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ON ISOMETRIES OF $\mathbb{R}^2_{\pi n}$

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Abstract

In this work, we introduce a family of distance functions and show that the group of isometries of the plane associated with the induced metrics is the semi-direct product of the Dihedral group D_{2n} and the translation group $T(2)$.

Keywords: Group, Isometry, Distance function, Metric.

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1. Introduction

Isometries can be viewed as the transformations preserving normed vector spaces. Characterizing the isometries will enable our understanding of the geometry of the space, which is useful in the study of approximation problems, optimization problems, etc. Thus this study also stimulates interactions among different areas: group theory, numerical range, error analysis, [1,2,3,4,8,9,13].

We recall that the symmetry group of a regular $2n$-gon is called the dihedral group and denoted by D_{2n}. It has $4n$ elements, namely $2n$ rotations and $2n$ reflections.

The group of isometries of the Euclidean plane with the usual metric is the semi-direct product of the symmetry group of the unit circle, $O(2)$, and the translation group consisting of all translations of the plane, $T(2)$ [5,6,14]. The groups of isometries of the Taxicab and CC-planes, including the symmetry group of the square and a regular octagon, were given in [10] and [7], respectively.

Here, we introduce a family of distances, $d_{\pi n}$, that includes the Taxicab, Chinese-Checker and Isotaxi distances, [7,8,11], as special cases and then show that the group of isometries of the plane with the $d_{\pi n}$-metric is the semi-direct product of D_{2n} and $T(2)$.

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2. $d_{\pi n}$-distances

Now, we define a family of distances $d_{\pi n}$ in the analytic plane $\mathbb{R}^2$ and show that every $d_{\pi n}$-distance gives a metric for every $n \geqslant 2$, $n \in \mathbb{Z}$. Here,the Chinese Checkers distance, Taxicab distance and iso-taxi distance are contained as special cases.

2.1. Definition. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$, a family of distances $d_{\pi n}$ is defined by

$$\begin{aligned} &d_{\pi n}(A,B) \\ &= \frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n}\right||x_1 - x_2| + \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right||y_1 - y_2|\right), \end{aligned}$$

provided that

$$\begin{cases} 1 \leq k \leq \left[\frac{n-1}{2}\right],\ k \in \mathbb{Z} & \text{if } \tan\frac{(k-1)\pi}{n} \leqslant \left|\dfrac{y_2 - y_1}{x_2 - x_1}\right| < \tan\frac{k\pi}{n}, \\ k = \left[\frac{n+1}{2}\right] & \text{if } \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n} \leqslant \left|\dfrac{y_2 - y_1}{x_2 - x_1}\right| < \infty \text{ or } x_1 = x_2 \end{cases}$$

where $\left[\frac{n-1}{2}\right]$ and $\left[\frac{n+1}{2}\right]$ *are the integer part of* $\frac{n-1}{2}$ *and* $\frac{n+1}{2}$, $n \geqslant 2$, $n \in \mathbb{Z}$.

From now on, the plane with the distance $d_{\pi n}$ will be denoted by $\mathbb{R}^2_{\pi n}$.

2.2. Proposition. *Every distance $d_{\pi n}$ determines a metric.*

Proof. It will be sufficient to prove that $\|\cdot\|_{\pi n}$ defined by

$$\|u\|_{\pi n} := \frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n}\right||x| + \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right||y|\right),$$

$$\text{provided that } \begin{cases} 1 \leq k \leq \left[\frac{n-1}{2}\right],\ k \in \mathbb{Z} & \text{if } \tan\frac{(k-1)\pi}{n} \leqslant \left|\dfrac{y}{x}\right| < \tan\frac{k\pi}{n}, \\ k = \left[\frac{n+1}{2}\right] & \text{if } \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n} \leqslant \left|\dfrac{y}{x}\right| < \infty \text{ or } x = 0, \end{cases}$$

where $u = (x, y)$, is a norm in $\mathbb{R}^2$, since then $d_{\pi n}(A,B) = \|u - v\|_{\pi n}$, where $u = \overrightarrow{OA}$ and $v = \overrightarrow{OB}$ (O is the origin), is a metric. Also, if the vector u lies in the sector determined by v_k and v_{k+1}, $\|u\|_{\pi n}$ can be defined by

$$\|u\|_{\pi n} := u_k.u, \quad u_k = \left(\frac{\sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n}}{\sin\frac{\pi}{n},},\ \frac{\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}}{\sin\frac{\pi}{n}}\right),$$

$$v_k = (\cos(k-1)\frac{\pi}{n}, \sin(k-1)\frac{\pi}{n}),\ k = \{1, 2, 3, \dots, 2n\}.$$

