad \left.\left.+m\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

where we use the facts that

$$\int_0^{\frac{b-x}{b-a}} t\,dt = \frac{1}{2}\left(\frac{b-x}{b-a}\right)^2,$$

$$\begin{aligned}
&\int_0^{\frac{b-x}{b-a}} t\left|f'(ta+(1-t)b)\right|^q dt \\
&\qquad \leq \frac{1}{3}\left(\frac{b-x}{b-a}\right)^3|f'(a)|^q + m\frac{(b-x)^2(b-3a+2x)}{6(b-a)^3}\left|f'\left(\frac{b}{m}\right)\right|^q,
\end{aligned}$$

$$\int_{\frac{b-x}{b-a}}^1 (1-t)\,dt = \frac{1}{2}\left(\frac{x-a}{b-a}\right)^2,$$

and

$$\begin{aligned}
&\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt \\
&\qquad \leq \left[\frac{1}{6}+\frac{(b-x)^2(3a-2x-b)}{6(b-a)^3}\right]|f'(a)|^q + m\frac{1}{3}\left(\frac{x-a}{b-a}\right)^3\left|f'\left(\frac{b}{m}\right)\right|^q.
\end{aligned}$$

The proof is completed. □

2.10. Remark. Suppose that all the assumptions of Theorem 2.9 are satisfied. If we choose $x = \frac{a+b}{2}$, we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq (b-a)\left(\frac{3^{1-\frac{1}{q}}}{8}\right)\left(\left|f'(a)\right|+m^{\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|\right),$$

which is the inequality (1.2).

3. Applications to special means

Let us recall the following means for two positive numbers.

(AM) The *arithmetic mean*

$$A = A(a,b) = \frac{a+b}{2};\ a,b>0,$$

(p-LM) The *p-logarithmic mean*

$$L_p = L_p(a,b) = \begin{cases} a & \text{if } a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b \end{cases};\ a,b>0,\ p\in\mathbb{R}\setminus\{-1,0\},$$

(IM) The *identric mean*

$$I = I(a,b) = \begin{cases} a & \text{if } a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b \end{cases};\ a,b>0.$$

The following propositions hold:

3.1. Proposition. *Let $a,b\in[0,\infty)$, and $a<b$, $n\geq 2$ with $m\in(0,1]$. Then we have*

$$|A^n(a,b) - L_n^n(a,b)| \leq n\frac{b-a}{8}\min\left\{2A\left(a^{n-1}, m\Big(\frac{b}{m}\Big)^{n-1}\right),\ 2A\left((b)^{n-1}, m\left(\frac{a}{m}\right)^{n-1}\right)\right\}.$$

Proof. The proof follows by Remark 2.3 on choosing $f:[0,\infty)\to[0,\infty)$, $f(x)=x^n$, $n\in\mathbb{Z}$, $n\geq 2$, which is m-convex on $[0,\infty)$. □

3.2. Proposition. *Let $a,b\in[0,\infty)$, with $a<b$, and $m\in(0,1]$. Then we have*

$$\left|\ln\frac{I(a+1,b+1)}{A(a,b)+1}\right| \leq \frac{b-a}{4}\left(\eta_1^{\frac{1}{q}}+\eta_2^{\frac{1}{q}}\right),$$

where

$$\eta_1^{\frac{1}{q}} = \min\left\{\frac{\left(\frac{1}{b+1}\right)^q + m\left(\frac{2m}{a+b+2m}\right)^q}{2},\ \frac{\left(\frac{2}{a+b+2}\right)^q + m\left(\frac{m}{b+m}\right)^q}{2}\right\},$$

$$\eta_2^{\frac{1}{q}} = \min\left\{\frac{\left(\frac{1}{a+1}\right)^q + m\left(\frac{2m}{a+b+2m}\right)^q}{2},\ \frac{\left(\frac{2}{a+b+2}\right)^q + m\left(\frac{m}{a+m}\right)^q}{2}\right\}.$$

Proof. The proof follows by Corollary 2.7 on choosing $f:[0,\infty)\to(-\infty,0]$, $f(x)=-\ln(x+1)$, which is m-convex on $[0,\infty)$, $p>1$. □

4. Applications to the midpoint formula for 1-convex functions

Let d be a division $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ of the interval $[a,b]$, and consider the quadrature formula

$$\int_a^b f(x)\,dx = M(f,d)+E(f,d), \tag{4.1}$$

where

$$M(f,d) = \sum_{i=1}^{n-1}(x_{i+1}-x_i)\,f\left(\frac{x_{i+1}+x_i}{2}\right)$$

is the midpoint formula and $E(f,d)$ denotes the associated approximation error (see [12]).

Here, we obtain some error estimates for the midpoint formula.

4.1. Proposition. *Let I be an open real interval such that $[0,\infty)\subset I$. Let $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([a,b])$, where $0\le a<b<\infty$. If $|f'|^q$ is 1-convex on $[a,b]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then in (4.1), for every division d of $[a,b]$, the midpoint error satisfies*

$$|E(f,d)|\le \frac{1}{4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right),$$

where

$$\begin{aligned}\mu_1&=\min\left\{\frac{|f'(x_i)|^q+\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q}{2},\ \frac{\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q+|f'(x_i)|^q}{2}\right\}\\&=\frac{\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q+|f'(x_i)|^q}{2},\\ \mu_2&=\min\left\{\frac{|f'(x_{i+1})|^q+\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q}{2},\ \frac{\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q+|f'(x_{i+1})|^q}{2}\right\}\\&=\frac{\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q+|f'(x_{i+1})|^q}{2}.\end{aligned}$$

Proof. Applying Corollary 2.7 for $m=1$ to the subinterval $[x_i,x_{i+1}]$, $(i=0,1,2,\dots,n-1)$ of the division, we have

$$\left|f\left(\frac{x_{i+1}+x_i}{2}\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx\right|\le\frac{x_{i+1}-x_i}{4}\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right),$$

where

$$\begin{aligned}\mu_1&=\frac{\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q+|f'(x_i)|^q}{2}\\ \mu_2&=\frac{\left|f'(\frac{x_i+x_{i+1}}{2})\right|^q+|f'(x_{i+1})|^q}{2}.\end{aligned}$$

Hence, in (4.1) we have

$$\begin{aligned}\left|\int_a^b f(x)dx-M(f,d)\right|&=\left|\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}f(x)\,dx-(x_{i+1}-x_i)\,f\left(\frac{x_{i+1}+x_i}{2}\right)\right]\right|\\&\le\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)\,dx-(x_{i+1}-x_i)\,f\left(\frac{x_{i+1}+x_i}{2}\right)\right|\\&\le\frac{1}{4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right),\end{aligned}$$

which completes the proof. □

4.2. Proposition. *Let I be an open real interval such that $[0,\infty)\subset I$. Let $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([a,b])$, where $0\le a<b<\infty$. If $|f'|^q$ is*

1-*convex on* $[a,b]$*, and* $q\in[1,\infty)$*,* $x\in[a,b]$*, then in (4.1), for every division* d *of* $[a,b]$*, the midpoint error satisfies*

$$|E(f,d)|\le\left(\frac{3^{1-\frac{1}{q}}}{8}\right)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left(\left|f'(x_i)\right|+\left|f'(x_{i+1})\right|\right).$$

Proof. Similar to that of Proposition 4.1 on using Remark 2.10 with $m=1$. □

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