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COVERING GROUPOIDS OF CATEGORICAL GROUPS

O. Mucuk * and T. Şahan †

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Abstract

If X is a topological group, then its fundamental groupoid $\pi_1(X)$ is a group-groupoid which is a group object in the category of groupoids. Further if X is a path connected topological group which has a simply connected cover, then the category of covering groups of X and the category of covering groupoids of $\pi_1(X)$ are equivalent. In this paper we prove that if (X, x_0) is an H-group, then the fundamental groupoid $\pi_1(X)$ is a weak categorical group. This enables one to prove that the category of the covering spaces of an H-group (X, x_0) is equivalent to the category of covering groupoids of the weak categorical group $\pi_1(X)$.

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Introduction

Covering spaces are studied in algebraic topology, but they have important applications in many other branches of mathematics including differential topology, the theory of topological groups and the theory of Riemann surfaces.

One of the ways of expressing the algebraic content of the theory of covering spaces is using groupoids and the fundamental groupoids. The latter functor gives an equivalence of categories between the category of covering spaces of a reasonably nice space X and the category of covering groupoids of $\pi_1(X)$.

If X is a connected topological group with identity e and $p\colon (\widetilde{X}, \tilde{e}) \to (X, e)$ is a covering map of pointed spaces such that $\widetilde{X}$ is simply connected, then $\widetilde{X}$ becomes a topological group with identity $\tilde{e}$ such that p is a morphism of topological groups (see for example [6, Proposition 5] and [11, Theorem 10.42]).

The problem of universal covers of non-connected topological groups was first studied by Taylor in [12]. He proved that a topological group X determines an obstruction class

*Department of Mathematics, Erciyes University Kayseri 38039, TURKEY E-mail: (O. Mucuk) `mucuk@erciyes.edu.tr`

†Department of Mathematics, Erciyes University Kayseri 38039, TURKEY E-mail: (T. Şahan) `tsahan@erciyes.edu.tr`

k_X in $H^3(\pi_0(X), \pi_1(X, e))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure to a universal cover. In [8] an analogous algebraic result is given in terms of crossed modules and group objects in the category of groupoids (see also [3] for a revised version, which generalizes these results and shows the relation with the theory of obstructions to extension for groups).

For a topological group X, the fundamental groupoid $\pi_1(X)$ becomes a *group-groupoid* which is a group object in the category of groupoids [4]. This notion is also known as an internal category in the category of groups [10]. This functor gives an equality of the category of the covering groups of a topological group X whose underlying space is locally nice, and the category of the covering groupoids of $\pi_1(X)$ [3, Proposition 2.3] (see also [8]). Recently the notion of monodromy for topological group-groupoids was developed by the authors in [9].

In this paper we prove that if (X, x_0) is an H-group (see Definition 1.3), then the fundamental groupoid $\pi_1(X)$ is a weak categorical group. This enables us to prove that the category of the covering spaces of an H-group (X, x_0) is equivalent to the category of covering groupoids of the weak categorical group $\pi_1(X)$. We also prove that the categorical group structure lifts to some kinds of the covering groupoids.

1. Covering Spaces and H-groups

We assume the usual theory of covering maps. All spaces X are assumed to be locally path connected and semi locally simply connected, so that each path component of X admits a simply connected cover.

Recall that a covering map $p\colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every cover of X in the sense that if $q\colon \widetilde{Y} \to X$ is another cover of X then there exists a map $r\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a cover). A covering map $p\colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. Note that a simply connected cover is a universal cover.

1.1. Definition. We call a subset U of X *liftable* if it is open, path connected and U lifts to each cover of X, that is, if $p\colon \widetilde{X} \to X$ is a covering map, $\imath\colon U \to X$ is the inclusion map, and $\tilde{x} \in \widetilde{X}$ satisfies $p(\tilde{x}) = x \in U$, then there exists a map (necessarily unique) $\hat{\imath}\colon U \to \widetilde{X}$ such that $p\hat{\imath} = \imath$ and $\hat{\imath}(x) = \tilde{x}$. □

It is an easy application that U is liftable if and only if it is open, path connected and for all $x \in U$, the fundamental group $\pi_1(U, x)$ is mapped to the singleton by the morphism $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $\imath\colon U \to X$. Remark that if X is a semi locally simply connected topological space, then each point $x \in X$ has a liftable neighbourhood.

The following result, which is very useful for the proof of Theorem 3.11, is known as Covering Homotopy Theorem [11, Theorem 10.6]. In Theorem 3.12 we prove a parallel result for covering groupoids.

1.2. Theorem. *Let $p\colon \widetilde{X} \to X$ be a covering map and Z a connected space. Consider the commutative diagram of continuous maps*

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & \widetilde{X} \\ {\scriptstyle j}\downarrow & \nearrow_{\widetilde{F}} & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow[F]{} & X \end{array}$$

where $j\colon Z \to Z \times I, j(z) = (z, 0)$ for all $z \in Z$. Then there is a unique continuous map $\widetilde{F}\colon Z \times I \to \widetilde{X}$ such that $p\widetilde{F} = F$ and $\widetilde{F}j = \tilde{f}$. □

As a corollary of this theorem if the maps $f, g\colon Z \to X$ are homotopic, then their respective liftings $\tilde{f}$ and $\tilde{g}$ are homotopic. If $f \simeq g$, there is a continuous map $F\colon Z \times I \to X$ such that $F(z,0) = f(z)$ and $F(z,1) = g(z)$. So there is a continuous map $\widetilde{F}\colon Z \times I \to \widetilde{X}$ as in Theorem 1.2. Here $p\widetilde{F}(z,0) = F(z,0) = f(z)$ and $p\widetilde{F}(z,1) = F(z,1) = g(z)$. By the uniqueness of the liftings we have that $\widetilde{F}(z,0) = \tilde{f}(z)$ and $\widetilde{F}(z,1) = \tilde{g}(z)$. Therefore $\tilde{f}$ and $\tilde{g}$ are homotopic.

