Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series

G. Murugusundaramoorthy*, K.Vijaya† and S.Porwal‡

Abstract

The purpose of the present paper is to investigate some characterization for Poisson distribution series to be in the new subclasses \( G(\lambda, \alpha) \) and \( K(\lambda, \alpha) \) of analytic functions.

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1. Introduction and Preliminaries

Let \( A \) be the class of functions \( f \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). As usual, we denote by \( S \) the subclass of \( A \) consisting of functions which are normalized by \( f(0) = 0 = f'(0) - 1 \) and also univalent in \( U \). Denote by \( T \) \[19\] the subclass of \( A \) consisting of functions of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, \ldots
\]
Also, for functions \( f \in A \) given by (1.1) and \( g \in A \) given by
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n,
\]
we define the Hadamard product (or convolution) of \( f \) and \( g \) by
\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U).
\]
The class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) may be defined as
\[
S^*(\alpha) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in U \right\}.
\]
The class \( S^*(\alpha) \) and the class \( \mathcal{K}(\alpha) \) of convex functions of order \( \alpha \) (\( 0 \leq \alpha < 1 \))
\[
\mathcal{K}(\alpha) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in U \right\} = \{ f \in A : zf' \in S^*(\alpha) \}
\]
were introduced by Robertson in [17]. We also write \( S^*(0) = S^* \), where \( S^* \) denotes the class of functions \( f \in A \) that \( f(U) \) is starlike with respect to the origin. Further, \( \mathcal{K}(0) = \mathcal{K} \) is the well-known standard class of convex functions. It is an established fact that \( f \in \mathcal{K}(\alpha) \iff zf' \in S^*(\alpha) \).

A function \( f \in A \) is said to be in the class \( f \in \Re^*(A, B) \) if it satisfies the inequality
\[
\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1.
\]
where \( z \in U, \tau \in \mathbb{C} \setminus \{0\}, -1 \leq B < A \leq 1 \). The class \( \Re^*(A, B) \) was introduced earlier by Dixit and Pal [6]. If we put
\[
\tau = 1, \ A = \alpha \text{ and } B = -\alpha \quad (0 < \alpha \leq 1),
\]
we obtain the class of functions \( f \in A \) satisfying the inequality
\[
\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \alpha \quad (z \in U; 0 < \alpha \leq 1)
\]
which was studied by (among others) Padmanabhan [12] and Caplinger and Causey [4].

Very recently, Porwal [13] introduce a power series whose coefficients are probabilities of Poisson distribution
\[
K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (z \in U)
\]
and we note that, by ratio test the radius of convergence of above series is infinity. In [13], Porwal also defined the series
\[
F(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (z \in U).
\]
Now, we considered the linear operator
\[
\mathcal{J}(m) : A \to A
\]
defined by
\[
(1.4) \quad \mathcal{J}(m)f = K(m, z) \ast f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n.
\]
Motivated by results on connections between various subclasses of analytic univalent functions by using generalized Bessel functions by Srivastava et al. [20] (see [5, 7, 9, 10, 18]) we obtain necessary and sufficient condition for functions \( F(m,z) \) in \( \mathcal{S}^*(\lambda,\alpha) \) and \( \mathcal{K}^*(\lambda,\alpha) \). Further due to the works of Ramesha et al. [16], Padmanabhan [12], we estimate certain inclusion relations between the classes \( \mathcal{R}^*(A,B) \), and \( \mathcal{S}^*(\lambda,\alpha) \) and \( \mathcal{K}^*(\lambda,\alpha) \).

For \( 0 \leq \lambda < 1 \) and \( 0 \leq \alpha < 1 \), we let \( \mathcal{G}(\lambda,\alpha) \) the subclass of functions \( f \in A \) which satisfy the condition
\[
R \left( \frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \right) > \alpha, \quad (z \in \mathbb{U}).
\]
and also let \( \mathcal{K}(\lambda,\alpha) \) the subclass of functions \( f \in A \) which satisfy the condition
\[
R \left( \frac{z[f'(z) + \lambda z^2 f''(z)]'}{zf'(z)} \right) > \alpha, \quad (z \in \mathbb{U}).
\]
Also denote \( \mathcal{G}^*(\lambda,\alpha) = \mathcal{G}(\lambda,\alpha) \cap \mathcal{T} \) and \( \mathcal{K}^*(\lambda,\alpha) = \mathcal{K}(\lambda,\alpha) \cap \mathcal{T} \).

