$\oplus$-CO-COATOMICALLY SUPPLEMENTED AND CO-COATOMICALLY SEMIPERFECT MODULES

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Abstract

In this paper it is shown that a factor module of an $\oplus$-co-coatomically supplemented module is not in general $\oplus$-co-coatomically supplemented. If M is $\oplus$-co-coatomically supplemented and U is a fully invariant submodule of M, then M/U is $\oplus$-co-coatomically supplemented. A ring R is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented R-module. A projective module M is co-coatomically semiperfect if and only if M is $\oplus$-co-coatomically supplemented. A ring is semiperfect if and only if every finitely generated free R-module is co-coatomically semiperfect.

Keywords: Co-coatomic submodule, $\oplus$-co-coatomically supplemented module, co-coatomically semiperfect module.

1. Introduction

Throughout this paper $\mathbb{N}$ is the set of all positive integers, R is an associative ring with identity and all modules are *left* unitary R-modules (${}_RM$) unless otherwise stated. For any module M, $\mathrm{Rad}(M)$ denotes the radical of M. The Jacobson radical of ${}_RR$ is denoted by $\mathrm{Jac}(R)$. Let U be a submodule of M. A submodule V of M is called a *supplement* of U in M if V is minimal element in the set of submodules $L \leq M$ with $U + L = M$. V is a supplement of U in M if and only if $U + V = M$ and $U \cap V \ll V$. A module M is called *supplemented* if every submodule of M has a supplement in M (see [15, Section 41] or [5, Chapter 4]). Semisimple, artinian and hollow (in particular local) modules are supplemented. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule (see [18]). Semisimple, finitely generated and hollow modules are coatomic modules. Let N be a submodule of a module M. We say that N is a *co-coatomic* submodule in M if M/N is coatomic. Since factor module of a coatomic module is coatomic, every submodule of semisimple, finitely generated and hollow modules is co-coatomic. A module M is said to be *co-coatomically supplemented* if every co-coatomic submodule of M has a supplement in M. A submodule N of M is called *cofinite* if M/N is finitely generated. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M (see [1]). Clearly a co-coatomically supplemented module is cofinitely

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supplemented and a coatomic module is co-coatomically supplemented if and only if it is a supplemented module. A module is said to be $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M. A module M is called $\oplus$-*co-coatomically supplemented* if every co-coatomic submodule of M has a supplement that is a direct summand of M. Obviously an $\oplus$-supplemented module is $\oplus$-co-coatomically supplemented. Hollow modules (in particular local modules) are $\oplus$-supplemented, so $\oplus$-co-coatomically supplemented. A module M is called $\oplus$-cofinitely supplemented if every cofinite submodule of M has a supplement that is a direct summand of M (see [4]). Clearly an $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented.

In Section 2 , we show that a factor module of an $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented by Example 2.1. If M is $\oplus$-co-coatomically supplemented and U is a fully invariant submodule of M, then M/U is $\oplus$-co-coatomically supplemented. For any ring R, any finite direct sum of $\oplus$-co-coatomically supplemented R-modules is $\oplus$-co-coatomically supplemented, but any direct sum of $\oplus$-co-coatomically supplemented modules need not be $\oplus$-co-coatomically supplemented. We show that a ring R is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented R-module.

In Section 3, we define co-coatomically semiperfect module. For an R-module M, a pair (P, π) is a projective cover of M in case P is a projective R-module and $\pi : P \to M$ is a small epimorphism (see [2]). A module M is called *semiperfect* if every factor module of M has a projective cover (see [15]). A projective module is semiperfect if and only if it is $\oplus$-supplemented (see [10, Lemma 1.2.]). A module M is called *co-coatomically semiperfect* if every coatomic factor module of M has a projective cover. A projective module M is co-coatomically semiperfect if and only if M is $\oplus$-co-coatomically supplemented. An R-module M is *cofinitely semiperfect* or briefly *cof-semiperfect* if every finitely generated factor module of M has a projective cover (see [4]). Clearly a semiperfect module is co-coatomically semiperfect and a co-coatomically semiperfect module is cof-semiperfect. We show that a ring R is semiperfect if and only if every finitely generated free R-module is co-coatomically semiperfect.