1.3. Definition. [11, p.324] A pointed space (X, x_0) is called an $H-group$ if there are continuous pointed maps

$$m\colon (X \times X, (x_0, x_0)) \to (X, x_0), (x, x') \mapsto xx'$$

$$n\colon (X, x_0) \to (X, x_0), x \mapsto x^{-1}$$

and pointed homotopies

(i) associativity : $m(1_X \times m) \simeq m(m \times 1_X)$;
(ii) unit: $m\,\imath_1 \simeq 1_X \simeq m\,\imath_2$;
(iii) inverse: $m(1_X, n) \simeq c \simeq m(n, 1_X)$

where $\imath_1, \imath_2\colon X \to X \times X$ are injections defined by $\imath_1(x) = (x, x_0)$ and $\imath_2(x) = (x_0, x)$; and $c\colon X \to X$ is the constant map at x_0. ☐

We remark that these axioms of the associativity, the unit and the inverse can be respectively stated that the following diagrams are commutative up to homotopy.

$$\begin{array}{ccc} (X \times X \times X, (x_0, x_0, x_0)) & \xrightarrow{m \times 1_X} & (X \times X, (x_0, x_0)) \\ {\scriptstyle 1_X \times m}\downarrow & & \downarrow{\scriptstyle m} \\ (X \times X, (x_0, x_0)) & \xrightarrow[m]{} & (X, x_0) \end{array}$$

$$\begin{array}{ccccc} (X \times X, (x_0, x_0)) & \xleftarrow{\imath_1} & (X, x_0) & \xrightarrow{\imath_2} & (X \times X, (x_0, x_0)) \\ & {\scriptstyle m}\searrow & \downarrow{\scriptstyle 1_X} & \swarrow{\scriptstyle m} & \\ & & (X, x_0) & & \end{array}$$

and

$$\begin{array}{ccccc} (X \times X, (x_0, x_0)) & \xleftarrow{(n, 1_X)} & (X, x_0) & \xrightarrow{(1_X, n)} & (X \times X, (x_0, x_0)) \ . \\ & {\scriptstyle m}\searrow & \downarrow{\scriptstyle c} & \swarrow{\scriptstyle m} & \\ & & (X, x_0) & & \end{array}$$

Let (X, x_0) and (Y, y_0) be H-groups. A continuous map $f\colon (X, x_0) \to (Y, y_0)$ such that $f(xx') = f(x)f(x')$ for $x, x' \in X$, is called a *morphism* of H-groups. So we have a category of H-groups denoted by HGrp.

1.4. Example. A topological group X with identity e is an H-group since the group operation

$$m\colon (X \times X, (e, e)) \to (X, e)$$

and the inverse map $n\colon (X, x_0) \to (X, x_0)$ are continuous; and the following axioms are satisfied:

(i) the associativity: $m(1_X \times m) = m(m \times 1_X)$;
(ii) the unit: $m\,\imath_1 = 1_X = m\,\imath_2$;

(iii) the inverse: $m(1_X, n) = c = m(n, 1_X)$

where $c\colon X \to X$ is the constant map at e. □

1.5. Theorem. [11, Theorem 11.9] *If (X, x_0) is a pointed space, then the loop space $\Omega(X, x_0)$ is an H-group.*

We give a definition before we prove a result on H-groups.

1.6. Definition. Let (X, x_0) and (Y, y_0) be H-groups and U an open neighbourhood of x_0 in X. A continuous map $\phi\colon (U, x_0) \to (Y, y_0)$ is called a *local morphism* of H-groups if $\phi(xy) = \phi(x)\phi(y)$ for $x, y \in U$ such that $xy \in U$. □

1.7. Theorem. *Let (X, x_0) and $(\widetilde{X}, \tilde{x}_0)$ be H-groups and $q\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a morphism of H-groups which is a covering map on the underlying spaces. Let U be an open and path connected neighbourhood of x_0 in X such that U^2 is contained in a liftable neighbourhood V of x_0 in X. Then the inclusion map $\imath\colon (U, x_0) \to (X, x_0)$ lifts to a local morphism $\hat{\imath}\colon (U, x_0) \to (\widetilde{X}, \tilde{x}_0)$ of H-groups.*

