

APPROXIMATION BY q -PHILLIPS OPERATORS

İsmet Yüksel*

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Abstract

In this study, we introduce a q -analogue of the Phillips operators and investigate approximation properties. We establish direct and local approximation theorems. We give a weighted approximation theorem. We estimate the rate of convergence of these operators for functions of polynomial growth on the interval $[0, \infty)$.

Keywords: Phillips operators, q -type operators, Rate of convergence, Weighted approximation, q -integral.

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1. Introduction

Phillips firstly introduced the q -analogue of Bernstein polynomials based on q -integer and q -binomial coefficients in [12]. Gupta and Finta obtain some direct results on certain q -Durrmeyer type operators in [6]. Recently, Aral and Gupta introduced Durrmeyer type modification of the q -Baskakov type operators in [1]. We aim to introduce a q -analogue of Phillips operators and to study approximation properties. Before this, we mention the following notations and formulas, which can be founded in [2, 8, 9] and [10]: For $n \in \mathbb{N}$, $0 < q < 1$ and $a, b \in \mathbb{R}$,

$$(1.1) \quad [n]_q = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N} \setminus \{0\}; \quad [0]_q = 0,$$

$$(1.2) \quad [n]_q! = [1]_q [2]_q \cdots [n]_q, \quad n \in \mathbb{N} \setminus \{0\}; \quad [0]_q! = 1,$$

$$(1.3) \quad (a + b)_q^n = \prod_{j=1}^n (a + q^{j-1}b),$$

and

$$(1.4) \quad (1 + a)_q^\infty = \prod_{j=1}^{\infty} (1 + q^{j-1}a).$$

*Gazi University, Faculty of Science, Department of Mathematics, Teknikokullar, 06500 Beşevler, Ankara, Turkey. E-mail: iyuksel@gazi.edu.tr

The q -binomial coefficients are given by

$$(1.5) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad 0 \leq k \leq n.$$

The q -derivative $D_q f$ of the function f is given by

$$(1.6) \quad (D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{for } x \neq 0$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The two q -analogues of the exponential function are defined by

$$(1.7) \quad e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1 - (1-q)x)_q^{\infty}}$$

and

$$(1.8) \quad E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (1 + (1-q)x)_q^{\infty}.$$

The q -Jackson integrals and the q -improper integrals are defined as

$$(1.9) \quad \int_0^a f(x) d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad a > 0$$

and

$$(1.10) \quad \int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0,$$

respectively. The q -Gamma function is given by

$$(1.11) \quad \Gamma_q(s) = K(A, s) \int_0^{\infty/A(1-q)} t^{s-1} e_q^{-t} d_q t,$$

where

$$(1.12) \quad K(A, s) = \frac{A^s}{1+A} \left(1 + \frac{1}{A}\right)_q^s (1+A)_q^{1-s}.$$

In particular, for $s \in \mathbb{Z}$, $K(A, s) = q^{s(s-1)/2}$ and $K(A, 0) = 1$.

2. Construction of the operators

Let f be a real valued continuous function on the interval $[0, \infty)$. Using the formulas and notations in (1.1)–(1.12), we now define the q -Phillips operators as

$$(2.1) \quad \mathcal{P}_n^q(f; x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}(x; q) \int_0^{\infty/A(1-q)} q^{k(k-1)} f(t) p_{n,k-1}(t; q) d_q t + e_q^{-[n]_q x} f(0),$$

where

$$(2.2) \quad p_{n,k}(x; q) := \frac{([n]_q x)^k}{[k]_q!} e_q^{-[n]_q x}.$$

In the case $q = 1$, these operators are reduced to the Phillips operators studied in [11] and [13].

Now we give an auxiliary lemma for the Korovkin monomial functions.