2. $\oplus$-Co-coatomically Supplemented Modules

2.1. Example. [8, Example 2.2] Let R be a commutative local ring which is not a valuation ring and let $n \geq 2$. By [14, Theorem 2], there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than n elements. By [7, Corollary 1], $R^{(n)}$ is $\oplus$-co-coatomically supplemented. However, M is not $\oplus$-cofinitely supplemented so it is not $\oplus$-co-coatomically supplemented (see [7, Proposition 2] and [13, Example 2.1]).

The above example shows that a factor module of an $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented.

Let M be a nonzero module and let U be a fully invariant submodule of M, i.e. $f(U) \leq U$ for each $f \in End_R(M)$. If $M = M_1 \oplus M_2$, then $U = (U \cap M_1) \oplus (U \cap M_2)$ (see [6, Lemma 9.3] for abelian groups).

2.2. Proposition. *Let M be a nonzero module and U be a fully invariant submodule of M. If M is $\oplus$-co-coatomically supplemented, then M/U is $\oplus$-co-coatomically* *supplemented. Furthermore, if U is a co-coatomic direct summand of M, then U is also $\oplus$-co-coatomically supplemented.*

Proof. Suppose that M is an $\oplus$-co-coatomically supplemented module and L/U is a co-coatomic submodule of M/U. Therefore $M/L \cong (M/U)/(L/U)$ is coatomic. Since M is $\oplus$-co-coatomically supplemented, there exist submodules N and N' of M such that $M = N \oplus N'$, $M = N + L$ and $N \cap L \ll N$. Then $(N + U)/U$ is a supplement of L/U in M/U (see [15, 41.1(7)]). By hypothesis, U is fully invariant, therefore $U = (U \cap N) \oplus (U \cap N')$ (see [6, Lemma 9.3]). Thus $U = (N + U) \cap (N' + U)$ and $M/U = ((N + U)/U) \oplus ((N' + U)/U)$. Hence M/U is $\oplus$-co-coatomically supplemented.

Now suppose that U is a co-coatomic direct summand of M. Then there exists a submodule U' of M such that $M = U \oplus U'$ and U' is coatomic. Let V be a co-coatomic submodule of U. Therefore $M/V = (U \oplus U')/V \cong (U/V) \oplus U'$ is coatomic as it is direct sum of two coatomic modules. Since M is $\oplus$-co-coatomically supplemented, there exist submodules K and K' of M such that $M = K \oplus K'$, $M = V + K$ and $V \cap K \ll K$. Thus $U = V + (U \cap K)$. Since U is fully invariant, $U = (U \cap K) \oplus (U \cap K')$, and so $U \cap K$ is direct summand of U. Furthermore, $V \cap (U \cap K) = V \cap K \ll K$. Then $V \cap (U \cap K) \ll U \cap K$ (see [15, 19.3(5)]). Therefore $U \cap K$ is a supplement of V in U and it is a direct summand of U. Hence U is $\oplus$-co-coatomically supplemented. □

2.3. Corollary. *Let M be an $\oplus$-co-coatomically supplemented R-module. Then $M/\operatorname{Rad}(M)$ and $M/\operatorname{Soc}(M)$ are also $\oplus$-co-coatomically supplemented modules.*

Property (D3) for an R-module M is the following: If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

2.4. Proposition. *Let M be an $\oplus$-co-coatomically supplemented module with property (D3). Then every co-coatomic direct summand of M is $\oplus$-co-coatomically supplemented.*

Proof. Let N be a co-coatomic direct summand of M that is there exists a submodule N' of M such that $M = N \oplus N'$ and $N' \cong M/N$ is coatomic. Let U be a co-coatomic submodule of N. Then

$$M/U = (N \oplus N')/U \cong (N/U) \oplus N'$$

is coatomic as it is a direct sum of two coatomic modules. Since M is $\oplus$-co-coatomically supplemented, there exists a direct summand V of M such that

$$M = U + V \text{ and } U \cap V \ll V.$$

Hence

$$N = N \cap M = N \cap (U + V) = U + (N \cap V)$$

Since M has property (D3), $N \cap V$ is a direct summand of M. Furthermore $N \cap V$ is a direct summand of N because N is a direct summand of M. Then $U \cap (N \cap V) = U \cap V$ is small in $N \cap V$ by [15, 19.3(5)]. Hence N is $\oplus$-co-coatomically supplemented. □

A ring R is called a left V-ring if every simple R-module is injective (see [15, p. 192]). A commutative ring R is a V-ring if and only if R is a von Neumann regular ring (see [15, 23.5]).