It is obvious that $\|\cdot\|_{\pi n}$ satisfies

i) $\|u\|_{\pi n} \geqslant 0$, with the equality iff $u = 0$, and

ii) $\|\alpha u\|_{\pi n} = |\alpha|\, \|u\|_{\pi n}\, (\alpha \in \mathbb{R})$.

Thus, we will verify the triangle inequality

iii) $\|u + v\|_{\pi n} \leqslant \|u\|_{\pi n} + \|v\|_{\pi n}$ for the vectors u and v.

This can easily be obtained from the equivalence of the convexities of the closed unit ball $\left\{u \in \mathbb{R}^2 \mid \|u\|_{\pi n} \leqslant 1\right\}$ and the norm function on $\mathbb{R}^2$.

The unit circle is the set of vectors x in $\mathbb{R}^2$ satisfying

$$u_k \cdot x = 1,$$

which is a regular $2n$-gon with vertex vectors $v_k = (\cos(k-1)\frac{\pi}{n}, \sin(k-1)\frac{\pi}{n})$, $k \in \{1, 2, \dots, 2n\}$.

If the vector v lies in the sector determined by v_k and v_{k+1}, then $\|v\|_{\pi n} = u_k \cdot v$. Equivalently, if $v = t_k v_k + t_{k+1} v_{k+1}$, where t_k and $t_{k+1}$are nonnegative, $\|v\|_{\pi n} = t_k + t_{k+1}$. The vectors inside the unit ball have norm smaller than 1 and the vectors outside the unit circle have norm greater than 1. For u and v on the unit circle and for $0 \leqslant t \leqslant 1$, the convexity of the unit ball shows that $tu + (1-t)v$ is inside or on the unit circle, so $\|tu + (1-t)v\|_{\pi n} \leqslant 1$. For $a, b > 0$, set $t = \dfrac{a}{a+b}$ to obtain the triangle inequality

$$\frac{\|au + bv\|_{\pi n}}{a+b} = \left\| \frac{a}{a+b} u + (1 - \frac{a}{a+b}) v \right\|_{\pi n} \leqslant 1$$

$$\implies \|au + bv\|_{\pi n} \leqslant a + b = a\|u\|_{\pi n} + b\|v\|_{\pi n} = \|au\|_{\pi n} + \|bv\|_{\pi n}.$$

So, the triangle inequality is satisfied since au and bv are arbitrary (nonzero) vectors. □

If u and v lie in the same sector, then $\|u + v\|_{\pi n} = \|u\|_{\pi n} + \|v\|_{\pi n}$.

According to the definition of the $d_{\pi n}$-distance function, the shortest path between the points A and B is the union of line segments with the same slopes as v_k and v_{k+1}, $k \in \{1, 2, \ldots, 2n\}$, when the vector AB is in the sector obtained by extending the vectors v_k and v_{k+1}.

If the slope of the segment AB is equal to the slope of v_k, $k \in \{1, 2, 3, \ldots, 2n\}$, then the $d_{\pi n}$-distance is equal to the Euclidean distance between A and B.

3. Inequalities among the $d_{\pi n}$-distances

The following lemma gives a functional relation between $d_{\pi n}$-distance and d_E-distance (Euclidean distance).