2.1. Lemma. Let $e_m(t) = t^m$, $m = 0, 1, 2, 3, 4$. We have

$$\begin{aligned}
 \text{(i)} \quad & \mathcal{P}_n^q(e_0; x) = 1, \\
 \text{(ii)} \quad & \mathcal{P}_n^q(e_1; x) = \frac{x}{q}, \\
 \text{(iii)} \quad & \mathcal{P}_n^q(e_2; x) = \frac{x^2}{q^4} + \frac{[2]_q}{q^3[n]_q}x, \\
 \text{(iv)} \quad & \mathcal{P}_n^q(e_3; x) = \frac{x^3}{q^9} + \frac{([2]_q q + [4]_q)}{q^8[n]_q}x^2 + \frac{[2]_q [3]_q}{q^6[n]_q^2}x, \\
 \text{(v)} \quad & \mathcal{P}_n^q(e_4; x) = \frac{x^4}{q^{16}} + \frac{([2]_q q^2 + [4]_q q + [6]_q)}{q^{15}[n]_q}x^3 \\
 & \quad + \frac{([2]_q [3]_q q^2 + [2]_q [5]_q q + [4]_q [5]_q)}{q^{13}[n]_q^2}x^2 + \frac{[2]_q [3]_q [4]_q}{q^{10}[n]_q^3}x.
 \end{aligned}$$

Proof. (i) Using the formulas (1.10), (1.11), (1.12) and (2.2), we can calculate the following integral:

$$\begin{aligned}
 \int_0^{\infty/A(1-q)} t^m p_{n,k}(t; q) d_q t &= \int_0^{\infty/A(1-q)} t^m \frac{([n]_q t)^k}{[k]_q!} e_q^{-[n]_q t} d_q t \\
 &= \frac{1}{[n]_q^{m+1} [k]_q!} \int_0^{\infty/(A/[n]_q)(1-q)} u^{k+m} e_q^{-u} d_q u \\
 &= \frac{\Gamma_q(k+m+1)}{[n]_q^{m+1} [k]_q! K(A/[n]_q, k+m+1)} \\
 \text{(2.3)} \quad &= \frac{[k+m]_q!}{[n]_q^{m+1} [k]_q! q^{(k+m+1)(k+m)/2}}.
 \end{aligned}$$

Using (1.7), (1.8) and (2.3) we obtain

$$\begin{aligned}
 \mathcal{P}_n^q(e_0; x) &= \sum_{k=1}^{\infty} q^{k(k-1)/2} p_{n,k}(x; q) + e_q^{-[n]_q x} \\
 &= \left(\sum_{k=1}^{\infty} q^{k(k-1)/2} p_{n,k}(x; q) + 1 \right) e_q^{-[n]_q x} \\
 &= E_q^{[n]_q x} e_q^{-[n]_q x} \\
 &= 1,
 \end{aligned}$$

which completes the proof of (i).

(ii) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_1; x) = \sum_{k=1}^{\infty} \frac{[k]_q}{[n]_q} q^{(k^2-3k)/2} p_{n,k}(x; q).$$

Thus, we obtain

$$\begin{aligned}
 \mathcal{P}_n^q(e_1; x) &= \frac{1}{[n]_q} \sum_{k=1}^{\infty} q^{(k^2-3k)/2} \frac{([n]_q x)^k}{[k-1]_q!} e_q^{-[n]_q x} \\
 &= \frac{x}{q} \mathcal{P}_n^q(e_0; x),
 \end{aligned}$$

as required.

(iii) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_2; x) = \sum_{k=1}^{\infty} \frac{[k]_q [k+1]_q}{[n]_q^2} q^{(k^2-5k-2)/2} p_{n,k}(x; q).$$

Using the equality $[k]_q [k+1]_q = [k]_q [k-1]_q + [2]_q q^{k-1} [k]_q$, we obtain

$$\begin{aligned} \mathcal{P}_n^q(e_2; x) &= \frac{1}{[n]_q^2} \sum_{k=2}^{\infty} q^{(k^2-5k-2)/2} \frac{([n]_q x)^k}{[k-2]_q!} e_q^{-[n]_q x} \\ &\quad + \frac{[2]_q}{[n]_q^2} \sum_{k=1}^{\infty} q^{(k^2-3k-4)/2} \frac{([n]_q x)^k}{[k-1]_q!} e_q^{-[n]_q x} \\ &= \frac{x^2}{q^4} \mathcal{P}_n^q(e_0; x) + \frac{[2]_q x}{q^3 [n]_q} \mathcal{P}_n^q(e_0; x), \end{aligned}$$

which is the required result.