2.5. Proposition. *Over a V-ring R, a module M is $\oplus$-co-coatomically supplemented if and only if M is semisimple.*

Proof. ($\Leftarrow$) Clear.
($\Rightarrow$) Since M is $\oplus$-co-coatomically supplemented, M is $\oplus$-cofinitely supplemented and so cofinitely supplemented. Therefore $M/\operatorname{Soc}(M)$ has no maximal submodule by [1, Theorem 2.8] and [1, Proposition 3.6]. Since R is a V-ring, $M/\operatorname{Soc}(M) = \operatorname{Rad}(M/\operatorname{Soc}(M)) = 0$ (see [15, 23.1]). Thus M is semisimple. □

Obviously an $\oplus$-supplemented module is supplemented and $\oplus$-co-coatomically supplemented module is co-coatomically supplemented. An $\oplus$-supplemented module is $\oplus$-co-coatomically supplemented module, but an $\oplus$-co-coatomically supplemented module need not be $\oplus$-supplemented in general by the following example.

2.6. Example. The $\mathbb{Z}$-module $\mathbb{Q}$ does not have any proper co-coatomic submodule. Thus $\mathbb{Q}$ is $\oplus$-co-coatomically supplemented. But $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented so it is not $\oplus$-supplemented (see [16, Theorem 3.1]).

An $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented but the example below shows that an $\oplus$-cofinitely supplemented module need not be $\oplus$-co-coatomically supplemented.

2.7. Example. [9, p. 282] Let R denote the ring $K[[x]]$ of all power series $\sum_{i=0}^{\infty} k_i x^i$ in an indeterminate x and with coefficients from a field K. R is a local ring. The R-module R is supplemented so R is semiperfect (see [15, 42.6]). Note that

$$\operatorname{Jac}(R) = \left\{ \sum_{i=1}^{\infty} k_i x^i \mid k_i \in K \right\} = Rx$$

is not t-nilpotent. Thus R is not perfect (see [15, 43.9]). Since R is semiperfect, $R/\operatorname{Jac}(R)$ is semisimple. Therefore ${}_RR^{(\mathbb{N})}/\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is semisimple, so $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is a co-coatomic submodule of ${}_RR^{(\mathbb{N})}$. By [3, Theorem 1], $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ does not have a supplement. Thus ${}_RR^{(\mathbb{N})}$ is not co-coatomically supplemented so it is not $\oplus$-co-coatomically supplemented. On the other hand, since R is local, R-module R is $\oplus$-supplemented and so $\oplus$-cofinitely supplemented. Any direct sum of ${}_RR$, in particular ${}_RR^{(\mathbb{N})}$ is $\oplus$-cofinitely supplemented by [4, Theorem 2.6].

By the example above, it is seen that arbitrary direct sum of $\oplus$-co-coatomically supplemented modules need not be $\oplus$-co-coatomically supplemented.

To prove that a finite sum of $\oplus$-co-coatomically supplemented modules is an $\oplus$-co-coatomically supplemented module, we will use the following lemma.

2.8. Lemma. *Let M be an R-module and N, U be submodules of M such that N is co-caotomically supplemented, U is co-coatomic and $N + U$ has a supplement A in M. Then $N \cap (U + A)$ has a supplement B in N and $A + B$ is a supplement of U in M.*

