A CLASS OF FUNCTIONS AND SEPARATION AXIOMS WITH RESPECT TO AN OPERATION

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Abstract

In this paper, a new kind of set called a $\gamma$-$\beta$-open set is introduced and investigated using the $\gamma$-operator due to Ogata (H. Ogata, Operations on topological spaces and associated topology, Math. Japonica 36 (1), 175–184, 1991). Such sets are used for studying new types of mappings, viz. $\gamma$-continuous, $\gamma$-$\beta$-continuous, $\gamma$-$\beta$-open, $\gamma$-$\beta$-closed, $\gamma$-$\beta$-generalized mappings, etc. A decomposition theorem for $\gamma$-continuous mappings, as well as a characterization of continuous mapping are obtained in terms of $\gamma$-$\beta$-continuous mappings. Finally, new separation axioms: $\gamma$-$\beta$-$T_i$ $(i = 0, \frac{1}{2}, 1, 2)$, $\gamma$-$\beta$-regularity and $\gamma$-$\beta$-normality are investigated along with the result that every topological space is $\gamma$-$\beta T_\frac{1}{2}$.

Keywords: $\beta$-open, $\gamma$-open, $\gamma$-$\beta$-open, $\gamma$-regular, $\gamma$-$\beta$g-closed, $\gamma$-continuous, $\gamma$-$\beta$-continuous, $\gamma$-$\beta T_i$ $(i = 0, \frac{1}{2}, 1, 2)$, $\gamma$-$\beta$-regular, $\gamma$-$\beta$-normal.

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1. Introduction

In [9] Monsef et al. introduced $\beta$-open sets (semi-preopen sets [1]), Kasahara [5] defined an operation $\alpha$ on a topological space to introduce $\alpha$-closed graphs, which were further investigated by Janković [4]. Following the same technique, Ogata [10] defined an operation $\gamma$ on a topological space and introduced $\gamma$-open sets. Recently Krishnan et al. [6] used the operation $\gamma$ for the introduction of $\gamma$-semi open sets.

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In this paper, we introduce $\gamma$-$\beta$-open sets which are shown to be independent of $\beta$-open sets but are a generalization of $\gamma$-open sets.

Section 3 deals with the introduction of $\gamma$-$\beta$-open sets, $\gamma$-$\beta$-generalized open sets and their various characterizations in terms of $\gamma$-$\beta$-interior, $\gamma$-$\beta$-closure, $\gamma$-$\beta$-frontier and $\gamma$-$\beta$-derived sets.

$\gamma$-continuity, $\gamma$-$\beta$-continuity, $\gamma'$-closedness, $\gamma'$-$\beta$-closedness, $\gamma'$-$\beta$-openness, $\gamma'$-$\beta$g-closedness, $\gamma$-$\gamma'$-irresoluteness and their interrelations are studied in section 4.

Section 5 is concerned with new separation axioms, viz. $\gamma$-$\beta$-$T_i$ ($i = 0, 1, 2$), $\gamma$-$\beta$-regular and $\gamma$-$\beta$-normal spaces. Such spaces, specially $\gamma$-$\beta$-regular spaces and $\gamma$-$\beta$-normal spaces, are characterized in various ways. Although $\gamma$-$\beta$-regularity and $\gamma$-$\beta$-normality are independent of regularity and normality, respectively, we have been able to achieve them in terms of regularity and normality respectively. Finally, the result that every topological space is $\gamma$-$\beta$-$T_2$ is obtained.

Throughout this paper, unless otherwise stated, $X$ or $Y$ denotes a topological space without any separation axioms. We use $cl(A)$ and $int(A)$ to denote respectively the closure and interior of a subset $A$ of $X$.

2. Preliminaries

A subset $A$ of $X$ is called $\beta$-open [9] if $A \subseteq cl(int(A))$. The complement of a $\beta$-open set is called a $\beta$-closed set. The family of all $\beta$-open sets of $X$ is denoted by $\beta O(X)$. For a subset $A$ of $X$, the union of all $\beta$-open sets of $X$ contained in $A$ is called the $\beta$-interior [9] (in short $\beta int(A)$) of $A$, and the intersection of all $\beta$-open sets of $X$ containing $A$ is called the $\beta$-closure [9] (in short $\beta cl(A)$) of $A$. A subset $A$ of $X$ is called generalized closed [7] (in short g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

An operation $\gamma$ [10] on a topology $\tau$ on $X$ is a mapping $\gamma : \tau \rightarrow P(X)$, such that $V \subseteq V' \subseteq \tau$ for each $V \subseteq \tau$, where $P(X)$ is the power set of $X$ and $V'$ denotes the value of $\gamma$ at $V$. A subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is called $\gamma$-open [10] if for each $x \in A$, there exists an open set $U$ such that $x \in U$ and $U' \subseteq A$. Then, $\tau_\gamma$ denotes the set of all $\gamma$-open sets in $X$. Clearly $\tau_\gamma \subseteq \tau$.$\gamma$-open sets are called $\gamma$-closed. The $\gamma$-closure [10] of a subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is denoted by $\gamma-$cl$(A)$ and is defined to be the intersection of all $\gamma$-closed sets containing $A$, and the $\gamma$-interior [6] of $A$ is denoted by $\gamma-$int$(A)$ and defined to be the union of all $\gamma$-open sets of $X$ contained in $A$. A subset $A$ of $X$ with an operation $\gamma$ on $\tau$ is called a $\gamma$-semiopen set [6] if and only if there exists a $\gamma$-open set $U$ such that $U \subseteq A \subseteq \gamma-$cl$(U)$. The family of all $\gamma$-semiopen sets in $X$ is denoted by $\gamma-$SO$(X)$. A topological $X$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-regular [5] if for each $x \in X$ and open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $U' \subseteq V$ is contained in $V$. It is also to be noted that $\tau_\gamma = \tau$ if and only if $X$ is a $\gamma$-regular space [10].

3. $\gamma$-$\beta$-open sets

3.1. Definition. Let $(X, \tau)$ be a topological space, $\gamma$ an operation on $\tau$ and $A \subseteq X$. Then $A$ is called a $\gamma$-$\beta$-open set if $A \subseteq \tau_\gamma-$cl$(\tau_\gamma-$int$(\tau_\gamma-$cl$(A)))$.

$\gamma$-$\beta O(X)$ (resp. $\gamma O(X)$ or $\tau_\gamma$) denotes the collection of all $\gamma$-$\beta$-open (resp. $\gamma$-open) sets of $(X, \tau)$, and $\gamma$-$\beta O(X, x)$ (resp. $\gamma O(X, x)$) is the collection of all $\gamma$-$\beta$-open (resp. $\gamma$-open) sets containing the point $x$ of $X$. 
A subset $A$ of $X$ is called $\gamma$-$\beta$-closed if and only if its complement is $\gamma$-$\beta$-open. Moreover, $\gamma$-$\beta C(X)$ (resp. $\gamma C(X)$) denotes the collection of all $\gamma$-$\beta$-closed (resp. $\gamma$-closed) sets of $(X, \tau)$.

It can be shown that a subset $A$ of $X$ is $\gamma$-$\beta$-closed if and only if $\tau_{\gamma}$-$\text{int} (\tau_{\gamma}$-$\text{cl} (\tau_{\gamma}$-$\text{int} (A))) \subset A$.

3.2. Remark.

(a) The concepts of $\beta$-open and $\gamma$-$\beta$-open sets are independent, while in a $\gamma$-regular space [5] these concepts are equivalent.

(b) A $\gamma$-open set is $\gamma$-$\beta$-open but the converse may not be true.

(c) A $\gamma$-semiopen set [6] is $\gamma$-$\beta$-open and it is quite clear that the converse is true when $A$ is $\gamma$-closed.

3.3. Example. (a) Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1, 2\}, \{1\}\}$. Define an operation $\gamma$ on $\tau$ by

$$A^\gamma = \begin{cases} \{1\} & \text{if } A = \{1\} \\ A \cup \{3\} & \text{if } A \neq \{1\} \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, X, \{1\}\}$. Then $\{2\}$ is $\beta$-open but not $\gamma$-$\beta$-open.

Again, if we define $\gamma$ on $\tau$ by $A^\gamma = \text{cl}(A)$, then $\{3\}$ is $\gamma$-$\beta$-open but not $\beta$-open.

(b) Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, \{b, d, a\}\}$ be a topology on $X$. Define an operation $\gamma$ on $\tau$ by

$$A^\gamma = \begin{cases} \{a\} & \text{if } A = \{a\} \\ A \cup \{b\} & \text{if } A \neq \{a\} \end{cases}$$

Then, $\tau_\gamma = \tau$. But $\gamma$-$\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, \{b, d, a\}\}$.

3.4. Theorem. An arbitrary union of $\gamma$-$\beta$-open sets is $\gamma$-$\beta$-open.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of $\gamma$-$\beta$-open sets. Then for each $\alpha$, $A_\alpha \subset \tau_\gamma$-$\text{int} (\tau_\gamma$-$\text{cl} (\tau_\gamma$-$\text{int} (A_\alpha)))$ and so

$$\bigcup_\alpha A_\alpha \subset \bigcup_\alpha \tau_\gamma$-$\text{int} (\tau_\gamma$-$\text{cl} (A_\alpha)))$$

$$\subset \bigcup_\alpha \tau_\gamma$-$\text{cl} (\bigcup_\alpha \tau_\gamma$-$\text{int} (\tau_\gamma$-$\text{cl} (A_\alpha)))$$

$$\subset \tau_\gamma$-$\text{cl} (\bigcup_\alpha (\tau_\gamma$-$\text{int} (\tau_\gamma$-$\text{cl} (\bigcup_\alpha A_\alpha))))$$

Thus, $\bigcup_\alpha A_\alpha$ is $\gamma$-$\beta$-open. \square

3.5. Remark.

(a) An arbitrary intersection of $\gamma$-$\beta$-closed sets is $\gamma$-$\beta$-closed.