(iv) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_3; x) = \sum_{k=1}^{\infty} \frac{[k]_q [k+1]_q [k+2]_q}{[n]_q^3} q^{(k^2-7k-6)/2} p_{n,k}(x; q).$$

Using the equality

$$\begin{aligned} [k]_q [k+1]_q [k+2]_q &= [k]_q [k-1]_q [k-2]_q + ([2]_q q + [4]_q) q^{k-2} [k]_q [k-1]_q \\ &\quad + [2]_q [3]_q q^{2k-2} [k]_q, \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{P}_n^q(e_3; x) &= \frac{1}{[n]_q^3} \sum_{k=3}^{\infty} q^{(k^2-7k-6)/2} \frac{([n]_q x)^k}{[k-3]_q!} e_q^{-[n]_q x} \\ &\quad + \frac{([2]_q q + [4]_q)}{[n]_q^3} \sum_{k=2}^{\infty} q^{(k^2-5k-10)/2} \frac{([n]_q x)^k}{[k-2]_q!} e_q^{-[n]_q x} \\ &\quad + \frac{[2]_q [3]_q}{[n]_q^3} \sum_{k=1}^{\infty} q^{(k^2-3k-10)/2} \frac{([n]_q x)^k}{[k-1]_q!} e_q^{-[n]_q x} \\ &= \frac{x^3}{q^9} \mathcal{P}_n^q(e_0; x) + \frac{([2]_q q + [4]_q) x^2}{q^8 [n]_q} \mathcal{P}_n^q(e_0; x) + \frac{[2]_q [3]_q x}{q^6 [n]_q^2} \mathcal{P}_n^q(e_0; x), \end{aligned}$$

which is the required result.

(v) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_4; x) = \sum_{k=1}^{\infty} \frac{[k]_q [k+1]_q [k+2]_q [k+3]_q}{[n]_q^4} q^{(k^2-9k-12)/2} p_{n,k}(x; q).$$

Using the equality

$$\begin{aligned} [k]_q [k+1]_q [k+2]_q [k+3]_q &= [k]_q [k-1]_q [k-2]_q [k-3]_q \\ &\quad + ([2]_q q^2 + [4]_q q + [6]_q) q^{k-3} [k]_q [k-1]_q [k-2]_q \\ &\quad + ([2]_q [3]_q q^2 + [2]_q [5]_q q + [4]_q [5]_q) q^{2k-4} [k]_q [k-1]_q \\ &\quad + [2]_q [3]_q [4]_q q^{3k-3} [k]_q, \end{aligned}$$

so we can write,

$$\begin{aligned} \mathcal{P}_n^q(e_4; x) &= \frac{1}{[n]_q^4} \sum_{k=4}^{\infty} q^{(k^2-9k-12)/2} \frac{([n]_q x)^k}{[k-4]_q!} e_q^{-[n]_q x} \\ &+ \frac{[2]_q q^2 + [4]_q q + [6]_q}{[n]_q^4} \sum_{k=3}^{\infty} q^{(k^2-7k-18)/2} \frac{([n]_q x)^k}{[k-3]_q!} e_q^{-[n]_q x} \\ &+ \frac{[2]_q [3]_q q^2 + [2]_q [5]_q q + [4]_q [5]_q}{[n]_q^4} \sum_{k=2}^{\infty} q^{(k^2-5k-20)/2} \frac{([n]_q x)^k}{[k-2]_q!} e_q^{-[n]_q x} \\ &+ \frac{[2]_q [3]_q [4]_q}{[n]_q^4} \sum_{k=1}^{\infty} q^{(k^2-3k-18)/2} \frac{([n]_q x)^k}{[k-1]_q!} e_q^{-[n]_q x}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{P}_n^q(e_4; x) &= \frac{x^4}{q^{16}} \mathcal{P}_n^q(e_0; x) + \frac{([2]_q q^2 + [4]_q q + [6]_q) x^3}{q^{15} [n]_q} \mathcal{P}_n^q(e_0; x) \\ &+ \frac{([2]_q [3]_q q^2 + [2]_q [5]_q q + [4]_q [5]_q) x^2}{q^{13} [n]_q^2} \mathcal{P}_n^q(e_0; x) \\ &+ \frac{([2]_q [3]_q [4]_q) x}{q^{10} [n]_q^3} \mathcal{P}_n^q(e_0; x), \end{aligned}$$

as required. □

2.2. Remark. Take a fixed number $q \in (0, 1)$. Since

$$\lim_{n \rightarrow \infty} [n]_q = \frac{1}{1-q},$$

in Lemma 2.1, $\mathcal{P}_n^q(t^m, x)$, $m \in \mathbb{N}$, does not tend to x^m as $n \rightarrow \infty$. From this result, we have to consider the condition $q := (q_n)$ as a sequence with $\lim_{n \rightarrow \infty} q_n = 1$ for approximation properties of the operators $\mathcal{P}_n^q(f, x)$ defined by (2.1)

For shortness, q denotes the n^{th} term of the sequence $(q_n) \subset (0, 1)$ with $\lim_{n \rightarrow \infty} q_n = 1$ after this section.