1.1. Remark. It is of interest to note that for \( \lambda = 0 \), we have \( \mathcal{G}(\lambda,\alpha) \equiv \mathcal{S}^*(\alpha) \) and \( \mathcal{K}(\lambda,\alpha) \equiv \mathcal{K}(\alpha) \).

To prove the main results, we need the following Lemmas.

1.2. Lemma. [21] A function \( f \in A \) belongs to the class \( \mathcal{G}(\lambda,\alpha) \) if
\[
\sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha) |a_n| \leq 1 - \alpha.
\]

1.3. Lemma. [21] A function \( f \in A \) belongs to the class \( \mathcal{K}(\lambda,\alpha) \) if
\[
\sum_{n=2}^{\infty} n(n + \lambda n(n-1) - \alpha) |a_n| \leq 1 - \alpha.
\]

Further we can easily prove that the conditions are also necessary if \( f \in \mathcal{T} \).

1.4. Lemma. A function \( f \in \mathcal{T} \) belongs to the class \( \mathcal{G}^*(\lambda,\alpha) \) if and only if
\[
\sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha) |a_n| \leq 1 - \alpha.
\]

1.5. Lemma. A function \( f \in \mathcal{T} \) belongs to the class \( \mathcal{K}^*(\lambda,\alpha) \) if and only if
\[
\sum_{n=2}^{\infty} n(n + \lambda n(n-1) - \alpha) |a_n| \leq 1 - \alpha.
\]

2. Main Results

2.1. Theorem. If \( m > 0 \), then \( F(m,z) \) is in \( \mathcal{S}^*(\lambda,\alpha) \) if and only if
\[
e^m \left[ \lambda m^2 + (1 + 2\lambda)m \right] \leq 1 - \alpha.
\]

Proof. Since \( F(m,z) = z - \sum_{n=2}^{\infty} \frac{m^n-1}{(n-1)!} e^{-m} z^n \) and by virtue of Lemma 1.4, it suffices to show that
\[
\sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha) \frac{m^n-1}{(n-1)!} e^{-m} \leq 1 - \alpha.
\]
Let
\[ L_1(m, \lambda, \alpha) = \sum_{n=2}^{\infty} \frac{(n^2 \lambda + n(1 - \lambda) - \alpha) m^{n-1}}{(n-1)!} e^{-m} \]
Writing \( n^2 = (n-1)(n-2) + 3(n-1) + 1 \) and \( n = (n-1) + 1 \), and by simple computation, we get
\[ L_1(m, \lambda, \alpha) = \sum_{n=2}^{\infty} \frac{\lambda(n-1)(n-2) m^{n-1}}{(n-1)!} e^{-m} \]
\[ + (1 + 2\lambda) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} + (1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \]
\[ = \lambda \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} e^{-m} + (1 + 2\lambda) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} e^{-m} \]
\[ + (1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} e^{-m} \]
\[ = e^{-m} [\lambda m^2 e^m + (1 + 2\lambda) me^m + (1 - \alpha)(e^m - 1)] \]
\[ = \lambda m^2 + (1 + 2\lambda)m + (1 - \alpha)(1 - e^{-m}). \]
But, this last expression is bounded above by \( 1 - \alpha \) if and only if (2.1) is satisfied. \( \square \)

2.2. Theorem. If \( m > 0 \), then \( F(m, z) \) is in \( \mathcal{K}^*(\lambda, \alpha) \) if and only if
\[ (2.2) \quad e^m [\lambda m^3 + (1 + 5\lambda)m^2 + (3 + 4\lambda - \alpha)m] \leq 1 - \alpha. \]

Proof. Since \( F(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n \) and by virtue of Lemma 1.5, it suffices to show that
\[ \sum_{n=2}^{\infty} \frac{(n^2 \lambda + n^2(1 - \lambda) - n\alpha) m^{n-1}}{(n-1)!} e^{-m} \leq 1 - \alpha. \]
Let
\[ L_2(m, \lambda, \alpha) = \sum_{n=2}^{\infty} \frac{(n^2 \lambda + n^2(1 - \lambda) - n\alpha) m^{n-1}}{(n-1)!} e^{-m} \]
Writing \( n^3 = (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1 \),
\( n^2 = (n-1)(n-2) + 3(n-1) + 1 \) and \( n = (n-1) + 1 \), we can rewrite the above terms as
\[ L_2(m, \lambda, \alpha) = \lambda \sum_{n=2}^{\infty} \frac{(n-1)(n-2)(n-3) m^{n-1}}{(n-1)!} e^{-m} \]
\[ + (1 + 5\lambda) \sum_{n=2}^{\infty} \frac{(n-1)(n-2) m^{n-1}}{(n-1)!} e^{-m} \]
\[ + (3 + 4\lambda - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \]
\[ + (1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \]
\[ \sum_{n=2}^{\infty} n(n^2\lambda + n(1 - \lambda) - \alpha)^{m \lambda}\frac{n-1}{(n-1)!} e^{-m} |a_n| \leq 1 - \alpha. \]