Proof. Since V lifts to $\widetilde{X}$, then U lifts to $\widetilde{X}$ by $\hat{\imath}\colon (U, x_0) \to (\widetilde{X}, \tilde{x}_0)$. We now prove that $\hat{\imath}$ is a local morphism of H-groups. By the lifting theorem the map $\hat{\imath}\colon (U, x_0) \to (\widetilde{X}, \tilde{x}_0)$ is continuous. We have to prove that $\hat{\imath}\colon (U, x_0) \to (\widetilde{X}, \tilde{x}_0)$ preserves the multiplication. Let $x, y \in U$ such that $xy \in U$. Let a and b be paths from x and y to x_0 in U respectively. By the continuity of

$$m\colon (X \times X, (x_0, x_0)) \to (X, x_0)$$

$c = ab$ defined by $c(t) = a(t)b(t)$ for $t \in [0, 1]$ is a path from xy to x_0. Since $U^2 \subseteq V$, the path c is in V. So the paths a, b and c lift to $\widetilde{X}$. Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be the liftings of the paths a, b and c with end points x_0 chosen respectively as above. Since q is a morphism of H-spaces, we have that

$$q(\tilde{c}) = c = ab = q(\tilde{a})q(\tilde{b}).$$

and

$$q(\tilde{a}\tilde{b}) = q(\tilde{a})q(\tilde{b}).$$

Since $\tilde{c}$ and $\tilde{a}\tilde{b}$ end at $\widetilde{x}_0 \in \widetilde{X}$, by the unique path lifting, we have that

$$\tilde{c} = \tilde{a}\tilde{b}.$$

By evaluating these paths at $0 \in I$ we have that

$$\hat{\imath}(xy) = \hat{\imath}(x)\hat{\imath}(y).$$

Hence $\hat{\imath}\colon (U, x_0) \to (\widetilde{X}, \tilde{x}_0)$ is a local morphism of H-groups. □

2. Covering Groupoids

A *groupoid* G on $\mathrm{Ob}(G)$ is a small category in which each morphism is an isomorphism. Thus G has a set of morphisms, a set $\mathrm{Ob}(G)$ of *objects* together with functions $s, t\colon G \to \mathrm{Ob}(G)$, $\epsilon\colon \mathrm{Ob}(G) \to G$ such that $s\epsilon = t\epsilon = 1_{\mathrm{Ob}(G)}$, the identity map. The functions s, t are called *initial* and *final* point maps respectively. If $a, b \in G$ and $t(a) = s(b)$, then the *product* or *composite* ba exists such that $s(ba) = s(a)$ and $t(ba) = t(b)$. Further, this composite is associative, for $x \in \mathrm{Ob}(G)$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and each element a has an inverse $\bar{a}$ such that $s(\bar{a}) = t(a)$, $t(\bar{a}) = s(a)$, $a\bar{a} = (\epsilon t)(a)$, $\bar{a}a = (\epsilon s)(a)$. The map $G \to G$, $a \mapsto \bar{a}$, is called the *inversion*. So a group is a groupoid with only one object.

In a groupoid G for $x, y \in \mathrm{Ob}(G)$ we write $G(x,y)$ for the set of all morphisms with initial point x and final point y. We say G is *connected* if for all $x, y \in \mathrm{Ob}(G)$, $G(x,y)$ is not empty and *simply connected* if $G(x,y)$ has only one morphism. For $x \in \mathrm{Ob}(G)$ we denote the star $\{a \in G \mid s(a) = x\}$ of x by G_x. The *object group* at x is $G(x) = G(x,x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f\colon H \to G$ and $O_f\colon \mathrm{Ob}(H) \to \mathrm{Ob}(G)$ such that $s \circ f = O_f \circ s$, $t \circ f = O_f \circ t$ and $f(ba) = f(b)f(a)$ for all $(a,b) \in H_t \times_s H$. For such a morphism we simply write $f\colon H \to G$.

2.1. Definition. Let $p\colon \widetilde{G} \to G$ be a morphism of groupoids. Then p is called a *covering morphism* and $\widetilde{G}$ *a covering groupoid* of G if for each $\tilde{x} \in \mathrm{Ob}(\widetilde{G})$ the restriction of p

$$p_x\colon (\widetilde{G})_{\tilde{x}} \to G_{p(\tilde{x})}$$

is bijective. A covering morphism $p\colon \widetilde{G} \to G$ is called *connected* if both $\widetilde{G}$ and G are connected. □

A group homomorphism $f\colon G \to H$ is a covering morphism if and only if it is an isomorphism.

A connected covering morphism $p\colon \widetilde{G} \to G$ is called *universal* if $\widetilde{G}$ covers every cover of G, i.e., if for every covering morphism $q\colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\tilde{p}\colon \widetilde{G} \to \widetilde{H}$ such that $q\tilde{p} = p$ (and hence $\tilde{p}$ is also a covering morphism), this is equivalent to that for $\tilde{x}, \tilde{y} \in \mathrm{Ob}(\widetilde{G})$ the set $\widetilde{G}(\tilde{x}, \tilde{y})$ has not more than one element.

For any groupoid morphism $p\colon \widetilde{G} \to G$ and an object $\tilde{x}$ of $\widetilde{G}$ we call the subgroup $p(\widetilde{G}(\tilde{x}))$ of $G(p\tilde{x})$ the *characteristic group* of p at $\tilde{x}$.