2.3. Lemma. For the operators $\mathcal{P}_n^q(f, x)$ defined by (2.1), we have the inequality

$$\mathcal{P}_n^q((t-x)^2; x) \leq \frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q} \right) x(1+x).$$

Proof. From the linearity of the \mathcal{P}_n^q operators, and Lemma 2.1, we have the second moment

$$\begin{aligned} \mathcal{P}_n^q((t-x)^2; x) &= \frac{x^2}{q^4} + \frac{[2]_q}{q^3 [n]_q} x - 2x \frac{x}{q} + x^2 \\ &= \left(\frac{1}{q^4} - \frac{2}{q} + 1 \right) x^2 + \frac{[2]_q}{q^3 [n]_q} x \\ &\leq \frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q} \right) x(1+x). \end{aligned}$$

Therefore, The proof is completed. □

2.4. Lemma. If we make a slight modification to the operators $\mathcal{P}_n^q(f; x)$ defined in (2.1) as follows:

$$(2.4) \quad \overline{\mathcal{P}}_n^q(f; x) = \mathcal{P}_n^q(f; x) - f\left(\frac{x}{q}\right) + f(x),$$

then we have

$$\overline{\mathcal{P}}_n^q(t-x; x) = 0.$$

Proof. From Lemma 2.1,

$$\overline{\mathcal{P}}_n^q(1; x) = \mathcal{P}_n^q(1; x)$$

and

$$\overline{\mathcal{P}}_n^q(t; x) = \mathcal{P}_n^q(t; x) - \frac{x}{q} + x = x.$$

Therefore, we obtain the result stated in the Lemma. \square

3. Local approximation

In this section, let $C_B[0, \infty)$ be the space of all real valued continuous bounded functions on $[0, \infty)$ and let $f \in C_B[0, \infty)$ be equipped with the norm $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$.

We denote the first modulus of continuity on the finite interval $[0, a]$, $a > 0$, by

$$(3.1) \quad \omega_{[0, a]}(f; \delta) = \sup_{0 < h \leq \delta, x \in [0, a]} |f(x+h) - f(x)|.$$

Peetre's K -functional is defined by

$$K_2(f; \delta) = \inf \{ \|f - g\| + \delta \|g''\| : g \in W_\infty^2 \}, \quad \delta > 0$$

where $W_\infty^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By [3, Theorem 2.4, p. 177] there exists a positive constant M such that

$$(3.2) \quad K_2(f; \delta) \leq M\omega_2(f; \sqrt{\delta}),$$

where

$$\omega_2(f; \sqrt{\delta}) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) - f(x)|.$$

3.1. Theorem. For every $x \in [0, \infty)$ and $f \in C_B[0, \infty)$ we have the inequality

$$|\mathcal{P}_n^q(f; x) - f(x)| \leq M\omega_2\left(f; \sqrt{\delta_{n,q}(x)}\right) + \omega_{[0, a]} \left(f; \frac{1-q}{q}x\right),$$

where

$$\delta_{n,q}(x) := \frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q}\right) x(1+x).$$

Proof. Let $g \in W_\infty^2$ and $x \in [0, \infty)$. Using Taylor's expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u) du,$$

and from Lemma 2.4, we have

$$\overline{\mathcal{P}}_n^q(g; x) = g(x) + \overline{\mathcal{P}}_n^q \left(\int_x^t (t-u)g''(u) du; x \right).$$

Then, we get

$$\begin{aligned} \left| \overline{\mathcal{P}}_n^q(g; x) - g(x) \right| &= \left| \mathcal{P}_n^q \left(\int_x^t (t-u)g''(u) du; x \right) - \int_x^{x/q} \left(\frac{x}{q} - u \right) g''(u) du \right| \\ &\leq \mathcal{P}_n^q \left(\left| \int_x^t (t-u)g''(u) du \right|; x \right) + \int_x^{x/q} \left| \frac{x}{q} - u \right| |g''(u)| du. \end{aligned}$$