(b) The intersection of two $\gamma$-$\beta$-open sets may not be $\gamma$-$\beta$-open.

3.6. Example. In Example 3.3(b), take $M = \{b, c\}$ and $N = \{a, c\}$. Then $M \cap N = \{c\}$, which is not a $\gamma$-$\beta$-open set.

3.7. Definition. Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ an operation on $\tau$. The union of all $\gamma$-$\beta$-open sets contained in $A$ is called the $\gamma$-$\beta$-interior of $A$ and denoted by $\gamma$-$\beta$-$\text{int} (A)$.

3.8. Definition. Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ an operation on $\tau$. The intersection of all $\gamma$-$\beta$-closed sets containing $A$ is called the $\gamma$-$\beta$-closure of $A$ and denoted by $\gamma$-$\beta$-$\text{cl} (A)$. 
3.9. Definition. Let \((X, \tau)\) be a topological space with an operation \(\gamma\) on \(\tau\).

(a) The set denoted by \(\gamma\beta D(A)\) (resp. \(\gamma D(A)\)) and defined by
\[
\{x : \text{for every } \gamma\beta\text{-open (resp. } \gamma\text{-open) set } U \text{ containing } x, U \cap (A - \{x\}) \neq \emptyset \}
\]
is called the \(\gamma\beta\)-derived (resp. \(\gamma\)-derived) set of \(A\).

(b) The \(\gamma\beta\) (resp. \(\gamma\)-frontier) frontier of \(A\), denoted by \(\gamma\beta Fr(A)\) (resp. \(\gamma Fr(A)\)) is defined as \(\gamma\beta cl(A) \cap (X - A)\) (resp. \(\tau_\gamma cl(A) \cap \tau_\gamma cl (X - A)\)).

We now state the following theorem without proof.

3.10. Theorem. Let \((X, \tau)\) be a topological space and \(\gamma\) an operation on \(\tau\). For any subsets \(A, B\) of \(X\) we have the following:

(i) \(A\) is \(\gamma\beta\)-open if and only if \(A = \gamma\beta int (A)\).
(ii) \(A\) is \(\gamma\beta\)-closed if and only if \(A = \gamma\beta cl (A)\).
(iii) If \(A \subset B\) then \(\gamma\beta int (A) \subset \gamma\beta int (B)\) and \(\gamma\beta cl (A) \subset \gamma\beta cl (B)\).
(iv) \(\gamma\beta cl (A) \cup \gamma\beta cl (B) \subset \gamma\beta cl (A \cup B)\).
(v) \(\gamma\beta cl (A \cap B) \subset \gamma\beta cl (A) \cap \gamma\beta cl (B)\).
(vi) \(\gamma\beta int (A) \cup \gamma\beta int (B) \subset \gamma\beta int (A \cup B)\).
(vii) \(\gamma\beta int (A \cap B) \subset \gamma\beta int (A) \cap \gamma\beta int (B)\).
(viii) \(\gamma\beta int (X - A) = X - \gamma\beta cl (A)\).
(ix) \(\gamma\beta cl (X - A) = X - \gamma\beta int (A)\).
(x) \(\gamma\beta int (A) = A \cap \gamma\beta\beta D(X - A)\).
(xi) \(\gamma\beta cl (A) = A \cup \gamma\beta\beta D(A)\).
(xii) \(\tau_\gamma int (A) \subset \gamma\beta int (A)\).
(xiii) \(\gamma\beta cl (A) \subset \tau_\gamma cl (A)\).

3.11. Remark. The reverse inclusions of (iii) to (vii) in Theorem 3.10 are not true, in general.

3.12. Example. In Example 3.3(b), let \(A = \{a, b\}, B = \{c\}, C = \{a, c\},\) and \(D = \{b, c\}\). Then \(\gamma\beta cl (A) = X, \gamma\beta cl (B) = \{c\}\) but \(\gamma\beta cl (A \cap B) = \emptyset\). Also, \(\gamma\beta cl (C) = \{a, c\}, \gamma\beta cl (D) = \{b, c\}\) but \(\gamma\beta cl (C \cup D) = X\).

Again, \(\gamma\beta int (A) = \{a, b\}, \gamma\beta int (B) = \emptyset\) but \(\gamma\beta int (A \cup B) = \{a, b, c\}\) and \(\gamma\beta int (C) = \{a, c\}, \gamma\beta int (D) = \{b, c\}\) but \(\gamma\beta int (C \cap D) = \emptyset\).

Also we note that \(\gamma\beta int (B) \subset \gamma\beta int (A)\) but \(B \notin A\) and \(\gamma\beta cl (B) \subset \gamma\beta cl (A)\) but \(B \notin A\).

3.13. Theorem. Let \(A\) be a subset of a topological space \((X, \tau)\) and \(\gamma\) be an operation on \(\tau\). Then \(x \in \gamma\beta cl (A)\) if and only if for every \(\gamma\beta\)-open set \(U\) of \(X\) containing \(x, A \cap U \neq \emptyset\).

Proof. First suppose that \(x \in \gamma\beta cl(A)\) and \(U\) is any \(\gamma\beta\)-open set containing \(x\) such that \(A \cap U = \emptyset\). Then \((X - U)\) is a \(\gamma\beta\)-closed set containing \(A\). Thus \(\gamma\beta cl (A) \subset (X - U)\). Then \(x \notin \gamma\beta cl (A)\), which is a contradiction.

Conversely, suppose \(x \notin \gamma\beta cl (A)\). Then there exists a \(\gamma\beta\)-closed set \(V\) such that \(A \subset V\) and \(x \notin V\). Hence \((X - V)\) is a \(\gamma\beta\)-open set containing \(x\) such that \(A \cap (X - V) = \emptyset\).

\(\square\)

3.14. Theorem. Let \(U, V\) be subsets of a topological space \((X, \tau)\) and \(\gamma\) an operation on \(\tau\). If \(V \in \gamma\beta\beta O(X)\) is such that \(U \subset V \subset \tau_\gamma cl(U)\), then \(U\) is a \(\gamma\beta\)-open set.

Proof. Since \(V\) is \(\gamma\beta\)-open, \(V \subset \tau_\gamma cl (\tau_\gamma int (\tau_\gamma cl (V)))\). Again, \(\tau_\gamma cl (V) \subset \tau_\gamma cl (U)\) for \(V \subset \tau_\gamma cl (U)\). Then \(U \subset V \subset \tau_\gamma cl (\tau_\gamma int (\tau_\gamma cl (V))) \subset \tau_\gamma cl (\tau_\gamma int (\tau_\gamma cl (U)))\).

\(\square\)
3.15. Definition. A subset $A$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is called $\tau$-$\gamma$-$\beta$-open (resp. $\gamma$-$\gamma$-$\beta$-open) if $\text{int}(A) = \gamma\beta\text{int}(A)$ (resp. $\tau\gamma$-$\text{int}(A) = \gamma\beta\text{int}(A)$).

3.16. Definition. A subset $A$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is called a $\gamma$-$\beta$ generalized closed set ($\gamma$-$\beta$g closed, for short) if $\gamma\beta\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is a $\gamma$-$\beta$-open set in $X$.

The complement of a $\gamma$-$\beta$g closed set is called a $\gamma$-$\beta$g open set. Clearly, $A$ is $\gamma$-$\beta$g open if and only if $F \subset \gamma\beta\text{int}(A)$ whenever $F \subset A$ and $F$ is $\gamma$-$\beta$ closed in $X$.

3.17. Theorem. A subset $A$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$, is $\gamma$-$\beta$g closed if and only if $\gamma\beta\text{cl}\{x\} \cap A \neq \emptyset$ for every $x \in \gamma\beta\text{cl}(A)$.

Proof. Let $A$ be a $\gamma$-$\beta$g closed set in $X$ and suppose if possible there exists an $x \in \gamma\beta\text{cl}(A)$ such that $\gamma\beta\text{cl}\{x\} \cap A = \emptyset$. Therefore $A \subset X - \gamma\beta\text{cl}\{x\}$, and so $\gamma\beta\text{cl}(A) \subset X - \gamma\beta\text{cl}\{x\}$. Hence $x \notin \gamma\beta\text{cl}(A)$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let $U$ be any $\gamma$-$\beta$ open set containing $A$. Let $x \in \gamma\beta\text{cl}(A)$. Then $\gamma\beta\text{cl}\{x\} \cap A \neq \emptyset$, so there exists $z \in \gamma\beta\text{cl}\{x\} \cap A$ and so $z \in A \subset U$. Thus by the Theorem 3.13, $\{x\} \subset U \neq \emptyset$. Hence $x \in U$, which implies $\gamma\beta\text{cl}(A) \subset U$.

3.18. Theorem. Let $A$ be a $\gamma$-$\beta$g closed set in a topological space $(X, \tau)$ with operation $\gamma$ on $\tau$. Then $\gamma\beta\text{cl}(A) - A$ does not contain any nonempty $\gamma$-$\beta$ closed set.