Proof. Let A be a supplement of $N+U$ in M. Then $M = N+U+A$ and $(N+U)\cap A \ll A$. Note that

$$N/(N\cap(U+A)) \cong (N+U+A)/(U+A) = M/(U+A) \cong (M/U)/((U+A)/U)$$

is coatomic. Therefore $N\cap(U+A)$ is a co-coatomic submodule of N. Since N is co-coatomically supplemented, $N\cap(U+A)$ has a supplement B in N, i.e. $N\cap(U+A)+B = N$ and $B\cap(U+A) \ll B$. Then

$$M = N+U+A = U+A+B.$$

$$U\cap(A+B) \leq (A\cap(U+B))+(B\cap(U+A)) \leq (A\cap(U+N))+(B\cap(U+A))$$

$$\ll A+B$$

Hence $A+B$ is a supplement of U in M. □

2.9. Proposition. *For any ring R, any finite direct sum of $\oplus$-co-coatomically supplemented R-modules is $\oplus$-co-coatomically supplemented.*

Proof. Let n be a positive integer and $M = M_1 \oplus \cdots \oplus M_n$ where M_i is co-coatomically supplemented for each $1 \leq i \leq n$. To prove that M is $\oplus$-co-coatomically supplemented it is sufficient to prove the case when $n=2$. Therefore let $M = M_1 \oplus M_2$ and L be any co-coatomic submodule of M. Then $M = M_1 + M_2 + L$ such that $M_1 + M_2 + L$ has a supplement 0 in M. Consider the submodule $M_2 \cap (M_1 + L)$ of M_2.

$$M_2/(M_2\cap(M_1+L)) \cong (M_1+M_2+L)/(M_1+L) = M/(M_1+L).$$

Since M_1+L is a co-coatomic submodule of M, $M_2\cap(M_1+L)$ is a co-coatomic submodule of M_2. Since M_2 is an $\oplus$-co-coatomically supplemented module, $M_2 \cap (M_1+L)$ has a supplement H in M_2 such that H is a direct summand of M_2. H is a supplement of M_1+L in M by Lemma 2.8. Now consider the submodule $M_1\cap(L+H)$ of M_1.

$$M_1/(M_1\cap(L+H)) \cong (M_1+L+H)/(L+H) = M/(L+H).$$

Since $L+H$ is a co-coatomic submodule of M, $M_1\cap(L+H)$ is a co-coatomic submodule of M_1. Since M_1 is $\oplus$-co-coatomically supplemented, $M_1\cap(L+H)$ has a supplement K in M_1 such that K is a direct summand of M_1. Again by Lemma 2.8, we obtain $H+K$ is a supplement of L in M. It follows that $H+K = H\oplus K$ is a direct summand of M since H is a direct summand of M_2 and K is a direct summand of M_1. Thus $M = M_1\oplus M_2$ is $\oplus$-co-coatomically supplemented. □

2.10. Proposition. *Let M be an indecomposable R-module. The following are equivalent:*

(1) *Every co-coatomic submodule of M has a supplement that is a direct summand.*
(2) *Every maximal submodule of M has a supplement that is a direct summand.*
(3) *M is radical or M is local.*

Proof. (1) $\Rightarrow$ (2) Since every maximal submodule is co-coatomic it is clear.
(2) $\Rightarrow$ (3) If M is not radical, that is there is a maximal submodule N of M, then N has a supplement K that is a direct summand. Since M is indecomposable either $K = 0$ or $K = M$. If $K = 0$ then $M = N + K = N$. Contradiction. If $K = M$ then $N = N \cap K \ll M$, therefore N is the largest proper submodule of M, so M is local.
(3) $\Rightarrow$ (1) Let N be a co-coatomic submodule of M. If $N \neq M$ then M/N has a maximal submodule, therefore M has also a maximal submodule, that is M is not a radical module. Then M is local and therefore $\oplus$-supplemented. Thus every co-coatomic submodule has a supplement that is a direct summand in M. □

2.11. Corollary. *Let M be an indecomposable R-module such that* $\mathrm{Rad}(M) \neq M$. *M is $\oplus$-co-coatomically supplemented if and only if M is local.*

A module M is called Σ-*selfprojective* if for each index set I, the module $M^{(I)}$ is selfprojective (see [17]).

2.12. Remark. For an R-module M, if M is Σ-selfprojective and $U \leq \mathrm{Rad}(M)$, then the following holds: U has a supplement in M, so U is small in M [17, Satz 4.1]. Clearly ${}_RR^{(\mathbb{N})}$ is Σ-selfprojective and $\mathrm{Rad}({}_RR^{(\mathbb{N})}) \leq \mathrm{Rad}({}_RR^{(\mathbb{N})})$, therefore if $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$ then $\mathrm{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$.

