ON THE NORMS OF TOEPLITZ MATRICES INVOLVING FIBONACCI AND LUCAS NUMBERS

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Abstract

Let us define $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{ij} \equiv L_{i-j}$ where $F$ and $L$ denote the usual Fibonacci and Lucas numbers, respectively. We have found upper and lower bounds for the spectral norms of these matrices.

Keywords: Fibonacci numbers, Lucas numbers, Toeplitz matrix, Spectral norm, Euclidean norm.


1. Introduction

Let $\{t_n\}_{n=-\infty}^\infty$ be a doubly infinite sequence. A Toeplitz matrix is an $n \times n$ matrix $T_n = [t_{ij}]_{i,j=0}^{n-1}$ where $t_{ij} = t_{i-j}$, i.e., a matrix of the form

$$T_n = \begin{bmatrix}
    t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\
    t_1 & t_0 & t_{-1} & \cdots & t_{-(n-2)} \\
    t_2 & t_1 & t_0 & \cdots & t_{-(n-3)} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0
\end{bmatrix}.$$

The Fibonacci and Lucas sequences $F_n$ and $L_n$ are defined by the recurrence relations

(1) $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

and

(2) $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

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If we start from $n = 0$, then the Fibonacci and Lucas sequences are given by

\[
\begin{array}{ccccccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 \\
L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 \\
\end{array}
\]

The rules (1) and (2) can be used to extend the sequences backwards, thus

\[
F_{-1} = F_1 - F_0, \quad F_{-2} = F_0 - F_{-1},
\]

\[
L_{-1} = L_1 - L_0, \quad L_{-2} = L_0 - L_{-1}
\]

and so on [7]. These produce

\[
\begin{array}{ccccccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
F_{-n} & 0 & 1 & -1 & 2 & -3 & 5 & -8 & 13 \\
L_{-n} & 2 & -1 & 3 & -4 & 7 & -11 & 18 & -29 \\
\end{array}
\]

Generally,

\[
F_{-n} = (-1)^{n+1} F_n 
\]

and

\[
L_{-n} = (-1)^{n} L_n.
\]

In [6], Solak has defined $n \times n$ circulant matrices with Fibonacci and Lucas numbers of the forms

\[
A = \left[ F_{\text{mod}(j-i,n)} \right]_{i,j=1}^n
\]

and

\[
B = \left[ L_{\text{mod}(j-i,n)} \right]_{i,j=1}^n.
\]

He has given lower and upper bounds for the spectral norms of these matrices.

In [2], Kayabaş has defined Toeplitz and Hankel matrices given by

\[
T_k = \left( g_{r-s}^k \right)_{r,s=1}^n
\]

and

\[
H_k = \left( g_{r+s}^k \right)_{r,s=1}^n,
\]

where $g_i^k$ is the $i$th element of the $k$-Fibonacci sequence [1]. When $k = 2$, the usual Fibonacci sequence is obtained. Moreover she has found upper bounds for the Euclidean norms of $T_k^n$ and $H_k^n$ as follows:

\[
\|T_k^n\|_E \leq \sqrt{n (F_{n} F_{n+1} - 1)}
\]

and

\[
\|H_k^n\|_E \leq \sqrt{n (F_{2n} F_{2n+1} - F_{n+1} F_{n+2})}.
\]

In this study, we define Toeplitz matrices involving Lucas numbers of the form

\[
B = \left[ L_{i-j} \right]_{i,j=1}^n.
\]

We have found the Euclidean norms, and the upper and lower bounds for the spectral norms of the matrices (3) and (4).
2. Preliminaries

Let \( A = (a_{ij}) \) be an \( m \times n \) matrix. The \( \ell_p \) norm of the matrix \( A \) is defined by

\[
\| A \|_p = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} \quad (1 \leq p < \infty).
\]

If \( p = \infty \), then

\[
\| A \|_\infty = \lim_{p \to \infty} \| A \|_p = \max_{i,j} |a_{ij}|.
\]

The well-known Frobenius (Euclidean) norm of the matrix \( A \) is

\[
\| A \|_E = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2},
\]

and also the spectral norm of the matrix \( A \) is

\[
\| A \|_2 = \sqrt{\max_{1 \leq i \leq n} |\lambda_i|}
\]

where the numbers \( \lambda_i \) are the eigenvalues of the matrix \( A^H A \) and the matrix \( A^H \) is the conjugate transpose of the matrix \( A \).