Using the inequality

$$\left| \int_x^t (t-u)g''(u) du \right| \leq \|g''\| \frac{(t-x)^2}{2},$$

and from Lemma 2.3, we write

$$\begin{aligned} (3.3) \quad \left| \overline{\mathcal{P}}_n^q(g; x) - g(x) \right| &\leq \|g''\| \mathcal{P}_n^q \left(\frac{(t-x)^2}{2}; x \right) + \|g''\| \frac{\left(\frac{x}{q} - x \right)^2}{2} \\ &\leq \frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q} \right) x(1+x) \|g''\|. \end{aligned}$$

The operators $\overline{\mathcal{P}}_n^q(f, x)$ are bounded, that is

$$(3.4) \quad \left| \overline{\mathcal{P}}_n^q(f; x) \right| = \left| \mathcal{P}_n^q(f; x) - f \left(\frac{x}{q} \right) + f(x) \right| \leq \|f\| \mathcal{P}_n^q(1; x) + 2\|f\| \leq 3\|f\|.$$

From (2.4), (3.3) and (3.4), we get

$$\begin{aligned} \left| \overline{\mathcal{P}}_n^q(f; x) - f(x) \right| &= \left| \mathcal{P}_n^q(f - g; x) - (f - g)(x) + \mathcal{P}_n^q(g; x) - g(x) \right| \\ &\leq \left| \overline{\mathcal{P}}_n^q(f - g; x) - (f - g)(x) \right| + \left| \overline{\mathcal{P}}_n^q(g; x) - g(x) \right| \\ &\quad + \left| f \left(\frac{x}{q} \right) - f(x) \right| \\ &\leq 4\|f - g\| + \frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q} \right) x(1+x) \|g''\| \\ &\quad + \left| f \left(x + \frac{1-q}{q} x \right) - f(x) \right|. \end{aligned}$$

Now taking the infimum over $g \in W_\infty^2$ on the right hand side of the above inequality, and using the inequalities (3.1) and (3.2), we get the desired result. \square

4. Weighted approximation

Weighted Korovkin-type theorems were proved by Gadzhiev [4] and [5]. Now, we give Gadzhiev's results in weighted spaces. Let $\rho(x) = 1 + \varphi^2(x)$, where $\varphi(x)$ is a monotone increasing continuous function on the real axis and B_ρ is the set of all functions f defined on the real axis satisfying the growth condition $|f(x)| \leq M_f \rho(x)$, where M_f is a constant depending only on f . Then B_ρ is a normed space with norm

$$\|f\|_\rho = \sup\{|f(x)|/\rho(x) : x \in \mathbb{R}\}$$

for any $f \in B_\rho$. Let C_ρ denote the subspace of all continuous functions in B_ρ , and C_ρ^* the subspace of all functions $f \in C_\rho$ for which $\lim_{|x| \rightarrow \infty} (f(x)/\rho(x))$ exists finitely.

4.1. Theorem. (See [4] and [5])

(a) *There exists a sequence of linear positive operators $A_n(C_\rho \rightarrow B_\rho)$ such that*

$$(4.1) \quad \lim_{n \rightarrow \infty} \|A_n(\varphi^\nu) - \varphi^\nu\|_\rho = 0, \quad \nu = 0, 1, 2,$$

and a function $f^ \in C_\rho \setminus C_\rho^*$ with $\lim_{n \rightarrow \infty} \|A_n(f^*) - f^*\|_\rho \geq 1$.*

(b) *If a sequence of linear positive operators $A_n(C_\rho \rightarrow B_\rho)$ satisfies conditions (4.1), then*

$$\lim_{n \rightarrow \infty} \|A_n(f) - f\|_\rho = 0,$$

for every $f \in C_\rho^$.* □

Throughout this paper we take the growth condition as $\rho(x) = 1 + x^2$.