2.2. Example. [2, 10.2] If $p\colon \widetilde{X} \to X$ is a covering map of topological spaces, then the morphism $\pi_1(p)\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ of fundamental groupoids is a covering morphism of groupoids. □

2.3. Definition. Let $p\colon \widetilde{G} \to G$ be a covering morphism of groupoids and $q\colon H \to G$ a morphism of groupoids. If there exists a unique morphism $\tilde{q}\colon H \to \tilde{G}$ such that $q = p\tilde{q}$ we just say q *lifts* to $\tilde{q}$ by p. □

We recall the following theorem from Brown [2, 10.3.3] which gives an important criteria to have the lifting maps on covering groupoids. For a useful application of this theorem see for example the proof of Theorem 3.12.

2.4. Theorem. *Let $p\colon \widetilde{G} \to G$ be a covering morphism of groupoids, $x \in \mathrm{Ob}(G)$ and $\tilde{x} \in \mathrm{Ob}(\widetilde{G})$ such that $p(\tilde{x}) = x$. Let $q\colon K \to G$ be a morphism of groupoids such that K is connected and $z \in \mathrm{Ob}(K)$ such that $q(z) = x$. Then the morphism $q\colon K \to G$ uniquely lifts to a morphism $\tilde{q}\colon K \to \widetilde{G}$ such that $\tilde{q}(z) = \tilde{x}$ if and only if $q[K(z)] \subseteq p[\widetilde{G}(\tilde{x})]$, where $K(z)$ and $\widetilde{G}(\tilde{x})$ are the object groups.*

From Theorem 2.4 the following corollary follows.

2.5. Corollary. *Let $p\colon (\widetilde{G}(\tilde{x}) \to (G, x)$ and $q\colon (\widetilde{H}, \tilde{z}) \to (G, x)$ be connected covering morphisms with characteristic groups C and D respectively. If $C \subseteq D$, then there is a unique covering morphism $r\colon (\widetilde{G}, \tilde{x}) \longrightarrow (\widetilde{H}, \tilde{z})$ such that $p = qr$. If $C = D$, then r is an isomorphism.*

3. Homotopies of functors and weak categorical groups

In this section before main results we prove that the functors are homotopic if and only if they are naturally isomorphic. For the homotopies of functors we first need the following fact whose proof is straightforward.

3.1. Proposition. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ a functor. Then for $x\in \mathrm{Ob}(\mathcal{C})$ and $y\in\mathrm{Ob}(\mathcal{D})$ we have the induced functors*

$$F(x,-)\colon \mathcal{D}\to\mathcal{E}$$
$$F(-,y)\colon \mathcal{C}\to\mathcal{E}.$$

□

We write $\mathcal{J}$ for the simply connected groupoid whose objects are 0 and 1; and whose non-identity morphisms are $\iota\colon 0\to 1$ and $\bar{\iota}\colon 1\to 0$.

As similar to the homotopy of continuous functions, the homotopy of functors is defined as follows.

3.2. Definition. [2, p.228] Let $f,g\colon \mathcal{C}\to\mathcal{D}$ be functors. These functors are called *homotopic* and written $f\simeq g$ if there is a functor $F\colon \mathcal{C}\times\mathcal{J}\to\mathcal{D}$ such that $F(-,0)=f$ and $F(-,1)=g$ □

3.3. Proposition. [2, 6.5.10] *If the maps $f,g\colon X\to Y$ are homotopic, then the induced morphisms $\pi_1 f,\pi_1 g\colon \pi_1(X)\to\pi_1(Y)$ of the fundamental groupoids are homotopic.*

3.4. Definition. Let $f,g\colon \mathcal{C}\to\mathcal{D}$ be two functors. We call f and g are *naturally isomorphic* if there exists a natural isomorphism $\sigma\colon f\to g$. □

3.5. Theorem. *The functors $f,g\colon \mathcal{C}\to\mathcal{D}$ are homotopic in the sense of Definition 3.2 if and only if they are naturally isomorphic.*

Proof. If the functors $f,g\colon \mathcal{C}\to\mathcal{D}$ are homotopic there is a functor $F\colon \mathcal{C}\times\mathcal{J}\to\mathcal{D}$ such that $F(-,0)=f$ and $F(-,1)=g$. Since $(1_x,\iota)\colon (x,0)\to(x,1)$ is an isomorphism in $\mathcal{C}\times\mathcal{J}$ the morphism $F(1_x,\iota)\colon F(x,0)\to F(x,1)$ is an isomorphism in $\mathcal{D}$ where $F(x,0)=f(x)$ and $F(x,1)=g(x)$. We now define a natural transformation $\sigma\colon f\to g$ by $\sigma(x)=F(x,\iota)\colon f(x)\to g(x)$ for $x\in\mathrm{Ob}(\mathcal{C})$. To prove that for a morphism $\alpha\colon x\to y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} f(x) & \xrightarrow{\sigma(x)} & g(x) \\ {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle g(\alpha)} \\ f(y) & \xrightarrow[\sigma(y)]{} & g(y) \end{array}$$

is commutative we prove that the diagram

$$\begin{array}{ccc} F(x,0) & \xrightarrow{F(1_x,\iota)} & F(x,1) \\ {\scriptstyle F(\alpha,0)}\downarrow & & \downarrow{\scriptstyle F(\alpha,1)} \\ F(y,0) & \xrightarrow[F(1_y,\iota)]{} & F(y,1) \end{array}$$

is commutative. Since F is a functor

$$\begin{aligned} F(\alpha,1)\circ F(1_x,\iota) &= F((\alpha,1)\circ(1_x,\iota)) \\ &= F(\alpha\circ 1_x, 1\circ\iota) = F(\alpha,1) \end{aligned}$$

and

$$\begin{aligned} F(1_y,\iota)\circ F(\alpha,0) &= F((1_y,\iota)\circ(\alpha,0)) \\ &= F(1_y\circ\alpha, \iota\circ 0) = F(\alpha,1) \end{aligned}$$

and therefore the latter diagram is commutative. Therefore the functors f and g are naturally isomorphic.