3.1. Lemma. *Let l be the line through the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in the analytical plane, and let d_E denote the Euclidean metric. If l has slope m, then $d_{\pi n}(A, B) = \rho(m) d_E(A, B)$, with*

$$\rho(m) = \begin{cases} \dfrac{1}{\sin\frac{\pi}{n}\sqrt{1+m^2}} \left(\left| \sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n} \right| + \left| \cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n} \right| |m| \right), & y_1 \neq y_2, \\ \frac{1}{\sin\frac{\pi}{n}} \left| \cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n} \right|, & |m| \to \infty, \end{cases}$$

$$\textit{provided that } \begin{cases} 1 \leq k \leq \left[\frac{n-1}{2}\right], \ k \in \mathbb{Z}, & \textit{if } \tan\frac{(k-1)\pi}{n} \leqslant |m| < \tan\frac{k\pi}{n}, \\ k = \left[\frac{n+1}{2}\right], & \textit{if } \tan\frac{\left[\frac{n-1}{2}\right]}{n} \leqslant |m| < \infty \textit{ or } |m| \to \infty. \end{cases}$$

Proof. If l is parallel to the v_k vectors, then $d_{\pi n}(A, B) = d_E(A, B)$ and $\rho(m) = 1$. So $d_{\pi n}(A, B) = \rho(m) d_E(A, B)$. If l is not parallel to the x-axis or the y-axis, then $x_1 \neq x_2$ and $y_1 \neq y_2$, $m = (y_1 - y_2)/(x_1 - x_2)$, where m is the slope of l, and

$$\begin{aligned} d_{\pi n}(A, B) &= \frac{1}{\sin\frac{\pi}{n}} \left(\left| \sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n} \right| |x_1 - x_2| + \left| \cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n} \right| |y_1 - y_2| \right) \\ &= \frac{|x_1 - x_2|}{\sin\frac{\pi}{n}} \left(\left| \sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n} \right| + \left| \cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n} \right| |m| \right). \end{aligned}$$

Similarly,

$$d_E(A, B) = |x_1 - x_2| \sqrt{1 + m^2}, \text{ for all } m \in \mathbb{R},$$

and consequently the given equality is valid. If $m \to \infty$, then

$$\begin{aligned} d_{\pi n}(A,B) &= \frac{1}{\sin\frac{\pi}{n}} \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right| |y_1 - y_2| \\ &= \frac{1}{\sin\frac{\pi}{n}} \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right| d_E(A,B). \end{aligned} \qquad \square$$

The above proposition says that the $d_{\pi n}$-distance along any line is some positive constant multiple of the Euclidean distance along the same line. More precisely, the following inequality for this metric family is valid:

$$d_T(A,B) \geqslant d_I(A,B) \geqslant d_C(A,B) \geqslant d_{\pi 5}(A,B) \geqslant \cdots \geqslant d_{\pi n}(A,B) \geqslant d_E(A,B),$$

$n \geqslant 6$, $n \in \mathbb{Z}$, for every pair of points A and B in $\mathbb{R}^2$. Notice that $d_{\pi 2}$, $d_{\pi 3}$ and $d_{\pi 4}$ coincide with the Taxicab d_T, Isotaxi d_I and Chinese-Checkers d_C distance, respectively. That is, $d_{\pi n}(A,B)$ approaches to $d_E(A,B)$ as n gets greater.

Now, one can immediately obtain the following:

3.2. Corollary. *If A, B, X are any three collinear points in $\mathbb{R}^2$, then*

$$d_E(X,A) = d_E(X,B) \text{ iff } d_{\pi n}(X,A) = d_{\pi n}(X,B). \qquad \square$$

3.3. Corollary. *If A, B and X are any three distinct collinear points in the real plane then*

$$d_{\pi n}(X,A)/d_{\pi n}(X,B) = d_E(X,A)/d_E(X,B). \qquad \square$$

That is, the ratios of the Euclidean and $d_{\pi n}$-distances along a line are the same.

Notice that, the latter corollary gives us the validity of the *Theorems of Menelaus and Ceva* in $\mathbb{R}^2_{\pi n}$.

In the remaining part of this work, we will study the isometries of $\mathbb{R}^2_{\pi n}$, and determine its group of isometries.

4. Isometries of the plane $\mathbb{R}^2_{\pi n}$

An isometry of a plane is defined to be a transformation which preserves the distances in the plane. Therefore, an isometry of $\mathbb{R}^2_{\pi n}$ is an isometry of the real plane with respect to the $d_{\pi n}$ metric. Note that T is an isometry for $\|\cdot\|_{\pi n}$ if and only if T transforms the unit ball to the unit ball [12].