Let
\[ L_\alpha(m, \lambda, \alpha) = \sum_{n=2}^{\infty} n(n^2\lambda + n(1 - \lambda) - \alpha)^{m \lambda}\frac{n-1}{(n-1)!} e^{-m} |a_n|. \]

Since \( f \in \mathcal{R}^*(A, B) \) by Lemma 3.1 we have \( |a_n| \leq (A - B)\frac{|\tau|}{n}, \ n \in \mathbb{N}\setminus\{1\} \), hence we get
\[ L_\alpha(m, \lambda, \alpha) \leq e^{-m} \sum_{n=2}^{\infty} (n^2\lambda + n(1 - \lambda) - \alpha)^{m \lambda}\frac{n-1}{(n-1)!} (A - B)|\tau| \]
\[ \leq (A - B)|\tau| e^{-m} \sum_{n=2}^{\infty} (n^2\lambda + n(1 - \lambda) - \alpha)^{m \lambda}\frac{n-1}{(n-1)!}. \]
Writing \( n^2 = (n - 1)(n - 2) + 3(n - 1) + 1 \) and \( n = (n - 1) + 1 \), and by using the similar arguments as in the proof of Theorem 2.1, we get

\[
\mathcal{L}_3(m, \lambda, \alpha) \leq (A - B)|\tau| \left[ \lambda m^2 + (1 + 2\lambda)m + (1 - \alpha)(1 - e^{-m}) \right].
\]

But the last expression is bounded above by \( 1 - \alpha \) if and only if (3.2) is satisfied. Hence the proof is completed. \( \square \)

3.3. Corollary. Let \( m > 0 \) and \( \lambda = 0 \). If \( f \in \mathcal{R}^+(A, B) \), then \( \mathcal{I}(m) \in \mathcal{K}(\alpha) \) if and only if

\[
(A - B)|\tau|m \left[ 1 - (A - B)|\tau|(1 - e^{-m})^{-1} \right] \leq 1 - \alpha
\]

where \( \tau \in \mathbb{C}\setminus\{0\} - 1 \leq B < A \leq 1 \).

3.4. Theorem. Let \( m > 0 \), then

\[
G(m, z) = \int_0^z \frac{F(m, t)}{t} \, dt
\]

is in \( \mathcal{K}^*(\lambda, \alpha) \) if and only if

\[
e^m \left[ \lambda m^2 + (1 + 2\lambda)m \right] \leq 1 - \alpha.
\]

Proof. Since

\[
G(m, z) = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{(n - 1)!} \frac{z^n}{n} = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{n!} z^n
\]

by Lemma 1.5, we need only to show that

\[
\sum_{n=2}^{\infty} n(n^2 \lambda + n(1 - \lambda) - \alpha) \frac{m^{n-1}}{n!} e^{-m} \leq 1 - \alpha.
\]

Now, let

\[
\mathcal{L}_4(m, \lambda, \alpha) = \sum_{n=2}^{\infty} n(n^2 \lambda + n(1 - \lambda) - \alpha) \frac{m^{n-1}}{n!} e^{-m}
\]

\[
= \sum_{n=2}^{\infty} (n^2 \lambda + n(1 - \lambda) - \alpha) \frac{m^{n-1}}{(n - 1)!} e^{-m}.
\]

Hence, writing \( n^2 = (n - 1)(n - 2) + 3(n - 1) + 1 \) and \( n = (n - 1) + 1 \), and by using the similar arguments as in the proof of Theorem 2.1, we have

\[
\mathcal{L}_4(m, \lambda, \alpha) \leq \lambda m^2 + (1 + 2\lambda)m + (1 - \alpha)(1 - e^{-m}),
\]

which is bounded above by \( 1 - \alpha \) if and only if (3.3) holds. \( \square \)

3.5. Theorem. Let \( m > 0 \), then \( G(m, z) = \int_0^z \frac{F(m, t)}{t} \, dt \) is in \( \mathcal{S}^*(\lambda, \alpha) \) if and only if

\[
m\lambda + \left( 1 - \frac{\alpha}{m} \right) (1 - e^{-m}) + \alpha e^{-m} \leq 1 - \alpha.
\]

Proof. The proof of theorem is similar to that of Theorem 3.4, hence we omit the details involved. \( \square \)

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References