Conversely suppose that the functors $f, g\colon \mathcal{C} \to \mathcal{D}$ are naturally isomorphic. So there is a natural transformation $\sigma\colon f \to g$ such that $\sigma_x\colon f(x) \to g(x)$ is an isomorphism for $x, y \in \mathrm{Ob}(\mathcal{C})$ and the following diagram is commutative

$$\begin{array}{ccccc} x & & f(x) & \xrightarrow{\sigma(x)} & g(x) \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle f(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle g(\alpha)} \\ y & & f(y) & \xrightarrow[\sigma(y)]{} & g(y). \end{array}$$

for $\alpha \in \mathcal{C}(x, y)$. We now define a homotopy of functors $F\colon \mathcal{C} \times \mathcal{J} \to \mathcal{D}$ as follows: Define F on objects by $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for $x \in \mathrm{Ob}(\mathcal{C})$. For $x, y, z \in \mathrm{Ob}(\mathcal{C})$ consider the following diagram of the morphisms in $\mathcal{C} \times \mathcal{J}$.

$$\begin{array}{ccc} (x,0) & \xrightarrow{(\alpha,0)} & (y,0) \\ {\scriptstyle(\alpha,\iota)}\big\downarrow & & \big\uparrow{\scriptstyle(\alpha,\bar\iota)} \\ (y,1) & \xleftarrow[(\alpha,1)]{} & (x,1) \end{array}$$

Define F on these morphisms as follow:

$$F(\alpha, 0) = f(\alpha)$$
$$F(\alpha, 1) = g(\alpha)$$
$$F(\alpha, \iota) = g(\alpha) \circ \sigma_x$$
$$F(\alpha, \bar\iota) = f(\alpha) \circ (\overline{\sigma_x}).$$

In this way a functor $F\colon \mathcal{C} \times \mathcal{J} \to \mathcal{D}$ is defined such that $F(-, 0) = f$ and $F(-, 1) = g$. Therefore the functors f and g are homotopic. □

A *group-groupoid* which is also known as 2-*group* in literature is a group object in the category of groupoids. The formal definition of a group-groupoid is given in [4] under the name *G-groupoid* as follows:

3.6. Definition. A *group-groupoid* G is a groupoid endowed with a group structure such that the following maps, which are called respectively product, inverse and unit are the morphisms of groupoids:

- $m\colon G \times G \to G$, $(a, b) \mapsto ab$;
- $u\colon G \to G$, $a \mapsto a^{-1}$;
- $e\colon \{\star\} \to G$, where $\{\star\}$ is singleton.

□

Here note that the axioms of the associativity, the unit and the inverse can be stated respectively in terms of functions as follows:

(i) $m(1 \times m) = m(m \times 1)$;
(ii) $m\,\imath_1 = 1_G = m\,\imath_2$;
(iii) $m(1, u) = m(u, 1) = e$

where $\imath_1, \imath_2\colon G \to G \times G$ are injections defined by $\imath_1(a) = (a, e)$ and $\imath_2(a) = (e, a)$; and $e\colon G \to G$ is the constant map at e.

In the definition of group-groupoid if we take these functors to be homotopic rather than equal, we obtain the definition of *weak categorical group*. There are various forms of the definitions of a categorical group in the literature (see [5] and [7]) and we will use the following one without coherence conditions and call *weak categorical group*.

3.7. Definition. Let $\mathcal{G}$ be a groupoid. Let $\otimes\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ and $u\colon \mathcal{G}\to\mathcal{G}, a\mapsto a^{-1}$ be functors called respectively product and inverse. Let $e\in \mathrm{Ob}(\mathcal{G})$ be an object. If the following conditions are satisfied then we call $(\mathcal{G},\otimes,u,e)$ a *weak categorical group* and write just $\mathcal{G}$.

(1) The functors $\otimes(1\times\otimes), \otimes(1\times\otimes)\colon \mathcal{G}\times\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ are homotopic.
(2) The functors $e\otimes 1, 1\otimes e\colon \mathcal{G}\to\mathcal{G}$ defined by $(e\otimes 1)(a)=e\otimes a$ and $(1\times e)(a)=a\otimes e$ for $a\in\mathcal{G}$ are homotopic to the identity functor $\mathcal{G}\to\mathcal{G}$.
(3) The functors $\otimes(1,u), \otimes(u,1)\colon \mathcal{G}\to\mathcal{G}$ defined by $\otimes(1,u)(a)=a\otimes u(a)$ and $\otimes(u,1)(a)=u(a)\otimes a$ are homotopic to the constant functor $e\colon \mathcal{G}\to\mathcal{G}$.

□

In this definition if these functors are equal rather than homotopic, then the weak categorical group is called a *strict categorical group* which is a group-groupoid.