The following inequality holds \cite{8}:

\[
\frac{1}{\sqrt{n}} \| A \|_E \leq \| A \|_2 \leq \| A \|_E.
\]

Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( m \times n \) matrices. Then, the Hadamard product of \( A \) and \( B \) is the entry-wise product given by \cite{5}

\[
A \odot B = (a_{ij}b_{ij}).
\]

Define the maximum column length norm \( c_1(\cdot) \) and maximum row length norm \( r_1(\cdot) \) on \( m \times n \) matrices \( A = (a_{ij}) \) by

\[
c_1(A) \equiv \max_j \sqrt{\sum_i |a_{ij}|^2} = \max_j \| [a_{ij}]_{i=1}^m \|_E
\]

and

\[
r_1(A) \equiv \max_i \sqrt{\sum_j |a_{ij}|^2} = \max_i \| [a_{ij}]_{j=1}^n \|_E,
\]

respectively. Let \( A = (a_{ij}) \), \( B = (b_{ij}) \) and \( C = (c_{ij}) \) be \( m \times n \) matrices. If \( C = A \odot B \) then \cite{4}

\[
\| C \|_2 \leq r_1(A)c_1(B).
\]

The following sum formulae for the Fibonacci and Lucas numbers are well known \cite{3, 7}:

\[
\sum_{k=1}^{n-1} F_k^2 = F_n F_{n-1},
\]

\[
\sum_{k=1}^{n-1} L_k^2 = L_n L_{n-1} - 2,
\]

\[
\sum_{k=1}^{n} F_k F_{k-1} = \begin{cases} F_n^2 & \text{n even}, \\ F_n^2 - 1 & \text{n odd}, \end{cases}
\]
and
\[
\sum_{k=1}^{n} L_k L_{k-1} = \begin{cases} 
L_n^2 - 4 & \text{n even}, \\
L_n^2 + 1 & \text{n odd}.
\end{cases}
\]

3. Main results

3.1. Theorem. Let the \(n \times n\) matrix \(A = [a_{ij}]\) satisfy \(a_{ij} \equiv F_{i-j}\) (as in (3) for \(k = 2\)). Then,
\[
\|A\|_2 \geq \begin{cases} 
\sqrt{\frac{2}{n} F_n^2} & \text{n even}, \\
\sqrt{\frac{2}{n} (F_n^2 - 1)} & \text{n odd},
\end{cases}
\]
and
\[
\|A\|_2 \leq \sqrt{(1 + F_n F_{n-1}) (F_n F_{n-1})},
\]
where \(\|\cdot\|_2\) is the spectral norm, and \(F_n\) denotes the \(n\)th Fibonacci number.

Proof. The matrix \(A\) is of the form
\[
A = \begin{bmatrix}
F_0 & F_{-1} & F_{-2} & \cdots & F_{2-n} & F_{1-n} \\
F_1 & F_0 & F_{-1} & \cdots & F_{3-n} & F_{2-n} \\
F_2 & F_1 & F_0 & \cdots & F_{4-n} & F_{3-n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_0 & F_{-1} \\
F_{n-1} & F_{n-2} & F_{n-3} & \cdots & F_1 & F_0
\end{bmatrix}.
\]

We deduce from (7) that
\[
\|A\|_E^2 = n F_n^2 + 2 \sum_{i=1}^{n-1} \sum_{k=1}^{i} F_k^2
\]
\[
= 2 \sum_{i=1}^{n-1} F_i F_{i+1}
\]
\[
= 2 \sum_{k=1}^{n} F_k F_{k-1}.
\]

We conclude from (9) that
\[
\|A\|_E = \begin{cases} 
\sqrt{\frac{2}{n} F_n^2} & \text{n even}, \\
\sqrt{\frac{2}{n} (F_n^2 - 1)} & \text{n odd},
\end{cases}
\]

Using inequality (5) we obtain
\[
\|A\|_2 \geq \begin{cases} 
\sqrt{\frac{2}{n} F_n^2} & \text{n even}, \\
\sqrt{\frac{2}{n} (F_n^2 - 1)} & \text{n odd},
\end{cases}
\]