Proof. If possible, let $F$ be a nonempty $\gamma$-$\beta$ closed set such that $F \subset \gamma\beta\text{cl}(A) - A$. Let $x \in F$ and so $x \in \gamma\beta\text{cl}(A)$. Again we observe that $F \cap A = \gamma\beta\text{cl}(F) \cap A \supset \gamma\beta\text{cl}\{x\} \cap A \neq \emptyset$, which gives $F \cap A \neq \emptyset$, a contradiction.

3.19. Theorem. In a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$, either $\{x\}$ is $\gamma$-$\beta$ closed or $X - \{x\}$ is $\gamma$-$\beta$g closed.

Proof. If $\{x\}$ is not $\gamma$-$\beta$ closed, then $X - \{x\}$ is not $\gamma$-$\beta$-open. Then $X$ is the only $\gamma$-$\beta$-open set such that $X - \{x\} \subset X$. Hence $X - \{x\}$ is a $\gamma$-$\beta$g closed set.

4. $\gamma$-$\beta$-functions

4.1. Definition. Let $(X, \tau)$ and $(Y, \tau')$ be two topological spaces and $\gamma$ an operation on $\tau$. Then a function $f : (X, \tau) \to (Y, \tau')$ is said to be $\gamma$-$\beta$-continuous (resp. $\gamma$-continuous) at $x$ if for each open set $V$ containing $f(x)$, there exists an $U \in \gamma\beta\text{O}(X, x)$ (resp. $\gamma\text{O}(X, x)$) such that $f(U) \subset V$.

If $f$ is $\gamma$-$\beta$-continuous (resp. $\gamma$-continuous) at each point $x$ of $X$, then $f$ is called $\gamma$-$\beta$-continuous (resp. $\gamma$-continuous) on $X$.

4.2. Theorem. Let $(X, \tau)$ be a topological space with an operation $\gamma$ on $\tau$. For a function $f : (X, \tau) \to (Y, \tau')$, the following are equivalent:

(a) $f$ is $\gamma$-$\beta$-continuous (resp. $\gamma$-continuous).
(b) For each open subset $V$ of $Y$, $f^{-1}(V) \in \gamma\beta\text{O}(X)$ (resp. $\gamma\text{O}(X)$).
(c) For each closed subset $V$ of $Y$, $f^{-1}(V) \in \gamma\beta\text{C}(X)$ (resp. $\gamma\text{C}(X)$).
(d) For any subset $V$ of $Y$, $\gamma\beta\text{cl}(f^{-1}(V)) \subset f^{-1}(\text{cl}(V))$ (resp. $\tau\gamma\text{cl}(f^{-1}(V)) \subset f^{-1}(\text{cl}(V))$).
(e) For any subset $U$ of $X$, $f(\gamma\beta\text{cl}(U)) \subset \text{cl}(f(U))$ (resp. $f(\tau\gamma\text{cl}(U)) \subset \text{cl}(f(U))$).
(f) For any subset $U$ of $X$, $f(\gamma\beta\text{D}(U)) \subset \text{cl}(f(U))$ (resp. $f(\tau\gamma\text{D}(U)) \subset \text{cl}(f(U))$).
(g) For any subset \( V \) of \( Y \), \( f^{-1}(\text{int}(V)) \subset \gamma\text{-int}(f^{-1}(V)) \) (resp. \( f^{-1}(\text{int}(V)) \subset \tau_\gamma\text{-int}(f^{-1}(V)) \)).

(h) For each subset \( V \) of \( Y \), \( \gamma\beta Fr(f^{-1}(V)) \subset f^{-1}(Fr(V)) \) (resp. \( \gamma Fr(f^{-1}(V)) \subset f^{-1}(Fr(V)) \)).

**Proof.** We prove the theorem for \( \gamma\beta \)-continuous functions only. The proof for \( \gamma \)-continuity is quite similar.

(a) \( \iff \) (b) \( \iff \) (c) Obvious.

(c) \( \implies \) (d) Let \( V \) be any subset of \( Y \). By (c), we have \( f^{-1}(cl(V)) \) is a \( \gamma\beta \)-closed set containing \( f^{-1}(V) \) and hence \( \gamma\beta cl(f^{-1}(V)) \subset f^{-1}(cl(V)) \).

(d) \( \implies \) (e) Obvious.

(e) \( \implies \) (c) Let \( V \) be a closed set in \( Y \). Then by (e), we obtain \( f(\gamma\beta cl(f^{-1}(V))) \subset cl(f(f^{-1}(V))) \subset cl(V) = \gamma \), which implies \( \gamma\beta cl(f^{-1}(V)) \subset f^{-1}(V) \). Thus \( f^{-1}(V) \in \gamma\beta C(X) \).

(f) \( \implies \) (c) Let \( U \) be any subset of \( X \). Since (c) implies (e), then by the fact that \( \gamma\beta cl(U) = U \cup \gamma\beta D(U) \), we get \( f(\gamma\beta cl(U)) \subset f(\gamma\beta D(U)) \subset cl(f(U)) \).

(g) \( \iff \) (h) Let \( V \) be any open in \( Y \). Then by (g), we get \( f^{-1}(V) = f^{-1}(\text{int}(V)) \subset \gamma\beta int(f^{-1}(V)) \). Hence \( f^{-1}(V) \in \gamma\beta O(X) \).

On the other hand by (b), we get \( f^{-1}(\text{int}(V)) \in \gamma\beta O(Y) \), for any subset \( V \) of \( Y \). Therefore, we obtain \( f^{-1}(\text{int}(V)) = \gamma\beta int(f^{-1}(\text{int}(V))) \subset \gamma\beta int(f^{-1}(V)) \).

(b) \( \implies \) (h) Let \( V \) be any subset of \( Y \). Since (a) implies (d), we have

\[
\begin{align*}
f^{-1}(Fr(V)) &= f^{-1}(cl(V) - \text{int}(V)) \\
 &= f^{-1}(cl(V)) - f^{-1}(\text{int}(V)) \supset \gamma\beta cl(f^{-1}(V)) - f^{-1}(\text{int}(V)) \\
 &= \gamma\beta cl(f^{-1}(V)) - \gamma\beta int(f^{-1}(V)) \\
 &= \gamma\beta cl(f^{-1}(V)) - \gamma\beta int(f^{-1}(V)) \\
 &= \gamma\beta Fr(f^{-1}(V)),
\end{align*}
\]

and hence \( f^{-1}(Fr(V)) \supset \gamma\beta Fr(f^{-1}(V)) \).

(h) \( \implies \) (b) Let \( U \) be open in \( Y \) and \( V = Y - U \). Then by (h), we obtain \( \gamma\beta Fr(f^{-1}(V)) \subset f^{-1}(Fr(V)) \subset f^{-1}(cl(V)) = f^{-1}(V) \) and hence

\[
\gamma\beta cl(f^{-1}(V)) = \gamma\beta int(f^{-1}(V)) \cup \gamma\beta Fr(f^{-1}(V)) \subset f^{-1}(V).
\]

Thus \( f^{-1}(V) \) is \( \gamma\beta \)-closed and hence \( f^{-1}(U) \) is \( \gamma\beta \)-open in \( X \).

\[\square\]

**4.3. Remark.** Every \( \gamma \)-continuous function is \( \gamma\beta \)-continuous, but the converse is not true.

**4.4. Example.** Let \( X \) be a topological space and \( \gamma \) an operation as in Example 3.3(b). Suppose that \( Y = \{1, 2, 3\} \). Let \( \tau' = \{\emptyset, Y, \{1, 2\}, \{1\}, \{2\} \} \). Define a map \( f : (X, \tau) \to (Y, \tau') \) as follows:

\[
f(x) = \begin{cases} 
1 & \text{if } x \in \{a,b\} \\
2 & \text{if } x \in \{a\} \\
3 & \text{if } x \in \{c\}
\end{cases}
\]
Then the mapping \( f : (X, \tau) \to (Y, \tau') \) is \( \gamma; \beta \)-continuous but not \( \gamma \)-continuous, since \( f^{-1}(1) = \{b, d\} \nsubseteq \tau \).

4.5. Remark. Let \( \gamma \) and \( \gamma' \) be operations on the topological spaces \((X, \tau)\) and \((Y, \tau')\) respectively. If the functions \( f : (X, \tau) \to (Y, \tau') \) and \( g : (Y, \tau') \to (Z, \tau'') \) are \( \gamma; \beta \)-continuous and continuous, respectively, then \( g \circ f \) is \( \gamma; \beta \)-continuous. But the composition of a \( \gamma; \beta \)-continuous and a \( \gamma' \beta \)-continuous function may not be \( \gamma; \beta \)-continuous.

4.6. Example. Let us consider the topological spaces \((X, \tau)\) and \((Y, \tau')\) as in Example 4.4. Also let \( Z = \{p, q, r\}, \tau'' = \{\emptyset, Z, \{p\}, \{q, r\}\} \).

We take \( \gamma \) on \( \tau \) as in Example 4.4, and define \( \gamma' \) on \( \tau' \) by

\[
B_{\gamma'}' = \begin{cases} 
1 & \text{if } B = \{1\}, \\
2 \cup B & \text{if } B \neq \{1\}.
\end{cases}
\]

Now we define \( g : X \to Y \) and \( h : Y \to Z \) as follows:

\[
g(x) = \begin{cases} 
1 & \text{if } x \in \{a, b\}, \\
2 & \text{if } x = c, \\
3 & \text{if } x = d
\end{cases}
\]

and

\[
h(y) = \begin{cases} 
p & \text{if } y = 1, \\
q & \text{if } y = 3, \\
r & \text{if } y = 2.
\end{cases}
\]

Then \( g \) and \( h \) are \( \gamma; \beta \)-continuous and \( \gamma' \beta \)-continuous, respectively, but \( h \circ g \) is not \( \gamma; \beta \)-continuous.