2.13. Theorem. *A ring R is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented R-module.*

Proof. ($\Rightarrow$) By [12, Proposition 4.8 and Theorem 4.41], ${}_RR^{(\mathbb{N})}$ is $\oplus$-supplemented and so $\oplus$-co-coatomically supplemented.
($\Leftarrow$) Let M denote the R-module $R^{(\mathbb{N})}$. Since M is $\oplus$-co-coatomically supplemented, it is $\oplus$-cofinitely supplemented and so cofinitely supplemented. Thus ${}_RR$ is cofinitely supplemented (see [1, Lemma 2.1]). Therefore ${}_RR$ is supplemented since it is finitely generated. Therefore $R/\mathrm{Jac}(R)$ is semisimple by [15, 42.6]. It follows that ${}_RR^{(\mathbb{N})}/\mathrm{Rad}({}_RR^{(\mathbb{N})})$ is semisimple. Thus $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ is co-coatomic in ${}_RR^{(\mathbb{N})}$. By the assumption, $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$ that is a direct summand. By Remark 2.12, $\mathrm{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$. Therefore, since $R/\mathrm{Jac}(R)$ is semisimple, R is left perfect by [15, 43.9]. □

2.14. Corollary. *The following are equivalent for a ring R:*

(1) *R is left perfect.*
(2) *The R-module $R^{(\mathbb{N})}$ is $\oplus$-supplemented.*
(3) *The R-module $R^{(\mathbb{N})}$ is $\oplus$-co-coatomically supplemented.*

Proof. (1) $\Leftrightarrow$ (2) By [11, Theorem 2.10].
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) By Theorem 2.13. □

Let R be a commutative ring. An R-module M is called a multiplication module if every submodule of M is of the form IM for some ideal I of R. Let M be an $\oplus$-cofinitely supplemented multiplication module with $\mathrm{Rad}(M) \ll M$, then M can be written as an irredundant sum of local direct summands of M (see [13, Theorem 2.7]).

2.15. Proposition. *Let R be a commutative ring and M be a multiplication R-module. If M is an $\oplus$-co-coatomically supplemented module with* $\mathrm{Rad}(M) \ll M$*, then M can be written as an irredundant sum of local direct summands of M.*

Proof. Since every $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented the proof is clear by [13, Theorem 2.7]. □

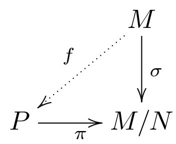
3. Co-coatomically Semiperfect Modules

3.1. Definition. Let M be an R-module. M is called co-coatomically semiperfect if every coatomic factor module of M has a projective cover.

The following proposition gives a characterization of a projective $\oplus$-co-coatomically supplemented module.

3.2. Proposition. *Let M be a projective R-module. Then M is co-coatomically semiperfect if and only if M is $\oplus$-co-coatomically supplemented.*

Proof. ($\Rightarrow$) Let N be a co-coatomic submodule of M. Then M/N is coatomic. By hypothesis, there exists a projective cover $\pi : P \to M/N$. Let $\sigma : M \to M/N$ be canonical epimorphism. Since M is projective there exists a homomorphism $f : M \to P$ such that the diagram



is commutative, i.e. $\pi \circ f = \sigma$. Since π is a small epimorphism, f is epic by [15, 19.2]. Since P is projective, f splits, i.e. there exists a homomorphism $g : P \to M$ such that $f \circ g = 1_P$ by [9, 3.9.3]. Thus $\pi = \pi \circ f \circ g = \sigma \circ g$. It follows that $M = \ker f \oplus g(P)$ and $\ker f \leq N$, so $M = N + g(P)$. Let $\mu = \sigma\mid_{g(P)} : g(P) \to M/N$. Then $\pi = \mu \circ g$ and therefore μ is epic since π is epimorphism. Furthermore, since π is a small epimorphism, μ is also a small epimorphism by [15, 19.3]. Therefore $\ker \mu = N \cap g(P) \ll g(P)$. Thus $g(P)$ is a supplement of N.
($\Leftarrow$) Let M/N be a coatomic factor module of M. Since M is $\oplus$-co-coatomically supplemented, there exists submodules K and K' such that $M = K \oplus K'$, $M = N + K$ and $N \cap K \ll K$. Since M is projective, K is projective. Therefore $\sigma \circ i : K \to M/N$ is an epimorphism and $\ker \sigma \circ i = N \cap K \ll K$ for the inclusion homomorphism $i : K \to M$ and the canonical epimorphism $\sigma : M \to M/N$. □