4.2. Theorem. *For every $f \in C_B[0, \infty)$ we have the following limit*

$$\lim_{n \rightarrow \infty} \|\mathcal{P}_n^q(f) - f\|_\rho = 0.$$

Proof. Since $\mathcal{P}_n^q(e_0; x) = 1$, it is obvious that

$$\|\mathcal{P}_n^q(e_0) - e_0\|_\rho = 0.$$

Considering Lemma 2.1 (ii), we get

$$\begin{aligned} \|\mathcal{P}_n^q(e_1) - e_1\|_\rho &= \sup_{x \in [0, \infty)} \frac{|\mathcal{P}_n^q(e_1; x) - x|}{1 + x^2} \\ &\leq \sup_{x \in [0, \infty)} \frac{\left| \frac{x}{q} - x \right|}{1 + x^2} \\ &\leq \left(\frac{1}{q} - 1 \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \\ &= o(1). \end{aligned}$$

Similarly, from Lemma 2.1 (iii) we get

$$\begin{aligned} \|\mathcal{P}_n^q(e_2) - e_2\|_\rho &= \sup_{x \in [0, \infty)} \frac{|\mathcal{P}_n^q(e_2; x) - x^2|}{1 + x^2} \\ &\leq \sup_{x \in [0, \infty)} \frac{\left| \frac{x^2}{q^4} + \frac{[2]_q}{q^3[n]_q} x - x^2 \right|}{1 + x^2} \\ &= \left(\frac{1 - q^4}{q^4} + \frac{[2]_q}{q^3[n]_q} \right) \sup_{x \in [0, \infty)} \frac{x + x^2}{1 + x^2} \\ &= o(1). \end{aligned}$$

Thus, from Theorem 4.1, we obtain the desired result. □

5. Rate of convergence

In this section, we want to estimate the rate of convergence for the sequence of the \mathcal{P}_n^q operators. As is known, if f is not uniformly continuous on the interval $[0, \infty)$, then the usual first modulus of continuity $\omega(f; \delta)$ does not tend to zero, as $\delta \rightarrow 0$. For every $f \in C_\rho^*[0, \infty)$, we would like to take a weighted modulus of continuity $\Omega(f; \delta)$ which tends to zero as $\delta \rightarrow 0$.

Let

$$(5.1) \quad \Omega(f; \delta) = \sup_{0 < h \leq \delta, x \geq 0} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}, \text{ for each every } f \in C_\rho^*[0, \infty).$$

The weighed modulus of continuity $\Omega(f; \delta)$ was defined by Ispir in [7]. It is known that $\Omega(f; \delta)$ has the following properties.

5.1. Lemma. [7] *Let $f \in C_\rho^*[0, \infty)$. Then:*

- (i) $\Omega(f; \delta)$ is a monotone increasing function of δ ,
- (ii) For each $f \in C_\rho^*[0, \infty)$, $\lim_{\delta \rightarrow 0^+} \Omega(f; \delta) = 0$,
- (iii) For each $m \in \mathbb{N} \setminus \{0\}$, $\Omega(f; m\delta) \leq m\Omega(f; \delta)$,
- (iv) For each $\lambda \in \mathbb{R}^+$, $\Omega(f; \lambda\delta) \leq (1 + \lambda)\Omega(f; \delta)$. □

Now we obtain a rate of convergence for the operators \mathcal{P}_n^q .

5.2. Theorem. *Let $f \in C_\rho^*[0, \infty)$. Then we have the inequality*

$$\|\mathcal{P}_n^q(f) - f\|_{\bar{\rho}} \leq M(q)\Omega\left(f; \sqrt{1 - q^3 + \frac{1}{[n]_q}}\right),$$

where $\bar{\rho}(x) = 1 + x^5$ and $M(q)$ is a positive real number dependent on q .

Proof. From the definition of $\Omega(f; \delta)$, and Lemma 5.1 (iv), we can write

$$|f(t) - f(x)| \leq (1 + (t - x)^2)(1 + x^2) \left(1 + \frac{|t - x|}{\delta}\right) \Omega(f; \delta).$$

Then, we have the inequality

$$(5.2) \quad \begin{aligned} |\mathcal{P}_n^q(f; x) - f(x)| &\leq (1 + x^2)\Omega(f; \delta)\mathcal{P}_n^q\left((1 + (t - x)^2) \left(1 + \frac{|t - x|}{\delta}\right); x\right) \\ &\leq (1 + x^2)\Omega(f; \delta) \left\{ \mathcal{P}_n^q((1 + (t - x)^2); x) \right. \\ &\quad \left. + \mathcal{P}_n^q\left((1 + (t - x)^2) \frac{|t - x|}{\delta}; x\right) \right\}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the second term, we get

$$(5.3) \quad \begin{aligned} \mathcal{P}_n^q\left((1 + (t - x)^2) \frac{|t - x|}{\delta}; x\right) \\ \leq \left\{ \mathcal{P}_n^q((1 + (t - x)^2)^2; x) \right\}^{1/2} \left\{ \mathcal{P}_n^q\left(\frac{|t - x|^2}{\delta^2}; x\right) \right\}^{1/2}. \end{aligned}$$