4.7. Theorem. Let \((X, \tau)\) be a topological space with an operation \( \gamma \) on \( \tau \) and let \( f : (X, \tau) \to (Y, \tau') \) be a function. Then

\[
X - \gamma; \beta C(f) = \bigcup\{\gamma; \beta Fr(f^{-1}(V)) : V \in \tau', \ f(x) \in V, \ x \in X\},
\]

where \( \gamma; \beta C(f) \) denotes the set of points at which \( f \) is \( \gamma; \beta \)-continuous.

Proof. Let \( x \in X - \gamma; \beta C(f) \). Then there exists \( V \in \tau' \) containing \( f(x) \) such that \( f(U) \nsubseteq V \), for every \( \gamma; \beta \)-open set \( U \) containing \( x \). Hence \( U \cap \{x - f^{-1}(V)\} \neq \emptyset \) for every \( \gamma; \beta \)-open set \( U \) containing \( x \). Therefore, by Theorem 3.13, \( x \in \gamma; \beta cl(X - f^{-1}(V)) \). Then \( x \in f^{-1}(V) \cap \gamma; \beta cl(X - f^{-1}(V)) \subseteq \gamma; \beta Fr(f^{-1}(V)) \). So,

\[
X - \gamma; \beta C(f) \subseteq \bigcup\{\gamma; \beta Fr(f^{-1}(V)) : V \in \tau', \ f(x) \in V, \ x \in X\}.
\]

Conversely, let \( x \notin X - \gamma; \beta C(f) \). Then for each \( V \in \tau' \) containing \( f(x) \), \( f^{-1}(V) \) is a \( \gamma; \beta \)-open set containing \( x \). Thus \( x \in \gamma; \beta int(f^{-1}(V)) \) and hence \( x \notin \gamma; \beta Fr(f^{-1}(V)) \), for every \( V \in \tau' \) containing \( f(x) \). Therefore,

\[
X - \gamma; \beta C(f) \supset \bigcup\{\gamma; \beta Fr(f^{-1}(V)) : V \in \tau', \ f(x) \in V, \ x \in X\}.
\]

\[\square\]

4.8. Theorem. Let \((X, \tau)\) be a topological space with an operation \( \gamma \) on \( \tau \) and let \( f : (X, \tau) \to (Y, \tau') \) be a function. Then \( X - \gamma C(f) = \bigcup\{\gamma Fr(f^{-1}(V)) : V \in \tau', \ f(x) \in V, \ x \in X\} \), where \( \gamma C(f) \) denotes the set of points at which \( f \) is \( \gamma \)-continuous.

4.9. Definition. Let \((X, \tau)\) be a topological space with an operation \( \gamma \) on \( \tau \). A function \( f : (X, \tau) \to (Y, \tau') \) is called \( \tau; \gamma \beta \)-continuous (resp. \( \gamma; \gamma \beta \)-continuous) if for each open set \( V \) in \( Y \), \( f^{-1}(V) \) is \( \tau; \gamma \beta \)-open (resp. \( \gamma; \gamma \beta \)-open) in \( X \).
4.10. Theorem. Let \( f : (X, \tau) \to (Y, \tau') \) be a mapping and \( \gamma \) an operation on \( \tau \). Then the following are equivalent:

(i) \( f \) is \( \gamma \)-continuous.
(ii) \( f \) is \( \gamma \)-\( \beta \) continuous and \( \gamma \)-\( \gamma \)-\( \beta \) continuous.

Proof. (i) \( \implies \) (ii) Let \( f \) be \( \gamma \)-continuous. Then \( f \) is \( \gamma \)-\( \beta \) continuous. Now, let \( G \) be any open set in \( Y \), then \( f^{-1}(G) \) is \( \gamma \)-open in \( X \). Then
\[
\tau_{\gamma} \text{-int } (f^{-1}(G)) = f^{-1}(G) = \gamma \text{-int } f^{-1}(G).
\]
Thus, \( f^{-1}(G) \) is \( \gamma \)-\( \gamma \)-\( \beta \) open in \( X \). Therefore \( f \) is \( \gamma \)-\( \gamma \)-\( \beta \) continuous.

(ii) \( \implies \) (i) Let \( f \) be \( \gamma \)-\( \beta \) continuous and \( \gamma \)-\( \gamma \)-\( \beta \) continuous. Then for any open set \( G \) in \( Y \), \( f^{-1}(G) \) is both \( \gamma \)-open and \( \gamma \)-\( \gamma \)-\( \beta \) open in \( X \). So
\[
f^{-1}(G) = \gamma \text{-int } f^{-1}(G) = \tau_{\gamma} \text{-int } (f^{-1}(G)).
\]
Thus \( f^{-1}(G) \in \tau_{\gamma} \) and hence \( f \) is \( \gamma \)-continuous. \( \Box \)

4.11. Theorem. Let \( f : (X, \tau) \to (Y, \tau') \) be \( \tau \)-\( \gamma \)-\( \beta \) continuous, where \( \gamma \) is an operation on \( \tau \). Then \( f \) is continuous if and only if \( f \) is \( \gamma \)-\( \beta \) continuous.

Proof. Let \( V \in \tau' \). Since \( f \) is continuous as well as \( \tau \)-\( \gamma \)-\( \beta \) continuous, \( f^{-1}(V) \) is open as well as \( \tau \)-\( \gamma \)-\( \beta \) open in \( X \) and hence \( f^{-1}(V) = \text{int } (f^{-1}(V)) = \gamma \text{-int } (f^{-1}(V)) \in \gamma \text{-BO}(X) \). Therefore, \( f \) is \( \gamma \)-\( \beta \) continuous.

Conversely, let \( V \in \tau' \). Then \( f^{-1}(V) \) is \( \gamma \)-\( \beta \) open and \( \tau \)-\( \gamma \)-\( \beta \) open. So \( f^{-1}(V) = \gamma \text{-int } f^{-1}(V) = f^{-1}(V) \). Hence \( f^{-1}(V) \) is open in \( X \). Therefore \( f \) is continuous. \( \Box \)

4.12. Definition. A function \( f : (X, \tau) \to (Y, \tau') \), where \( \gamma \) and \( \gamma' \) are operations on \( \tau \) and \( \tau' \), respectively, is called \( \gamma \)-closed (resp. \( \gamma \)-\( \beta \)-closed, \( \gamma \)-\( \beta \)-\( g \)-closed, \( \gamma \)-\( \beta \)-\( g \)-\( \gamma \)-\( \beta \)-closed) if for every closed (resp. closed, closed, \( \gamma \)-\( \beta \)-closed) set \( F \) in \( X \), \( f(F) \) is \( \gamma \)-closed (resp. \( \gamma \)-\( \beta \)-closed, \( \gamma \)-\( \beta \)-\( g \)-closed, \( \gamma \)-\( \beta \)-\( g \)-\( \gamma \)-\( \beta \)-closed) in \( Y \).

4.13. Theorem. A surjective function \( f : (X, \tau) \to (Y, \tau') \), where \( \gamma \) is an operation on \( \tau \), is \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed) if and only if for each subset \( A \subseteq Y \) and each open set \( V \subseteq X \) containing \( f^{-1}(A) \), there exists a \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed) set \( W \subseteq Y \) such that \( A \subseteq W \) and \( f^{-1}(W) \subseteq V \).

Proof. First we suppose that \( f : (X, \tau) \to (Y, \tau') \) is \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed), \( A \subseteq Y \) and \( V \subseteq X \) is open in \( X \) such that \( f^{-1}(A) \subseteq V \). Now we put \( X - V = G \), then \( f(G) \) is a \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed) set in \( Y \). If \( W = Y - f(G) \) then \( W \) is a \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed) set in \( Y \), \( A \subseteq W \) and \( f^{-1}(W) \subseteq V \).

Conversely, let \( B \) be any closed set in \( X \). Then \( A = Y - f(B) \) is a subset of \( Y \) and \( f^{-1}(A) \subseteq X - B \), where \( (X - B) \) is open in \( X \). Therefore, by hypothesis there exists a \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed) set \( W \subseteq Y \) such that \( A = Y - f(B) \subseteq W \) and \( f^{-1}(W) \subseteq X - B \). Now, \( f^{-1}(W) \subseteq X - B \) gives \( W \subseteq f(X - B) \subseteq Y - f(B) \). Therefore, \( Y - f(B) = W \) and hence \( f(B) \) is a \( \gamma \)-\( \beta \)-\( g \)-closed (resp. \( \gamma \)-\( \beta \)-closed) set. \( \Box \)

4.14. Theorem. A surjective function \( f : (X, \tau) \to (Y, \tau') \), where \( \gamma \) and \( \gamma' \) are operations on \( \tau \) and \( \tau' \), respectively, is \( \gamma \)-\( \beta \)-\( g \)-\( \gamma \)-\( \beta \)-closed if and only if for each subset \( B \subseteq Y \) and each \( \gamma \)-\( \beta \)-open set \( U \subseteq X \) containing \( f^{-1}(B) \), there exists a \( \gamma \)-\( \beta \)-\( g \)-open set \( V \subseteq Y \) such that \( B \subseteq V \) and \( f^{-1}(V) \subseteq U \).