4.1. Proposition. *Every Euclidean translation is an isometry of $\mathbb{R}^2_{\pi n}$.*

Proof. Let $T_a : \mathbb{R}^2_{\pi n} \to \mathbb{R}^2_{\pi n}$ be the translation $T_a(u) = a + u$ in the real plane $\mathbb{R}^2$, where a is a translation vector and u is any vector in $\mathbb{R}^2_{\pi n}$. For any vectors u and v in $\mathbb{R}^2_{\pi n}$, we have

$$\begin{aligned} d_{\pi n}\left(T_a(u), T_a(v)\right) &= d_{\pi n}\left(a+u, a+v\right) \\ &= \|a+u-(a+v)\|_{\pi n} \\ &= d_{\pi n}(u,v). \end{aligned}$$

That is, every translation T_a is an isometry of $\mathbb{R}^2_{\pi n}$. $\square$

$\mathbb{R}^2_{\pi n}$ plane geometry is the study of Euclidean points, lines and angles in $\mathbb{R}^2_{\pi n}$. The following proposition determines the reflections which preserves distance in $\mathbb{R}^2_{\pi n}$.

4.2. Proposition. *The set of reflections preserving the $d_{\pi n}$-distance is*

$$S_{\pi n} = \left\{ f \mid f \textit{ is defined by } \begin{pmatrix} \cos\frac{k\pi}{n} & \sin\frac{k\pi}{n} \\ \sin\frac{k\pi}{n} & -\cos\frac{k\pi}{n} \end{pmatrix},\ k \in \{1,2,\ldots,2n\} \right\}, n \geqslant 2,\ n \in \mathbb{Z}.$$

Proof. It is sufficient to study the reflections of $\mathbb{R}^2$ preserving the regular $2n$-gon, since every isometric reflection of $\mathbb{R}^2_{\pi n}$ preserves the unit ball of $\mathbb{R}^2_{\pi n}$. So, we must show that the set, $S_{\pi n}$ preserves the $d_{\pi n}$-distances. Let the vector $v = \overrightarrow{OA}$ be in the sector obtained by extending the vectors v_i and v_{i+1} in the plane $\mathbb{R}^2_{\pi n}$. Then $v = t_i v_i + t_{i+1} v_{i+1}$, $t_i, t_{i+1} > 0$ and $d_{\pi n}(O, A) = \|v\|_{\pi n} = t_i + t_{i+1}$. If we calculate $f(v)$, we obtain

$$\begin{aligned} f(v) &= \begin{pmatrix} \cos\frac{k\pi}{n} & \sin\frac{k\pi}{n} \\ \sin\frac{k\pi}{n} & -\cos\frac{k\pi}{n} \end{pmatrix} \begin{pmatrix} t_i \cos(i-1)\frac{\pi}{n} + t_{i+1}\cos i\frac{\pi}{n} \\ t_i \sin(i-1)\frac{\pi}{n} + t_{i+1}\sin i\frac{\pi}{n} \end{pmatrix} \\ &= t_i \begin{pmatrix} \cos(k-i+1)\frac{\pi}{n} \\ \sin(k-i+1)\frac{\pi}{n} \end{pmatrix} + t_{i+1} \begin{pmatrix} \cos(k-i)\frac{\pi}{n} \\ \sin(k-i)\frac{\pi}{n} \end{pmatrix} \\ &= t_i v_{k-i+2} + t_{i+1} v_{k-i+1}. \end{aligned}$$

We see that $f(v)$ is in the sector obtained by extending v_{k-i+2} and v_{k-i+1}. Now,

$$\begin{aligned} d_{\pi n}(f(O), f(A)) &= d_{\pi n}(O, f(v)) \\ &= \|f(v)\|_{\pi n} \\ &= u_{k-i+2}.(t_i v_{k-i+2} + t_{i+1} v_{k-i+1}) \\ &= t_i(u_{k-i+2}.v_{k-i+2}) + t_{i+1}(u_{k-i+2}.v_{k-i+1}) \\ &= t_i + t_{i+1}. \end{aligned}$$

This result completes the proof □

For the rotations, we claim that there are only $2n$ rotations that preserve $d_{\pi n}$-distances in $\mathbb{R}^2_{\pi n}$.