From Lemma 2.1 and Lemma 2.3, we get the following estimates

$$(5.4) \quad \begin{aligned} (\mathcal{P}_n^q((1 + (t - x)^2)^2; x)) &\leq 1 + \frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q}\right) x(1 + x) \\ &\leq \frac{2}{q^4} \left(2 - q^3 + \frac{1}{[n]_q}\right) (1 + x)^2 \\ &\leq M_1(q)(1 + x)^2, \end{aligned}$$

$$\begin{aligned}
(5.5) \quad & \mathcal{P}_n^q((1+(t-x)^2)^2; x) \\
&= 1 + 2\mathcal{P}_n^q((t-x)^2; x) + \mathcal{P}_n^q((t^4; x) - 4x\mathcal{P}_n^q((t^3; x) \\
&\quad + 6x^2\mathcal{P}_n^q(t^2; x) - 4x^3\mathcal{P}_n^q(t; x) + x^4\mathcal{P}_n^q(1; x) \\
&= x^4 \left(\frac{1}{q^{16}} - \frac{4}{q^9} + \frac{6}{q^4} - \frac{4}{q} + 1 \right) \\
&\quad + x^3 \left(\frac{([2]_q q^2 + [4]_q q + [6]_q)}{q^{15}[n]_q} - 4 \frac{([2]_q q + [4]_q)}{q^8[n]_q} + 6 \frac{[2]_q}{q^3[n]_q} \right) \\
&\quad + x^2 \left(\frac{([2]_q [3]_q q^2 + [2]_q [5]_q q + [4]_q [5]_q)}{q^{13}[n]_q^2} - 4 \frac{[2]_q [3]_q}{q^6[n]_q^2} + \frac{2}{q^4} - \frac{4}{q} + 2 \right) \\
&\quad + x \left(\frac{[2]_q [3]_q [4]_q}{q^{10}[n]_q^3} + 2 \frac{[2]_q}{q^3[n]_q} \right) + 1 \\
(5.6) \quad & \leq 8 \left(\frac{1-q^{15}}{q^{16}} \right) x^4 + \frac{48}{q^{15}[n]_q} x^3 + \frac{68}{q^{13}[n]_q^2} x^2 + \frac{28}{q^{10}[n]_q^3} x + 1 \\
& \leq M_2(q)(1+x^2)^2
\end{aligned}$$

and

$$\begin{aligned}
(5.7) \quad & \left\{ \mathcal{P}_n^q \left(\frac{|t-x|^2}{\delta^2}; x \right) \right\}^{1/2} \leq \frac{1}{\delta} \sqrt{\frac{2}{q^4} \left(1 - q^3 + \frac{1}{[n]_q} \right)} x(1+x) \\
& \leq \frac{M_3(q)}{\delta} \sqrt{1 - q^3 + \frac{1}{[n]_q}} (1+x).
\end{aligned}$$

Choosing $M(q) = (M_1(q) + \sqrt{M_2(q)M_3(q)})M_4$, where $M_4 = \sup_{x \geq 0} (1+x^2)^2(1+x)/(1+x^5)$ and $\delta = \sqrt{1 - q^3 + \frac{1}{[n]_q}}$, and combining the estimates between (5.2) and (5.7), we end up with

$$|\mathcal{P}_n^q(f, x) - f(x)| \leq (1+x^5)M(q)\Omega \left(f; \sqrt{1 - q^3 + \frac{1}{q^3[n]_q}} \right),$$

as required. \square

5.3. Remark. The weighted approximation theorem, Theorem 4.2, is obtained for the norm $\|\cdot\|_\rho$, where $\rho(x) = 1+x^2$. In Theorem 5.2, we estimated the rate of convergence for the operators \mathcal{P}_n^q for the norm $\|\cdot\|_{\bar{\rho}}$, where $\bar{\rho}(x) = 1+x^5$. It is an open problem to obtain the rate of convergence for the operators \mathcal{P}_n^q in the norm $\|\cdot\|_\rho$, where $\rho(x) = 1+x^2$, without adding an extra condition to the function $f \in C_\rho^*$.

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