4.15. Definition. A function \( f : (X, \tau) \to (Y, \tau') \), where \( \gamma \) is an operation on \( \tau \), is called \( \gamma \)-\( \beta \)-anti-closed (resp. \( \gamma \)-\( \beta \)-anti-open) if the image of each \( \gamma \)-\( \beta \)-closed (resp. \( \gamma \)-\( \beta \)-open) set in \( X \) is closed (resp. open) in \( Y \).
4.16. **Theorem.** A surjective function \( f : (X, \tau) \rightarrow (Y, \tau') \), where \( \gamma \) is an operation on \( \tau \), is \( \gamma' - \beta \)-anti-closed if for each subset \( A \) of \( Y \) and each \( \gamma' - \beta \)-open set \( V \) containing \( f^{-1}(A) \), there exists an open set \( W \) such that \( A \subset W \) and \( f^{-1}(W) \subset V \).

4.17. **Theorem.** Let \( f : (X, \tau) \rightarrow (Y, \tau') \) be a function and \( \gamma' \) an operation on \( \tau' \). Then the following conditions are equivalent:

(i) \( f \) is \( \gamma' - \beta \)-closed.

(ii) \( \gamma' - \beta \text{cl}(f(U)) \subset f(\text{cl}(U)), \) for each subset \( U \) of \( X \).

(iii) \( \gamma' - \beta D(f(U)) \subset f(\text{cl}(U)), \) for each subset \( U \) of \( X \).

**Proof.** (i) \( \implies \) (ii) Here, for any subset \( U \) of \( X \), \( f(\text{cl}(U)) \) is a \( \gamma' - \beta \)-closed set in \( Y \) and \( f(U) \subset f(\text{cl}(U)) \), hence \( \gamma' - \beta \text{cl}(f(U)) \subset f(\text{cl}(U)) \).

(ii) \( \implies \) (iii) For each \( U \subset X \), we have \( \gamma' - \beta D(f(U)) \subset \gamma' - \beta \text{cl}(f(U)) \subset f(\text{cl}(U)) \).

(iii) \( \implies \) (i) Let \( V \) be any closed set in \( X \). Then \( \gamma' - \beta D(f(V)) \subset f(\text{cl}(V)) = f(V) \). Hence \( f(V) \) is \( \gamma' - \beta \)-closed.

4.18. **Remark.** If \( f : (X, \tau) \rightarrow (Y, \tau') \) is a bijection and \( \gamma' \) an operation on \( \tau' \), then \( f \) is \( \gamma' - \beta \)-closed if and only if \( f^{-1} \) is \( \gamma' - \beta \)-continuous.

4.19. **Definition.** A function \( f : (X, \tau) \rightarrow (Y, \tau') \), where \( \gamma' \) is an operation on \( \tau' \), is said to be \( \gamma' - \beta \)-open if for each open set \( U \) in \( X \), \( f(U) \) is \( \gamma' - \beta \)-open in \( Y \).

4.20. **Theorem.** Let \( f : (X, \tau) \rightarrow (Y, \tau') \) be a mapping and \( \gamma' \) an operation on \( \tau' \). Then the following conditions are equivalent:

(i) \( f \) is \( \gamma' - \beta \)-open.

(ii) \( f(\text{int}(V)) \subset \gamma' - \beta \text{int}(f(V)) \).

4.21. **Remark.** If \( f \) is a bijection, then \( f \) is \( \gamma' - \beta \)-open if and only if \( f^{-1} \) is \( \gamma' - \beta \)-continuous.

4.22. **Remark.** The concepts of \( \gamma' - \beta \)-closedness and \( \gamma' - \beta \)-openness are independent.

4.23. **Example.** Let \( (X, \tau) \) be the topological space with operation \( \gamma \) on \( \tau \) as in Example 3.3(b), and let \( (Y, \tau') \) be a topological space where \( Y = \{1, 2, 3\}, \tau' = \{\emptyset, Y, \{1, 2\}, \{1\}, \{2\}\} \).

Let \( f : (Y, \tau') \rightarrow (X, \tau) \) be defined as follows:

\[
f(y) = \begin{cases} 
d & \text{if } y = 1, \\
c & \text{if } y = 2, \\
a & \text{if } y = 3.
\end{cases}
\]

Then \( f \) is \( \gamma - \beta \)-closed but not \( \gamma - \beta \)-open.

Let \( (Y, \tau') \) be the topological space with \( \gamma' \) an operation on \( \tau' \) as in Example 4.6. We define a function \( g : (X, \tau) \rightarrow (Y, \tau') \) as follows:

\[
g(x) = \begin{cases} 
1 & \text{if } x \in \{b, c\} \\
2 & \text{if } x \in \{a, d\}
\end{cases}
\]

Then \( g \) is \( \gamma' - \beta \)-open but not \( \gamma' - \beta \)-closed.

4.24. **Definition.** Let \( (X, \tau), (Y, \tau') \) be two topological spaces and \( \gamma, \gamma' \) operations on \( \tau, \tau' \), respectively. A mapping \( f : (X, \tau) \rightarrow (Y, \tau') \) is called \( (\gamma, \gamma') - \beta \) irresolute at \( x \) if and only if for each \( \gamma' - \beta \)-open set \( V \) in \( Y \) containing \( f(x) \), there exists a \( \gamma - \beta \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subset V \).

If \( f \) is \( (\gamma, \gamma') - \beta \) irresolute at each \( x \in X \) then \( f \) is called \( (\gamma, \gamma') - \beta \) irresolute on \( X \).
4.25. Theorem. Let \((X, \tau), (Y, \tau')\) be topological spaces and \(\gamma, \gamma'\) operations on \(\tau, \tau'\), respectively. If \(f : (X, \tau) \rightarrow (Y, \tau')\) is \((\gamma, \gamma')-\beta\) irresolute and \(\gamma'-\beta g\gamma-\beta\)-closed, and \(A\) is \(\gamma-\beta g\)-closed in \(X\), then \(f(A)\) is \(\gamma'-\beta g\)-closed in \(Y\).

Proof. Suppose \(A\) is a \(\gamma-\beta g\)-closed set in \(X\) and that \(U\) is a \(\gamma'-\beta\)-open set in \(Y\) such that \(f(A) \subseteq U\). Then \(A \subseteq f^{-1}(U)\). Since \(f\) is \((\gamma, \gamma')-\beta\) irresolute, \(f^{-1}(U)\) is \(\gamma-\beta\)-open set in \(X\).

Again \(A\) is a \(\gamma-\beta g\)-closed set, therefore \(\gamma-\beta cl(A) \subseteq f^{-1}(U)\) and hence \(f(\gamma-\beta cl(A)) \subseteq U\). Since \(f\) is a \(\gamma'-\beta g\gamma'-\beta\)-closed map, \(f(\gamma-\beta cl(A))\) is a \(\gamma'-\beta g\)-closed set in \(Y\). Therefore \(\gamma'-\beta cl(f(\gamma-\beta cl(A))) \subseteq U\), which implies \(\gamma'-\beta cl(f(A)) \subseteq U\).

\(\square\)

4.26. Theorem. Let \(f : (X, \tau) \rightarrow (Y, \tau')\) be a mapping and \(\gamma, \gamma'\) operations on \(\tau, \tau'\), respectively. Then the following are equivalent:

(i) \(f\) is \((\gamma, \gamma')-\beta\) irresolute.
(ii) The inverse image of each \(\gamma'-\beta\)-open set in \(Y\) is a \(\gamma-\beta\)-open set in \(X\).
(iii) The inverse image of each \(\gamma'-\beta\)-closed set in \(Y\) is a \(\gamma-\beta\)-closed set in \(X\).
(iv) \(\gamma-\beta cl(f^{-1}(V)) \subseteq f^{-1}(\gamma'-\beta cl(V))\), for all \(V \subseteq Y\).
(v) \(f(\gamma-\beta cl(U)) \subseteq \gamma'-\beta cl(f(U))\), for all \(U \subseteq X\).
(vi) \(\gamma-\beta Fr(f^{-1}(V)) \subseteq f^{-1}(\gamma'-\beta Fr(V))\), for all \(V \subseteq Y\).
(vii) \(f(\gamma-\beta D(U)) \subseteq \gamma'-\beta cl(f(U))\), for all \(U \subseteq X\).
(viii) \(f^{-1}(\gamma'-\beta int(V)) \subseteq \gamma-\beta int(f^{-1}(V))\), for all \(V \subseteq Y\).

4.27. Theorem. Let \((X, \tau), (Y, \tau')\) be topological spaces and \(\gamma, \gamma'\) operations on \(\tau, \tau'\), respectively. Also let \(f : (X, \tau) \rightarrow (Y, \tau')\) be a mapping. Then the set of points at which \(f\) is not \((\gamma, \gamma')-\beta\) irresolute is

\[
\bigcup \{\gamma'-\beta Fr(V) : V : \text{is a } \gamma'-\beta\text{ open set in } Y\text{ containing } f(x)\}.
\]

5. \(\gamma-\beta\)-separation axioms

5.1. Definition. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is called \(\gamma-\beta T_0\) if and only if for each pair of distinct points \(x, y\) in \(X\), there exists an \(\gamma-\beta\)-open set \(U\) such that either \(x \in U\) and \(y \notin U\) or \(x \notin U\) and \(y \in U\).