Note that the product $\otimes\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is a functor if and only if

$$(b\circ a)\otimes(d\circ c)=(b\otimes d)\circ(a\otimes c) \qquad (3.1)$$

for $a,b,c,d\in\mathcal{G}$ whenever the compositions $b\circ a$ and $d\circ c$ are defined. Since $u\colon\mathcal{G}\to\mathcal{G}, a\mapsto a^{-1}$ is a functor when the groupoid composition $b\circ a$ is defined $(b\circ a)^{-1}=b^{-1}\circ a^{-1}$ and ${1_x}^{-1}=1_{x^{-1}}$ for $x\in\mathrm{Ob}(\mathcal{G})$.

3.8. Theorem. *If (X,x_0) is an H-group, then the fundamental group $\pi_1(X)$ is a weak categorical group.*

Proof. Since (X,x_0) is an H-group by Definition 1.3 there are continuous maps

$$m\colon (X\times X,(x_0,x_0))\to(X,x_0)$$

$$n\colon (X,x_0)\to(X,x_0)$$

and there are the following homotopies of the maps:

(i) $m(1_X\times m)\simeq m(m\times 1_X)$;
(ii) $m\,\imath_1\simeq 1_X\simeq m\,\imath_2$;
(iii) $m(1_X,n)\simeq c\simeq m(n,1_X)$

where $\imath_1,\imath_2\colon X\to X\times X$ are injections defined by $\imath_1(x)=(x,x_0)$, $\imath_2(x)=(x_0,x)$ and $c\colon X\to X$ is the constant map at x_0. From m and n we have the following induced functors

$$\tilde{m}\colon \pi_1(X)\times\pi_1(X)\to\pi_1(X)$$

and

$$\tilde{n}\colon \pi_1(X)\to\pi_1(X).$$

By Proposition 3.3 from the above homotopies (i), (ii) and (iii), the following homotopies of the functors are obtained:

(i) $\tilde{m}(1\times\tilde{m})\simeq\tilde{m}(\tilde{m}\times 1)$;
(ii) $\tilde{m}\,\tilde{\imath}_1\simeq 1_{\pi_1X}\simeq\tilde{m}\,\tilde{\imath}_2$;
(iii) $\tilde{m}(1_{\pi_1X},\tilde{n})\simeq\pi_1c\simeq\tilde{m}(\tilde{n},1_{\pi_1X})$.

where $\tilde{\imath}_1$ and $\tilde{\imath}_2$ are respectively the morphisms induced by $\imath_1$ and $\imath_2$. Therefore $\pi_1(X)$ becomes a weak categorical group. □

3.9. Definition. Let $\mathcal{G}$ and $\mathcal{H}$ be two weak categorical groups. A morphism of weak categorical groups is a morphism $f\colon \mathcal{H} \to \mathcal{G}$ of groupoids such that the functors $f\otimes, \otimes(f\times f)\colon \mathcal{H}\times\mathcal{H} \to \mathcal{G}$ are homotopic and $f(e_{\mathcal{H}})$ is isomorphic to $e_{\mathcal{G}}$, where $e_{\mathcal{G}}$ and $e_{\mathcal{H}}$ are respectively the base points of $\mathcal{G}$ and $\mathcal{H}$. □

By Theorem 3.5 the condition that the functors $f\otimes, \otimes(f\times f)\colon \mathcal{H}\times\mathcal{H} \to \mathcal{G}$ are homotopic, is equivalent to that $f(x\otimes y)$ is naturally isomorphic to $f(x)\otimes f(y)$ for all $x, y \in \mathrm{Ob}(\mathcal{H})$.

So we have a category CatGrp of weak categorical groups.

The proof of the following proposition is immediate and therefore it is omitted.

3.10. Proposition. *If $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a morphism of H-groups, then the induced map $\pi_1 p\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ is a morphism of weak categorical groups.*

Let (X, x_0) be an H-group. Then we have a category $\mathsf{Cov_{HGrp}}/(\mathsf{X},\mathsf{x_0})$ of H-group morphisms $f\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ which are covering maps on the underlying spaces. So a morphism from $f\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ to $g\colon (\widetilde{Y}, \tilde{y}_0) \to (X, x_0)$ is a continuous map $p\colon (\widetilde{X}, \tilde{x}_0) \to (\widetilde{Y}, \tilde{y}_0)$ which becomes also a covering map, such that $f = gp$.

Similarly we have an other category $\mathsf{Cov_{CatGrp}}/\pi_1\mathsf{X}$ of morphisms $p\colon \widetilde{G} \to \pi_1(X)$ of weak categorical groups, which are covering morphisms on the underlying groupoids.