An R-module M is cofinitely semiperfect or briefly cof-semiperfect if every finitely generated factor module of M has a projective cover.

A co-coatomically semiperfect module is cof-semiperfect but converse need not be true by Example 2.7 since the projective R-module $R^{(\mathbb{N})}$ in that example is $\oplus$-cofinitely supplemented but not $\oplus$-co-coatomically supplemented.

A submodule U of an R-module M has *ample supplements* in M if, for every submodule V of M with $U+V = M$, there exists a supplement V' of U with $V' \leq V$ (see [5, p. 237]). A module M is called *co-coatomically amply supplemented* if every co-coatomic submodule of M has ample supplements in M. Clearly a co-coatomically amply supplemented module is co-coatomically supplemented.

Let M be an R-module and N be a submodule of M. N is called lie above a direct summand of M if there is a decomposition $M = K \oplus K'$ such that $K \leq N$ and $K' \cap N \ll K'$.

3.3. Proposition. *Let M be a projective module. Then the following are equivalent:*

(1) *M is co-coatomically semiperfect.*
(2) *M is $\oplus$-co-coatomically supplemented.*
(3) *Each co-coatomic submodule of M lies above a direct summand of M.*
(4) *M is co-coatomically amply supplemented by supplements which have projective covers.*
(5) *M is co-coatomically supplemented by supplements which have projective covers.*

Proof. (1) $\Leftrightarrow$ (2) By Proposition 3.2.
(2) $\Rightarrow$ (3) Let N be a co-coatomic submodule of M. Since M is $\oplus$-co-coatomically supplemented, there exist submodules K and K' of M such that $M = N + K$, $N \cap K \ll K$ and $M = K \oplus K'$. Since M is projective there exists a submodule $K'' \leq N$ such that $M = K'' \oplus K$ (see [15, 41.14]).
(3) $\Rightarrow$ (2) Clear.
(1) $\Rightarrow$ (4) Let N be a co-coatomic submodule of M and $M = N + L$ for some submodule L of M. Let (P, f) be a projective cover of M/N. Since P is projective and $M/N \cong L/(N \cap L)$, there exists a homomorphism $g : P \to L$. Since $\operatorname{Ker} f \ll P$ and $g(\operatorname{Ker} f) = \operatorname{Im} g \cap N \cap L = \operatorname{Im} g \cap N$, $\operatorname{Im} g \cap N = \operatorname{Im} g \cap N \cap L \ll \operatorname{Im} g$. $\operatorname{Im} g + (N \cap L) = L$ since f is an epimorphism. Therefore $\operatorname{Im} g$ is a supplement of $N \cap L$ in L. $M = N + L = N + \operatorname{Im} g + (N \cap L) = \operatorname{Im} g + N$ and $\operatorname{Im} g \cap N \ll \operatorname{Im} g$, i.e. $\operatorname{Im} g$ is a supplement of N in M and $\operatorname{Im} g$ is contained in L. Since $\operatorname{Ker} g \leq \operatorname{Ker} f$ and $\operatorname{Ker} f \ll P$, P is a projective cover of $\operatorname{Im} g$.
(4) $\Rightarrow$ (5) Clear.
(5) $\Rightarrow$ (1) Let N be a co-coatomic submodule of M and L be a supplement of N in M. Then L is a small cover of $L/(N \cap L)$. Therefore every projective cover of L is also a projective cover of $L/(N \cap L)$. Since $M/N \cong L/(N \cap L)$, M/N has a projective cover. Thus M is co-coatomically semiperfect. □