5.2. Definition. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is called \(\gamma-\beta T_1\) if and only if for each pair of distinct points \(x, y\) in \(X\), there exists two \(\gamma-\beta\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \notin U\) and \(y \in V\) and \(x \notin V\).

5.3. Definition. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is called \(\gamma-\beta T_2\) if and only if for each pair of distinct points \(x, y\) in \(X\), there exist two disjoint \(\gamma-\beta\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively.

5.4. Definition. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is called \(\gamma-\beta T^*_2\) if every \(\gamma-\beta g\)-closed set is \(\gamma-\beta\)-closed.

5.5. Theorem. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma-\beta T_0\) if and only if for every pair of distinct points \(x, y\) of \(X\), \(\gamma-\beta cl\{x\} \neq \gamma-\beta cl\{y\}\).

5.6. Theorem. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is \(\gamma-\beta T_1\) if and only if every singleton \(\{x\}\) is \(\gamma-\beta\)-closed.

5.7. Theorem. The following are equivalent for a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\):

(i) \(X\) is \(\gamma-\beta T_2\).
(ii) Let \(x \in X\). For each \(y \neq x\), there exists a \(\gamma-\beta\)-open set \(U\) containing \(x\) such that \(y \notin \gamma-\beta cl(U)\).
(iii) For each $x \in X$, $\bigcap \{\gamma^{-}\text{cl}(U) : U \in \gamma^{-}\text{O}(X, x)\} = \{x\}$.

Proof. (i) $\implies$ (ii) Since $X$ is $\gamma^{-}\beta T_2$, there exist disjoint $\gamma^{-}\beta$-open sets $U$ and $W$ containing $x$ and $y$ respectively. So $U \subset X - W$. Therefore, $\gamma^{-}\beta \text{cl}(U) \subset X - W$. So $y \notin \gamma^{-}\beta \text{cl}(U)$.

(ii) $\implies$ (iii) If possible for $y \neq x$, let $y \in \gamma^{-}\beta \text{cl}(U)$ for every $\gamma^{-}\beta$-open set $U$ containing $x$, which then contradicts (ii).

(iii) $\implies$ (i) Let $x, y \in X$ and $x \neq y$. Then there exists a $\gamma^{-}\beta$-open set $U$ containing $x$ such that $y \notin \gamma^{-}\beta \text{cl}(U)$. Let $V = X - \gamma^{-}\beta \text{cl}(U)$, then $y \in V$ and $x \in U$ and also $U \cap V = \emptyset$. \hfill $\square$

5.8. Theorem. The following are equivalent for a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$,

(i) $(X, \tau)$ is $\gamma^{-}\beta T_\frac{1}{2}$.

(ii) Each singleton $\{x\}$ of $X$ is either $\gamma^{-}\beta$-closed or $\gamma^{-}\beta$-open.

Proof. (i) $\implies$ (ii) Suppose $\{x\}$ is not $\gamma^{-}\beta$-closed. Then by Theorem 3.19, $X - \{x\}$ is $\gamma^{-}\beta g$ closed. Now since $(X, \tau)$ is $\gamma^{-}\beta T_\frac{1}{2}$, $X - \{x\}$ is $\gamma^{-}\beta$-closed i.e. $\{x\}$ is $\gamma^{-}\beta$-open.

(ii) $\implies$ (i) Let $A$ be any $\gamma^{-}\beta g$ closed set in $(X, \tau)$ and $x \in \gamma^{-}\beta \text{cl}(A)$. By (ii) we have $\{x\}$ is $\gamma^{-}\beta$-closed or $\gamma^{-}\beta$-open. If $\{x\}$ is $\gamma^{-}\beta$-closed then $x \notin A$ will imply $x \in \gamma^{-}\beta \text{cl}(A) - A$, which is not possible by Theorem 3.18. Hence $x \in A$. Therefore, $\gamma^{-}\beta \text{cl}(A) = A$ i.e. $A$ is $\gamma^{-}\beta$-closed. So, $(X, \tau)$ is $\gamma^{-}\beta T_\frac{1}{2}$. On the other hand if $\{x\}$ is $\gamma^{-}\beta$-open then as $x \in \gamma^{-}\beta \text{cl}(A)$, $\{x\} \cap A \neq \emptyset$. Hence $x \in A$. So $A$ is $\gamma^{-}\beta$-closed. \hfill $\square$

5.9. Theorem. Every topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is $\gamma^{-}\beta T_\frac{1}{2}$.

Proof. Let $x \in X$. To prove $(X, \tau)$ is $\gamma^{-}\beta T_\frac{1}{2}$, it is sufficient to show that $\{x\}$ is $\gamma^{-}\beta$-closed or $\gamma^{-}\beta$-open (by Theorem 5.8). Now if $\{x\}$ is $\gamma^{-}\beta$-open then it is obviously $\gamma^{-}\beta$-open. If $\{x\}$ is not $\gamma^{-}\beta$-open then $\gamma^{-}\text{int}(\{x\}) = \emptyset$ and hence $\tau^{-}\text{int}(\gamma^{-}\text{cl}(\gamma^{-}\text{int}(\{x\}))) = \emptyset \subset \{x\}$. Therefore, $\{x\}$ is $\gamma^{-}\beta$-closed. \hfill $\square$

5.10. Remark.

(a) Because of Theorem 5.9 above, the concepts $\gamma^{-}\beta$-closedness and $\gamma^{-}\beta g$-closedness are identical, and so also are $\gamma^{-}\beta T_\frac{1}{2}$ and $\gamma^{-}\beta T_0$.

(b) $\gamma^{-}\beta T_2 \implies \gamma^{-}\beta T_1 \implies \gamma^{-}\beta T_\frac{1}{2}$. But the reverse implications are not true in general.

5.11. Example. Let $X = \{a, b, c\}$ and let $\tau$ be the discrete topology on $X$. Then $\beta O(X) = \tau$. Define an operation $\gamma$ on $\tau$ by

$$\gamma^\tau = \begin{cases} \{a\} & \text{if } A = \{a\}, \\ A \cup \{c\} & \text{if } A \neq \{a\}. \end{cases}$$

Then $X$ is $\gamma^{-}\beta T_\frac{1}{2}$ but not $\gamma^{-}\beta T_1$.

5.12. Example. Let $X = \{a, b, c\}$ and let $\tau$ be the discrete topology on $X$. Define an operation $\gamma$ on $\tau$ by

$$\gamma^\tau = \begin{cases} A \cup \{b\} & \text{if } A = \{a\}, \\ A \cup \{c\} & \text{if } A = \{b\}, \\ A \cup \{a\} & \text{if } A = \{c\}, \\ A & \text{if } A \neq \{a\}, \{b\}, \{c\}. \end{cases}$$

Then $X$ is $\gamma^{-}\beta T_1$ but not $\gamma^{-}\beta T_2$. 

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5.13. Definition. A topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\) is called \(\gamma\)-\(\beta\)-regular if for each \(\gamma\)-\(\beta\)-closed set \(F\) of \(X\) not containing \(x\), there exist disjoint \(\gamma\)-\(\beta\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(F \subset V\).


5.14. Theorem. The following are equivalent for a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\):

(a) \(X\) is \(\gamma\)-\(\beta\)-regular.

(b) For each \(x \in X\) and each \(U \in \gamma\)-\(\beta\)O\((x, x)\), there exists a \(V \in \gamma\)-\(\beta\)O\((X, x)\) such that \(x \in V \subset \gamma\)-\(\beta\)cl\((V) \subset U\).

(c) For each \(\gamma\)-\(\beta\)-closed set \(F\) of \(X\), \(\bigcap\{\gamma\)-\(\beta\)cl\((V) : F \subset V, V \in \gamma\)-\(\beta\)O\((X)\} = F\).

(d) For each \(A\) subset of \(X\) and each \(U \in \gamma\)-\(\beta\)O\((X)\) with \(A \cap U \neq \emptyset\), there exists a \(V \in \gamma\)-\(\beta\)O\((X)\) such that \(A \cap V \neq \emptyset\) and \(\gamma\)-\(\beta\)cl\((V) \subset U\).

(e) For each nonempty subset \(A\) of \(X\) and each \(\gamma\)-\(\beta\)-closed subset \(F\) of \(X\) with \(A \cap F = \emptyset\), there exists \(V, W \in \gamma\)-\(\beta\)O\((X)\) such that \(A \cap V \neq \emptyset\), \(F \subset W\) and \(W \cap V = \emptyset\).

(f) For each \(\gamma\)-\(\beta\)-closed set \(F\) and \(x \notin F\), there exists \(U \in \gamma\)-\(\beta\)O\((X)\) and a \(\gamma\)-\(\beta\)g-open set \(V\) such that \(x \in U\), \(F \subset V\) and \(U \cap V = \emptyset\).

(g) For each \(A \subset X\) and each \(\gamma\)-\(\beta\)-closed set \(F\) with \(A \cap F = \emptyset\), there exists \(U \in \gamma\)-\(\beta\)O\((X)\) and a \(\gamma\)-\(\beta\)g-open set \(V\) such that \(A \cap U \neq \emptyset\), \(F \subset V\) and \(U \cap V = \emptyset\).

(h) For each \(\gamma\)-\(\beta\)-closed set \(F\) of \(X\), \(F \supseteq \bigcap\{\gamma\)-\(\beta\)cl\((V) : F \subset V, V \in \gamma\)-\(\beta\)O\((X)\) is \(\gamma\)-\(\beta\)g-open\}.