3.11. Theorem. *Let (X, x_0) be an H-group such that the underlying space has a simply connected cover. Then the categories $\mathsf{Cov_{HGrp}}/(\mathsf{X},\mathsf{x_0})$ and $\mathsf{Cov_{CatGrp}}/\pi_1\mathsf{X}$ are equivalent.*

Proof. Let $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a morphism of H-groups which is a covering map on the spaces. Then by Proposition 3.10 the induced morphism $\pi_1 p\colon (\pi_1\widetilde{X}) \to \pi_1(X)$ is a morphism of weak categorical groups which is a covering morphism of underlying groupoids. So in this way we have a functor

$$\pi_1\colon \mathsf{Cov_{HGrp}}/(\mathsf{X},\mathsf{x_0}) \to \mathsf{Cov_{CatGrp}}/\pi_1\mathsf{X}.$$

Conversely we define a functor

$$\eta\colon \mathsf{Cov_{CatGrp}}/\pi_1\mathsf{X} \to \mathsf{Cov_{HGrp}}/(\mathsf{X},\mathsf{x_0})$$

as follows:

Let $q\colon \widetilde{G} \to \pi_1(X)$ be a morphism of weak categorical groups which is a covering morphism on the underlying groupoids. Then by [2, 9.5.5] there is a topology on $\widetilde{X} = \mathrm{Ob}(\widetilde{G})$ and an isomorphism $\alpha\colon \widetilde{G} \to \pi_1(\widetilde{X})$ such that $p = O_q\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map and $q = \pi_1(p)\circ\alpha$. Hence weak categorical group structure on $\widetilde{G}$ transports via α to $\pi_1(\widetilde{X})$. So we have the morphisms of groupoids

$$\widetilde{m}\colon \pi_1(\widetilde{X}) \times \pi_1(\widetilde{X}) \longrightarrow \pi_1(\widetilde{X})$$

$$\widetilde{n}\colon \pi_1(\widetilde{X}) \longrightarrow \pi_1(\widetilde{X})$$

such that $\pi_1(p)\circ\widetilde{m} = m\circ(\pi_1(p)\times\pi_1(p))$ and $n\pi_1(p) = \pi_1(p)\widetilde{n}$. From these morphisms we obtain the maps

$$\widetilde{m}\colon \widetilde{X}\times\widetilde{X} \longrightarrow \widetilde{X}$$

$$\widetilde{n}\colon \widetilde{X} \longrightarrow \widetilde{X}.$$

Since (X, x_0) is an H-group with the maps

$$m\colon (X\times X, (x_0, x_0)) \to (X, x_0)$$

$$n\colon (X, x_0) \to (X, x_0)$$

we have the following homotopies of pointed maps:

(i) $m(1_X\times m) \simeq m(m\times 1_X)$;

(ii) $m\,\imath_1 \simeq 1_X \simeq m\,\imath_2$;
(iii) $m(1_X, n) \simeq c \simeq m(n, 1_X)$.

Then by Theorem 1.2 we have the following homotopies:

(i) $\widetilde{m}(1_{\widetilde{X}} \times \widetilde{m}) \simeq \widetilde{m}(\widetilde{m} \times 1_{\widetilde{X}})$;
(ii) $\widetilde{m}\,\imath_1 \simeq 1_{\widetilde{X}} \simeq \widetilde{m}\,\imath_2$;
(iii) $\widetilde{m}(1_{\widetilde{X}}, \tilde{n}) \simeq c \simeq \widetilde{m}(\tilde{n}, 1_{\widetilde{X}})$.

Therefore $(\widetilde{X}, \tilde{x}_0)$ is an H-group and $q = O_p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering morphism of H-groups.

If $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map on underlying spaces, then by [2, 9.5.5] the topology of $\widetilde{X}$ is that of X lifted by the covering morphism $\pi_1 p\colon \pi_1\widetilde{X} \to \pi_1 X$ and so $\eta\pi_1 = 1$. Further if $q\colon \widetilde{G} \to \pi_1(X)$ is a morphism of weak categorical groups, then for the lifted topology on $\widetilde{X}$, $\widetilde{G}$ is isomorph to $\pi_1\widetilde{X}$ and so $\eta\pi_1 \simeq 1$. Therefore these functors give an equivalence of the categories. □

As similar to Theorem 1.2, in the following theorem we prove that the liftings of homotopic functors are also homotopic.

3.12. Theorem. *Let $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ be a covering morphism of groupoids. Suppose that K is a simply connected groupoid, i.e., for each $x, y \in O_K$, $K(x, y)$ has only one morphism. Let $f, g\colon (K, z) \to (G, x)$ be the morphisms of groupoids such that f and g are homotopics. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g respectively. Then $\tilde{f}$ and $\tilde{g}$ are also homotopic.*

Proof. Since the functors f and g are homotopic, there is a functor $F\colon K \times \mathcal{J} \to G$ such that $F(-, 0) = f$ and $F(-, 1) = g$. Since K is a simply connected groupoid by Theorem 2.4 there is a functor $\widetilde{F}\colon (K \times \mathcal{J}, (z, 0)) \to (\widetilde{G}, \tilde{x})$ such that $p\widetilde{F} = F$. Hence $p\widetilde{F}(-, 0) = F(-, 0) = f$ and $p\widetilde{F}(-, 1) = F(-, 1) = g$. So by the uniqueness of the liftings we have that $\widetilde{F}(-, 0) = \tilde{f}$ and $\widetilde{F}(-, 1) = \tilde{g}$. Therefore $\tilde{f}$ and $\tilde{g}$ are homotopic. □

3.13. Definition. Let $\mathcal{G}$ be a weak categorical group, $e \in \mathrm{Ob}(\mathcal{G})$ the base point and let $\widetilde{G}$ be just a groupoid. Suppose $p\colon \widetilde{G} \to \mathcal{G}$ is a covering morphism of groupoids and $\tilde{e} \in \mathrm{Ob}(\widetilde{G})$ such that $p(\tilde{e}) = e$. We say that the weak categorical group structure of $\mathcal{G}$ *lifts* to $\widetilde{G}$ if there exists a weak categorical group structure on $\widetilde{G}$ with the base point $\tilde{e} \in \mathrm{Ob}(\widetilde{G})$ such that $p\colon \widetilde{G} \to \mathcal{G}$ is a morphism of weak categorical groups □