4.3. Proposition. *The set of isometric rotations in $\mathbb{R}^2_{\pi n}$ is*

$$R_\theta = \left\{ r_\theta \mid r_\theta \textit{ is a rotation about the origin and } \theta = \frac{k\pi}{n},\ k \in \mathbb{Z},\ k = 1, 2, \ldots, 2n \right\}.$$

Proof. It is enough to study the rotations preserving the regular $2n-$gon, since every isometric rotation of $\mathbb{R}^2_{\pi n}$ must preserve the unit ball of $\mathbb{R}^2_{\pi n}$. Therefore we will show that the set of rotations, R_θ, preserving the regular $2n$-gon, preserves $d_{\pi n}$-distances. When the vector $v = \overrightarrow{OA}$ is in the sector obtained by extending the vectors v_i and v_{i+1} in the plane $\mathbb{R}^2_{\pi n}$, we know that $v = t_i v_i + t_{i+1} v_{i+1}$, $t_i, t_{i+1} \geqslant 0$ and

$$d_{\pi n}(O, A) = \|v\|_{\pi n} = t_i + t_{i+1}.$$

If we calculate $r_\theta(v)$, we obtain

$$\begin{aligned} r_\theta(\overrightarrow{OA}) = r_\theta(v) &= \begin{pmatrix} \cos\frac{k\pi}{n} & -\sin\frac{k\pi}{n} \\ \sin\frac{k\pi}{n} & \cos\frac{k\pi}{n} \end{pmatrix} \begin{pmatrix} t_i \cos(i-1)\frac{\pi}{n} + t_{i+1}\cos i\frac{\pi}{n} \\ t_i \sin(i-1)\frac{\pi}{n} + t_{i+1}\sin i\frac{\pi}{n} \end{pmatrix} \\ &= t_i \begin{pmatrix} \cos(k+i-1)\frac{\pi}{n} \\ \sin(k+i-1)\frac{\pi}{n} \end{pmatrix} + t_{i+1} \begin{pmatrix} \cos(k-i)\frac{\pi}{n} \\ \sin(k-i)\frac{\pi}{n} \end{pmatrix} \\ &= t_i v_{k+i-1} + t_{i+1} v_{k+i}. \end{aligned}$$

Thus, $r_\theta(v)$ is in the sector obtained by extending v_{k+i-1} and v_{k-i}. Then,

$$\begin{aligned} d_{\pi n}(r_\theta(O), r_\theta(A)) &= d_{\pi n}(O, r_\theta(v)) \\ &= \|r_\theta(v)\|_{\pi n} \\ &= u_{k+i-1}.(t_i v_{k+i-1} + t_{i+1} v_{k+i}) \\ &= t_i(u_{k+i-1} \cdot v_{k+i-1}) + t_{i+1}(v_{k+i-1} \cdot v_{k+i}) \\ &= t_i + t_{i+1} \end{aligned}$$

This result completes the proof □

From Propositions 4.2 and 4.3, one obtains the orthogonal group, consisting of $2n$ reflections and $2n$ rotations:

$$O_{\pi n}(2) = R_\theta \cup S_{\pi n},$$

which gives us *Dihedral group* D_{2n}, that is, the Euclidean symmetry group of the regular $2n$-gon. Now, let us show that all isometries of $\mathbb{R}^2_{\pi n}$ are in $T(2) \cdot O_{\pi n}(2)$.

4.4. Definition. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$ be two points in $\mathbb{R}^2_{\pi n}$. The *minimum distance set* of A, B is defined by

$$\{X \mid d_{\pi n}(A, X) + d_{\pi n}(B, X) = d_{\pi n}(A, B)\},$$

and denoted by $\overset{\diamond}{AB}$.