On the other hand, let the matrices
\[C = (c_{ij}) = \begin{cases} 
c_{ij} = 1 & j = 1, \\
c_{ij} = F_{i-j} & j \neq 1,
\end{cases}\]
and
\[D = (d_{ij}) = \begin{cases} 
d_{ij} = 1 & j \neq 1, \\
d_{ij} = F_{i-j} & j = 1,
\end{cases}\]
satisfy \( A = C \circ D \). Then
\[
\begin{align*}
    r_1(C) &= \max_i \sqrt{\sum_j |c_{ij}|^2} = \sqrt{1 + \sum_{k=1}^{n-1} F_k^2} \\
    &= \sqrt{1 + \sum_{k=1}^{n-1} F_k^2} = \sqrt{1 + F_n F_{n-1}}
\end{align*}
\]
and
\[
\begin{align*}
    c_1(D) &= \max_j \sqrt{\sum_i |d_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} F_k^2} \\
    &= \sqrt{\sum_{k=1}^{n-1} F_k^2} = \sqrt{F_n F_{n-1}}.
\end{align*}
\]
From (6), we have
\[
\|A\|_2 \leq \sqrt{(1 + F_n F_{n-1}) (F_n F_{n-1})}.
\]
Thus, the proof is completed. \(\square\)

3.2. Theorem. Let the \( n \times n \) matrix \( B = [b_{ij}] \) satisfy \( b_{ij} \equiv L_{i-j} \). Then
\[
\|B\|_2 \geq \begin{cases} 
\sqrt{\frac{2 n}{\pi} (L_n^2 - 4)} & \text{n even,} \\
\sqrt{\frac{2 n}{\pi} (L_n^2 + 1)} & \text{n odd,}
\end{cases}
\]
and
\[
\|B\|_2 \leq \sqrt{(L_n L_{n-1} - 1) (L_n L_{n-1} + 2)},
\]
where \( \|\cdot\|_2 \) is the spectral norm, and \( L_n \) denotes the \( n \)th Lucas number.

Proof. The matrix \( B \) is of the form
\[
B = \begin{bmatrix}
L_0 & L_{-1} & L_{-2} & \cdots & L_{2-n} & L_{1-n} \\
L_1 & L_0 & L_{-1} & \cdots & L_{3-n} & L_{2-n} \\
L_2 & L_1 & L_0 & \cdots & L_{4-n} & L_{3-n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
L_{n-2} & L_{n-3} & L_{n-4} & \cdots & L_0 & L_{1-n} \\
L_{n-1} & L_{n-2} & L_{n-3} & \cdots & L_1 & L_0
\end{bmatrix}.
\]
We deduce from (8) that
\[
\|B\|_E^2 = n L_0^2 + 2 \sum_{i=1}^{n-1} \sum_{k=1}^{i} L_k^2
\]
\[
= 4n + 2 \sum_{i=1}^{n-1} (L_i L_{i+1} - 2)
\]
\[
= 2 \sum_{k=1}^{n} L_k L_{k-1}.
\]
We conclude from (10) that
\[
\|B\|_E = \begin{cases} 
\sqrt{2 (L_n^2 - 4)} & \text{n even,} \\
\sqrt{2 (L_n^2 + 1)} & \text{n odd.}
\end{cases}
\]
Using inequality (5) we obtain
\[
\|B\|_2 \geq \begin{cases} 
\sqrt{\frac{2}{n} (L_n^2 - 4)} & \text{if } n \text{ even}, \\
\sqrt{\frac{2}{n} (L_n^2 + 1)} & \text{if } n \text{ odd}.
\end{cases}
\]

On the other hand, let the matrices
\[
E = (e_{ij}) = \begin{cases} 
eq 1 & j = 1, \\
eq L_{i-j} & j \neq 1,
\end{cases}
\]
and
\[
F = (f_{ij}) = \begin{cases} 
eq 1 & j \neq 1, \\
eq L_{i-j} & j = 1,
\end{cases}
\]
satisfy \( B = E \circ F \). Then
\[
r_1(E) = \max_i \sum_j |e_{ij}|^2 = \sqrt{1 + \sum_{k=1}^{n-1} L_k^2}
\]
and
\[
c_1(F) = \max_j \sum_i |f_{ij}|^2 = \sqrt{1 + \sum_{k=1}^{n-1} L_k^2}
\]

From (6), we have
\[
\|B\|_2 \leq \sqrt{(L_n L_{n-1} - 1)(L_n L_{n-1} + 2)}.
\]
Thus, the proof is completed. \( \square \)

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References