3.4. Proposition. *Every homomorphic image of a co-coatomically semiperfect module is co-coatomically semiperfect.*

Proof. Let $f : M \to N$ be a homomorphism and let M be a co-coatomically semiperfect module. Let $f(M)/U$ be a coatomic factor module of $f(M)$. There is an epimorphism

$$\sigma : M \to f(M)/U,\ m \mapsto f(m) + U.$$

Since M is co-coatomically semiperfect,

$$M/f^{-1}(U) \cong f(M)/U$$

that is $f(M)/U$ has a projective cover. Thus $f(M)$ is co-coatomically semiperfect. □

3.5. Corollary. *Every factor module of a co-coatomically semiperfect module is co-coatomically semiperfect.*

3.6. Corollary. *Let M be a projective module. If M is $\oplus$-co-coatomically supplemented then every factor module of an $\oplus$-co-coatomically supplemented module is also $\oplus$-co-coatomically supplemented.*

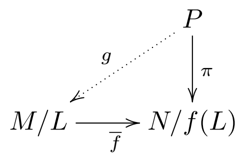
Proof. By Corollary 3.5. □

3.7. Proposition. *Every small cover of a co-coatomically semiperfect module is co-coatomically semiperfect.*

Proof. Let N be a co-coatomically semiperfect module, $f : M \to N$ be a small epimorphism and L be a co-coatomic submodule of M. Then $N/f(L)$ is an epimorphic image of M/L under the epimorphism

$$\overline{f} : M/L \to N/f(L),\ \overline{f}(m + L) = f(m) + f(L)$$

Note that $\ker \overline{f} \ll M/L$ since $\ker f \ll M$. Therefore $N/f(L)$ is coatomic since M/L is coatomic. By hypothesis, $N/f(L)$ has a projective cover, say $\pi : P \to N/f(L)$. Since P is projective there exists a homomorphism $g : P \to M/L$ such that the following diagram is commutative



i.e. $\overline{f} \circ g = \pi$. Thus g is epic by [15, 19.2] and since π is small, g is small by [15, 19.3]. Hence P is a projective cover of the module M/L. □

3.8. Corollary. *If $K \ll M$ and M/K is co-coatomically semiperfect then M is co-coatomically semiperfect.*

3.9. Corollary. *Let $\pi : P \to M$ be a projective cover of a module M. Then the following statements are equivalent:*

(1) *M is co-coatomically semiperfect.*
(2) *P is co-coatomically semiperfect.*
(3) *P is $\oplus$-co-coatomically supplemented.*

Proof. (1) $\Rightarrow$ (2) By Proposition 3.7.
(2) $\Rightarrow$ (1) By Proposition 3.4.
(2) $\Leftrightarrow$ (3) By Proposition 3.2. □

Let M be an R-module. Then an R-module N is called (finitely) M-generated if it is a homomorphic image of a (finite) direct sum of copies of M (see [5, 1.1.]).

3.10. Lemma. *Let M be a projective module. If M is semiperfect then every finitely M-generated module is co-coatomically semiperfect. The converse holds if M is finitely generated.*

Proof. Let N be a finitely M-generated module. Since M is semiperfect, M is $\oplus$-supplemented. Therefore M is $\oplus$-co-coatomically supplemented. By Proposition 2.9, a finite direct sum of M, i.e. for any finite set Λ, $M^{(\Lambda)}$ is also $\oplus$-co-coatomically supplemented. Therefore $M^{(\Lambda)}$ is co-coatomically semiperfect by Proposition 3.2. By Corollary 3.5, N is co-coatomically semiperfect. Conversely, suppose that M

is finitely generated and so it is coatomic. By hypothesis, M is co-coatomically semiperfect. Therefore M is semiperfect. □

3.11. Proposition. *For a ring R, the following statements are equivalent:*

(1) *R is semiperfect.*
(2) *Every finitely generated free R-module is semiperfect.*
(3) *Every finitely generated free R-module is co-coatomically semiperfect.*

Proof. (1) $\Leftrightarrow$ (2) By [10, Lemma 1.2] and [11, Theorem 2.1.].
(1) $\Leftrightarrow$ (3) By Lemma 3.10. □

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