Let m_{AB} denote the slope of the line through the points A and B. If the slope of AB is the same as the slope of v_k, $k \in \{1, 2, \ldots, 2n\}$, the set $\overset{\diamond}{AB}$ is the line segment joining A and B, that is, $\overset{\diamond}{AB} = \overline{AB}$. We say that $\overset{\diamond}{AB}$ is the *standard parallelogram with diagonal* $\overline{AB}$. If the vector AB is in the sector joining v_k and v_{k+1}, $\overset{\diamond}{AB}$ is the standard parallelogram with long diagonal $\overline{AB}$, and its sides are parallel to v_k and v_{k+1}.

4.5. Proposition. *Let $\phi : \mathbb{R}^2_{\pi n} \to \mathbb{R}^2_{\pi n}$ be an isometry and $\overset{\diamond}{AB}$ the standard parallelogram. Then*

$$\phi(\overset{\diamond}{AB}) = \phi(A)\overset{\diamond}{}\phi(B).$$

Proof. Let $Y \in \phi(\overset{\diamond}{AB})$. Then,

$$\begin{aligned} Y \in \phi(\overset{\diamond}{AB}) &\iff \exists X \in \overset{\diamond}{AB} \text{ such that } Y = \phi(X) \\ &\iff d_{\pi n}(A, X) + d_{\pi n}(X, B) = d_{\pi n}(A, B) \\ &\iff d_{\pi n}(\phi(A), \phi(X)) + d_{\pi n}(\phi(X), \phi(B)) = d_{\pi n}(\phi(A), \phi(B)) \\ &\iff Y = \phi(X) \in \phi(A)\overset{\diamond}{}\phi(B) \end{aligned}$$ □

4.6. Corollary. *Let $\phi : \mathbb{R}^2_{\pi n} \to \mathbb{R}^2_{\pi n}$ be an isometry and $\overset{\diamond}{AB}$ the standard parallelogram. Then ϕ maps vertices to vertices and preserves the lengths of the sides of $\overset{\diamond}{AB}$.* □

4.7. Proposition. *Let $\phi : \mathbb{R}^2_{\pi n} \to \mathbb{R}^2_{\pi n}$ be an isometry such that $\phi(O) = O$. Then $\phi \in R_\theta$ or $\phi \in S_{\pi n}$.*

Proof. Let $A_1 = (1, 0)$, $A_2 = (\cos\frac{\pi}{n}, \sin\frac{\pi}{n})$, $D = (1 + \cos\frac{\pi}{n}, \sin\frac{\pi}{n})$, and consider the standard parallelogram $\overset{\diamond}{OD}$. It is clear that $\phi(A_1) \in \overline{A_i A_{i+1}}$. Since ϕ is an isometry by Corollary 4.6, $\phi(A_1)$ and $\phi(A_2)$ must be the vertices of the standard parallelogram $O\overset{\diamond}{\phi}(D)$. Therefore, if $\phi(A_1) \in \overline{A_k A_{k+1}}$, then $\phi(A_1) = A_k$ or $\phi(A_1) = A_{k+1}$. Similarly, $\phi(A_2) = A_k$ or $\phi(A_2) = A_{k+1}$.

When $\phi(A_1) = A_k$ and $\phi(A_2) = A_{k+1}$, ϕ is an rotation with angle $\theta = \frac{(k-1)\pi}{n}$, $k \in \{1, 2, \ldots, 2n\}$. In case $\phi(A_1) = A_{k+1}$ and $\phi(A_2) = A_k$, ϕ is an reflection according to $y = mx$, with the angle of slope $(k - 1)\pi/2n$, $k \in \{1, 2, \ldots, 2n\}$, $k \in \mathbb{Z}$.

Consequently, $\phi \in R_\theta$ or $\phi \in S_{\pi n}$ □

4.8. Theorem. *Let $f : \mathbb{R}^2_{\pi n} \to \mathbb{R}^2_{\pi n}$ be an isometry. Then there exists a unique $T_a \in T(2)$ and $\phi \in O_{\pi n}(2)$ such that $f = T_a \circ \phi$.*

Proof. Let $f(O) = A$ where $A = (a_1, a_2)$. Define $\phi = T_{-a} \circ f$. We know that ϕ is an isometry and $\phi(O) = O$. Thus, $\phi \in O_{\pi n}(2)$ and $f = T_a \circ \phi$ by Proposition 4.7. The proof of uniqueness is trivial $\square$

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