3.14. Theorem. *Let $\widetilde{G}$ be a simply connected groupoid and $\mathcal{G}$ a weak categorical group. Suppose that $p\colon \widetilde{G} \to \mathcal{G}$ is a covering morphism on the underlying groupoids. Let $e \in$ $\mathrm{Ob}(G)$ be the base point of $\mathcal{G}$ and $\tilde{e} \in O_{\widetilde{G}}$ such that $p(\tilde{e}) = e$. Then the weak categorical group structure of $\mathcal{G}$ lifts to $\widetilde{G}$.*

Proof. Since $\mathcal{G}$ is weak categorical group, we have the following functors

- $\otimes\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, (a, b) \mapsto a \otimes b$;
- $u\colon \mathcal{G} \to \mathcal{G}, a \mapsto a^{-1}$;
- $e\colon \{\star\} \to \mathcal{G}$

such that the following functors are homotopic:

1. $\otimes(1 \times \otimes) \simeq \otimes(\otimes \times 1)$;
2. $\otimes\imath_1 \simeq \otimes\imath_2 \simeq 1_{\mathcal{G}}$;
3. $\otimes(1, u) \simeq c \simeq \otimes(u, 1)$.

Since $\widetilde{G}$ is a simply connected groupoid by Theorem 2.4 the functors $\otimes$ and u lift respectively to the morphisms of groupoids

$$\tilde{\otimes}\colon (\widetilde{G} \times \widetilde{G}, (\tilde{e}, \tilde{e})) \to (\widetilde{G}, \tilde{e})$$

and

$$\tilde{u}\colon (\widetilde{G}, \tilde{e}) \to (\widetilde{G}, \tilde{e})$$

By Theorem 3.12 we have the following homotopies of the functors:

1. $\tilde{\otimes}(1 \times \tilde{\otimes}) \simeq \tilde{\otimes}(\tilde{\otimes} \times 1)$;
2. $\tilde{\otimes}\, \imath_1 \simeq \tilde{\otimes}\, \imath_2 \simeq 1_{\tilde{G}}$;
3. $\tilde{\otimes}(1, \tilde{u}) \simeq c \simeq \tilde{\otimes}(\tilde{u}, 1)$.

Therefore $\widetilde{G}$ is a week categorical group as required. □

As it is stated in the introduction there is a well known result for topological groups that the group structure of a topological group lifts to its a simply connected covering space. See [1] for a lifting of R-module to the covering space. We now give a similar result for H-groups as a result of Theorem 3.14.

3.15. Corollary. *Let (X, x_0) be an H-group and $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map. If $\widetilde{X}$ is a simply connected topological space, then H-group structure of (X, x_0) lifts to $(\widetilde{X}, \tilde{x}_0)$, i.e., $(\widetilde{X}, \tilde{x}_0)$ is an H-group and $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a morphism of H-groups.*

Proof. Since $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, the induced morphism $\pi_1 p\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ is a covering morphism of groupoids. Since (X, x_0) is an H-group by Theoremn 3.8 $\pi_1(X)$ is a weak categorical group and since $\widetilde{X}$ is simpy connected the fundamental groupoid $\pi_1 \widetilde{X}$ is a simply connected groupoid. So by Theorem 3.14 the weak categorical group structure of $\pi_1(X)$ lifts to $\pi_1(\widetilde{X})$. So we have the groupoid morphisms

$$\tilde{m}\colon \pi_1(\widetilde{X}) \times \pi_1(\widetilde{X}) \longrightarrow \pi_1(\widetilde{X})$$

$$\tilde{n}\colon \pi_1(\widetilde{X}) \longrightarrow \pi_1(\widetilde{X})$$

such that $\pi_1(p) \circ \tilde{m} = m \circ (\pi_1(p) \times \pi_1(p))$ and $n\pi_1(p) = \pi_1(p)\tilde{n}$ and therefore we have the maps

$$\tilde{m}\colon \widetilde{X} \times \widetilde{X} \longrightarrow \widetilde{X}$$

$$\tilde{n}\colon \widetilde{X} \longrightarrow \widetilde{X}.$$

Since (X, x_0) is an H-group, we have the homotopies:

(i) $m(1_X \times m) \simeq m(m \times 1_X)$;
(ii) $m\, \imath_1 \simeq 1_X \simeq m\, \imath_2$;
(iii) $m(1_X, n) \simeq c \simeq m(n, 1_X)$

and so by Theorem 1.2 we have the homotopies:

(i) $\tilde{m}(1_X \times \tilde{m}) \simeq \tilde{m}(\tilde{m} \times 1_X)$;
(ii) $\tilde{m}\, \imath_1 \simeq 1_X \simeq \tilde{m}\, \imath_2$;
(iii) $\tilde{m}(1_X, \tilde{n}) \simeq c \simeq \tilde{m}(\tilde{n}, 1_X)$.

Therefore $(\widetilde{X}, \tilde{x_0})$ is an H-group and $q = \mathrm{Ob}(p)\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering morphism of H-groups. □

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