Proof. (a) \(\implies\) (b) Let \(x \notin X - U\), where \(U \in \gamma\)-\(\beta\)O\((X, x)\). Then there exists \(G, V \in \gamma\)-\(\beta\)O\((X)\) such that \((X - U) \subset G\), \(x \in V\) and \(G \cap V = \emptyset\). Therefore \(V \subset (X - G)\) and so \(x \in V \subset \gamma\)-\(\beta\)cl\((V) \subset (X - G) \subset U\).

(b) \(\implies\) (c) Let \(X - F \in \gamma\)-\(\beta\)O\((X, x)\). Then by (b) there exists an \(U \in \gamma\)-\(\beta\)O\((X, x)\) such that \(x \in U \subset \gamma\)-\(\beta\)cl\((U) \subset (X - F)\). So, \(F \subset X - \gamma\)-\(\beta\)cl\((U) = V\), \(V \in \gamma\)-\(\beta\)O\((X)\) and \(V \cap U = \emptyset\). Then by Theorem 3.13, \(x \notin \gamma\)-\(\beta\)cl\((V)\). Thus

\[ F \supseteq \bigcap\{\gamma\)-\(\beta\)cl\((V) : F \subset V, V \in \gamma\)-\(\beta\)O\((X)\}\].

(c) \(\implies\) (d) Let \(U \in \gamma\)-\(\beta\)O\((X)\) with \(x \in U \cap A\). Then \(x \notin (X - U)\) and hence by (c) there exists a \(\gamma\)-\(\beta\)-open set \(W\) such that \((X - U) \subset W\) and \(x \notin \gamma\)-\(\beta\)cl\((W)\). We put \(V = X - \gamma\)-\(\beta\)cl\((W)\), which is a \(\gamma\)-\(\beta\)-open set containing \(x\) and hence \(V \cap A \neq \emptyset\). Now \(V \subset (X - W)\) and so \(\gamma\)-\(\beta\)cl\((V) \subset (X - W) \subset U\).

(d) \(\implies\) (e) Let \(F\) be a set as in the hypothesis of (e). Then \((X - F)\) is \(\gamma\)-\(\beta\)-open and \((X - F) \cap A \neq \emptyset\). Then there exists \(V \in \gamma\)-\(\beta\)O\((X)\) such that \(A \cap V \neq \emptyset\) and \(\gamma\)-\(\beta\)cl\((V) \subset (X - F)\). If we put \(W = X - \gamma\)-\(\beta\)cl\((V)\), then \(F \subset W\) and \(W \cap V = \emptyset\).

(e) \(\implies\) (a) Let \(F\) be a \(\gamma\)-\(\beta\)-closed set not containing \(x\). Then by (e), there exist \(W, V \in \gamma\)-\(\beta\)O\((X)\) such that \(F \subset W\) and \(x \in V\) and \(W \cap V = \emptyset\).

(a) \(\implies\) (f) Obvious.

(f) \(\implies\) (g) For \(a \in A\), \(a \notin F\) and hence by (f) there exists \(U \in \gamma\)-\(\beta\)O\((X)\) and a \(\gamma\)-\(\beta\)g-open set \(V\) such that \(a \in U\), \(F \subset V\) and \(U \cap V = \emptyset\). So, \(A \cap U \neq \emptyset\).

(g) \(\implies\) (a) Let \(x \notin F\), where \(F\) is \(\gamma\)-\(\beta\)-closed. Since \(\{x\} \cap F = \emptyset\), by (g) there exists \(U \in \gamma\)-\(\beta\)O\((X)\) and a \(\gamma\)-\(\beta\)g-open set \(W\) such that \(x \in U\), \(F \subset W\) and \(U \cap W = \emptyset\). Now put \(V = \gamma\)-\(\beta\)int\((W)\). Using Definition 3.16 of \(\gamma\)-\(\beta\)g-open sets we get \(F \subset V\) and \(V \cap U = \emptyset\).
(c) \implies (h) We have

\[
F \subset \bigcap \{ \gamma - \beta \text{cl} (V) : F \subset V \text{ and } V \text{ is } \gamma - \beta \text{g-open} \}
\]
\[
\subset \bigcap \{ \gamma - \beta \text{cl} (V) : F \subset V \text{ and } V \text{ is } \gamma - \beta \text{-open} \}
\]
\[
= F.
\]

(h) \implies (a) Let \( F \) be a \( \gamma - \beta \)-closed set in \( X \) not containing \( x \). Then by (h) there exists a \( \gamma - \beta \text{g-open} \) set \( W \) such that \( F \subset W \) and \( x \in X - \gamma - \beta \text{cl} (W) \). Since \( F \) is \( \gamma - \beta \)-closed and \( W \) is \( \gamma - \beta \text{g-open} \), \( F \subset \gamma - \beta \text{int} (W) \). Take \( V = \gamma - \beta \text{int} (W) \). Then \( F \subset V \), \( x \in U = X - \gamma - \beta \text{cl} (V) \) and \( U \cap V = \emptyset \). \( \square \)

5.15. Definition. A mapping \( f : (X, \tau) \rightarrow (Y, \tau') \), where \( \gamma' \) is an operation on \( \tau' \), is called \( \gamma' - \beta \)-anti-continuous if the inverse image of each \( \gamma' - \beta \)-open set in \( Y \) is open in \( X \).

We now give examples which shows that regularity and \( \gamma - \beta \)-regularity are independent concepts:

5.16. Example. The topological space \((X, \tau)\) with the operation \( \gamma \) on \( \tau \) as defined in Example 5.11 is regular but not \( \gamma - \beta \)-regular.

5.17. Example. The topological space \((X, \tau)\) defined as in Example 3.3(b) is not regular but is \( \gamma - \beta \)-regular.

Although regularity and \( \gamma - \beta \)-regularity are independent concepts, we have been able to obtain \( \gamma - \beta \)-regularity from regularity and vice versa. The following theorems show these facts.

5.18. Theorem. Let \( f : (X, \tau) \rightarrow (Y, \tau') \) be a \( \gamma - \beta \)-continuous, \( \gamma - \beta \)-anti-closed and \( \gamma - \beta \)-anti-open surjective function on \((X, \tau)\) with an operation \( \gamma \) on \( \tau \). If \( X \) is \( \gamma - \beta \)-regular then \( Y \) is regular.

Proof. Let \( K \) be closed in \( Y \) and \( y \notin K \). Since \( f \) is \( \gamma - \beta \)-continuous and \( X \) is \( \gamma - \beta \)-regular, then for each point \( x \in f^{-1}(y) \), there exist disjoint \( V, W \in \gamma - \beta O(X) \) such that, \( x \in V \) and \( f^{-1}(K) \subset W \). Now since \( f \) is \( \gamma - \beta \)-anti-closed, there exists an open set \( U \) containing \( K \) such that \( f^{-1}(U) \subset W \). As \( f \) is a \( \gamma - \beta \)-anti-open map, we have \( y = f(x) \in f(V) \) and \( f(V) \) is open in \( Y \). Now, \( f^{-1}(U) \cap V = \emptyset \) and hence \( U \cap f(V) = \emptyset \). Therefore \( Y \) is regular. \( \square \)

5.19. Theorem. Let \( f : (X, \tau) \rightarrow (Y, \tau') \) be a \( \gamma' - \beta \)-anti-continuous, \( \gamma' - \beta \text{g-closed} \) and \( \gamma' - \beta \)-anti-open surjection, where \( \gamma' \) is an operation on \( \tau' \). If \( X \) is regular, then \( Y \) is \( \gamma' - \beta \)-regular.

Proof. Let \( y \in Y \) and \( F \) be any \( \gamma' - \beta \)-open set in \( Y \) containing \( y \). Since \( f \) is \( \gamma' - \beta \)-anti-continuous, \( f^{-1}(F) \) is open in \( X \) and contains \( x \), where \( y = f(x) \). Again since \( X \) is regular, there exists an open set \( V \) in \( X \) containing \( x \) such that \( x \in V \subset cl(V) \subset f^{-1}(F) \), which is equivalent to \( y \in f(V) \subset f(cl(V)) \subset F \). Since \( f \) is \( \gamma' - \beta \)-open and \( \gamma' - \beta \text{g-closed} \), \( f(V) \subset \gamma' - \beta \text{cl}(Y) \) and \( f(cl(V)) \subset \gamma' - \beta \text{cl}(F) \) and so,

\[
y \in f(V) \subset \gamma' - \beta \text{cl}(f(V)) \subset \gamma' - \beta \text{cl}(f(cl(V))) \subset F.
\]

Hence \( Y \) is \( \gamma' - \beta \)-regular by Theorem 5.14. \( \square \)

5.20. Definition. A topological space \((X, \tau)\) with an operation \( \gamma \) on \( \tau \), is said to be \( \gamma - \beta \)-normal if for any pair of disjoint \( \gamma - \beta \)-closed sets \( A, B \) of \( X \), there exist disjoint \( \gamma - \beta \)-open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

Tahiliani [11] characterized \( \beta \)-normal spaces. In a similar fashion we give several characterizations of \( \gamma - \beta \)-normal spaces.
5.21. Theorem. For a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\tau\), the following are equivalent:

(a) \(X\) is \(\gamma\)-\(\beta\)-normal.
(b) For each pair of disjoint \(\gamma\)-\(\beta\)-closed sets \(A, B\) of \(X\), there exist disjoint \(\gamma\)-\(\beta\)g-open sets \(U\) and \(V\) such that \(A \subset U\) and \(B \subset V\).
(c) For each \(\gamma\)-\(\beta\)-closed \(A\) and any \(\gamma\)-\(\beta\)-open set \(V\) containing \(A\), there exists a 
\(\gamma\)-\(\beta\)g-open set \(U\) such that \(A \subset U \subset \gamma\)-\(\beta\)cl \((U) \subset V\).
(d) For each \(\gamma\)-\(\beta\)-closed \(A\) and any \(\gamma\)-\(\beta\)g-open set \(B\) containing \(A\), there exists a
\(\gamma\)-\(\beta\)g-open set \(U\) such that \(A \subset U \subset \gamma\)-\(\beta\)cl \((U) \subset \gamma\)-\(\beta\)int \((B)\).  
(e) For each \(\gamma\)-\(\beta\)-closed \(A\) and any \(\gamma\)-\(\beta\)g-open set \(B\) containing \(A\), there exists a
\(\gamma\)-\(\beta\)-open set \(G\) such that \(A \subset G \subset \gamma\)-\(\beta\)cl \((G) \subset \gamma\)-\(\beta\)int \((B)\).
(f) For each \(\gamma\)-\(\beta\)g-closed \(A\) and any \(\gamma\)-\(\beta\)-open set \(B\) containing \(A\), there exists a
\(\gamma\)-\(\beta\)-open set \(U\) such that \(\gamma\)-\(\beta\)cl \((A) \subset U \subset \gamma\)-\(\beta\)cl \((G) \subset B\).

Proof. (a) \(\Rightarrow\) (b) Follows from the fact that every \(\gamma\)-\(\beta\)-open set is \(\gamma\)-\(\beta\)g-open.

(b) \(\Rightarrow\) (c) Let \(A\) be a \(\gamma\)-\(\beta\)-closed set and \(V\) any \(\gamma\)-\(\beta\)-open set containing \(A\). Since \(A\) and \((X - V)\) are disjoint \(\gamma\)-\(\beta\)-closed sets, there exist \(\gamma\)-\(\beta\)g-open sets \(U\) and \(W\) such that \(A \subset U\), \((X - V) \subset W\) and \(U \cap W = \emptyset\). By Definition 3.16, we get
\[(X - V) \subset \gamma\)-\(\beta\)int \((W)\).

Since \(U \cap \gamma\)-\(\beta\)int \((W) = \emptyset\), we have \(\gamma\)-\(\beta\)cl \((U) \cap \gamma\)-\(\beta\)int \((W) = \emptyset\), and hence
\[\gamma\)-\(\beta\)cl \((U) \subset X - \gamma\)-\(\beta\)int \((W) \subset V\).

Therefore \(A \subset U \subset \gamma\)-\(\beta\)cl \((U) \subset V\).

(c) \(\Rightarrow\) (a) Let \(A\) and \(B\) be any two disjoint \(\gamma\)-\(\beta\)-closed sets of \(X\). Since \((X - B)\) is an \(\gamma\)-\(\beta\)-open set containing \(A\), there exists a \(\gamma\)-\(\beta\)g-open set \(G\) such that
\[A \subset G \subset \gamma\)-\(\beta\)cl \((G) \subset (X - B)\).

Since \(G\) is a \(\gamma\)-\(\beta\)g-open set, using Definition 3.16, we have \(A \subset \gamma\)-\(\beta\)int \((G)\). Taking \(U = \gamma\)-\(\beta\)int \((G)\) and \(V = X - \gamma\)-\(\beta\)cl \((G)\), we have two disjoint \(\gamma\)-\(\beta\)-open sets \(U\) and \(V\) such that \(A \subset U\) and \(B \subset V\). Hence \(X\) is \(\gamma\)-\(\beta\)-normal.

(e) \(\Rightarrow\) (d) Obvious.

(d) \(\Rightarrow\) (c) Obvious.

(e) \(\Rightarrow\) (c) Let \(A\) be any \(\gamma\)-\(\beta\)-closed set and \(V\) any \(\gamma\)-\(\beta\)-open set containing \(A\). Since every \(\gamma\)-\(\beta\)-open set is \(\gamma\)-\(\beta\)g-open, there exists a \(\gamma\)-\(\beta\)-open set \(G\) such that
\[A \subset G \subset \gamma\)-\(\beta\)cl \((G) \subset \gamma\)-\(\beta\)int \((V)\).

Also we have a \(\gamma\)-\(\beta\)g-open set \(G\) such that \(A \subset G \subset \gamma\)-\(\beta\)cl \((G) \subset \gamma\)-\(\beta\)int \((V) \subset V\).

(f) \(\Rightarrow\) (g) Obvious.

(g) \(\Rightarrow\) (c) Obvious.

(c) \(\Rightarrow\) (e) Let \(A\) be a \(\gamma\)-\(\beta\)-closed set and \(B\) any \(\gamma\)-\(\beta\)g-open set containing \(A\). Using Definition 3.16 of a \(\gamma\)-\(\beta\)g-open set we get \(A \subset \gamma\)-\(\beta\)int \((B) = V\), say. Then applying (c), we get a \(\gamma\)-\(\beta\)g-open set \(U\) such that \(A = \gamma\)-\(\beta\)cl \((A) \subset U \subset \gamma\)-\(\beta\)cl \((U) \subset V\). Again, using the same Definition 3.16 we get \(A \subset \gamma\)-\(\beta\)int \((U)\), and hence
\[A \subset \gamma\)-\(\beta\)int \((U) \subset U \subset \gamma\)-\(\beta\)cl \((U) \subset V\);
which implies $A \subset \gamma\beta\text{int}(U) \subset \gamma\beta\text{cl} (\gamma\beta\text{int}(U)) \subset \gamma\beta\text{cl}(U) \subset V$, i.e.

$$A \subset G \subset \gamma\beta\text{cl}(G) \subset \gamma\beta\text{int}(B),$$

where $G = \gamma\beta\text{int}(U)$.

(c) $\implies$ (g) Let $A$ be a $\gamma\beta\gamma$-closed set and $B$ any $\gamma\beta$-open set containing $A$. Since $A$ is a $\gamma\beta\gamma$-closed set, we have $\gamma\beta\text{cl}(A) \subset B$, therefore by (c) we can find a $\gamma\beta\gamma$-open set $U$ such that $\gamma\beta\text{cl}(A) \subset U \subset \gamma\beta\text{cl}(U) \subset B$.

(g) $\implies$ (f) Let $A$ be a $\gamma\beta\gamma$-closed set and $B$ any $\gamma\beta$-open set containing $A$, then by (g) there exists a $\gamma\beta\gamma$-open set $G$ such that $\gamma\beta\text{cl}(A) \subset G \subset \gamma\beta\text{cl}(G) \subset B$. Since $G$ is a $\gamma\beta\gamma$-open set, then by Definition 3.16, we get $\gamma\beta\text{cl}(A) \subset \gamma\beta\text{int}(G)$. If we take $U = \gamma\beta\text{int}(G)$, the proof follows.

The following examples show that normality and $\gamma\beta$-normality are independent concepts:

5.22. Example. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Define an operation $\gamma$ on $\tau$ by $A^\gamma = \{\{b\}\}$ if $A = \{b\}$.

Then $X$ is normal but not $\gamma\beta$-normal.

Although from the above two examples we have seen that normality and $\gamma\beta$-normality are independent to each other, one can be obtained from the other. The following theorems show these facts.

5.24. Theorem. Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. If $f : (X, \tau) \to (Y, \tau')$ is a $\gamma\beta$-continuous, $\gamma\beta$-anti-closed surjective function and $X$ is $\gamma\beta$-normal, then $Y$ is normal.

Proof. Let $A$ and $B$ be two disjoint closed sets in $Y$. Since $f$ is $\gamma\beta$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\gamma\beta$-closed sets in $X$. Now as $X$ is $\gamma\beta$-normal, there exist disjoint $\gamma\beta$-open sets $V$ and $W$ such that $f^{-1}(A) \subset V$ and $f^{-1}(B) \subset W$. Since $f$ is $\gamma\beta$-anti-closed, there exist open sets $M$ and $N$ such that $A \subset M, B \subset N, f^{-1}(M) \subset V$ and $f^{-1}(N) \subset W$. Since $V \cap W = \emptyset$, we have $M \cap N = \emptyset$. So $Y$ is normal.

5.25. Theorem. Let $f : (X, \tau) \to (Y, \tau')$, where $\gamma$, $\gamma'$ are operations on $\tau$, $\tau'$, respectively, be a $\gamma'\beta$-anti-continuous and $\gamma'\beta$-open surjection. If $X$ is normal, then $Y$ is $\gamma'\beta$-normal.

Proof. For any pair of disjoint $\gamma'\beta$-closed sets $F_1$ and $F_2$ in $Y$, since $f$ is $\gamma'\beta$-anti continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets in $X$. Since $X$ is normal we have two disjoint open sets $V$ and $W$ such that $f^{-1}(F_1) \subset V$ and $f^{-1}(F_2) \subset W$. Since $f$ is $\gamma'\beta$-closed, by Theorem 4.13, we get $\gamma'\beta$-open sets $M$ and $N$ in $Y$ such that $F_1 \subset M$ and $F_2 \subset N$ with $f^{-1}(M) \subset V$ and $f^{-1}(N) \subset W$. Again $f^{-1}(M) \cap f^{-1}(N) \subset V \cap W = \emptyset$. So $M \cap N = \emptyset$. \[\square\]
References