EXACT MOMENTS OF GENERALIZED ORDER STATISTICS FROM TYPE II EXPONENTIATED LOG-LOGISTIC DISTRIBUTION

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Abstract
In this paper some new simple expressions for single and product moments of
generalized order statistics from type II exponentiated log-logistic distribution have
been obtained. The results for order statistics and record values are deduced from the
relations derived and some ratio and inverse moments of generalized order statistics
are also carried out. Further, a characterization result of this distribution by using the
conditional expectation of generalized order statistics is discussed.

Key words: Exact moments, ratio and inverse moments, generalized order
statistics, order statistics, upper record values, type II exponentiated
log-logistic distribution and characterization.

AMS Subject Classification: 62G30, 62E10

1 Introduction
A random variable \( X \) is said to have type II exponentiated log-logistic distribution if
its probability density function (pdf) is given by

\[
f(x) = \frac{\alpha \beta (x/\sigma)^{\beta-1}}{\sigma [1+(x/\sigma)^\beta]^{\alpha+1}}, \quad x \geq 0, \; \sigma > 0, \; \beta > 1, \; \alpha > 0
\]  

and the corresponding survival function is

\[
F(x) = \left(1+\left(\frac{x}{\sigma}\right)^\beta\right)^{-\alpha}, \quad x \geq 0, \; \sigma > 0, \; \beta > 1, \; \alpha > 0.
\]

It is easy to see that

\[
\alpha \beta F(x) = \sigma [1+1/(x/\sigma)^\beta] x f(x).
\]

Log-logistic distribution is considered as a special case of type II exponentiated log-logistic distribution when \( \alpha = 1 \). It is used in survival analysis as a parametric model where in the mortality rate first increases then decreases, for example in cancer diagnosis or any other type of treatment. It has also been used in hydrology to model stream flow and precipitation, and in economics to model the distribution of wealth or income.

Kamps (1995) introduced the concept of generalized order statistics (gos) as follows: Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed (iid) random

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variables \((rv)\) with absolutely continuous cumulative distribution function \((cdf)\) \(F(x)\) and \(pdf\) \(f(x), x \in (\alpha, \beta)\). Let \(n \in \mathbb{N}\), \(n \geq 2\), \(\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}, k > 0\), be the parameters such that

\[
\gamma_r = k + (n - r) + M_r > 0 \text{ for all } r \in \{1, 2, \ldots, n - 1\},
\]
where \(M_r = \sum_{j=r}^{n-1} m_j\). Then \(X(r, n, \tilde{m}, k)\), \(r = 1, 2, \ldots, n\) are called \(gos\) if their joint \(pdf\) is given by

\[
k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) \right) [1 - F(x_n)]^{k-1} f(x_n)
\] (1.4)

on the cone \(F^{-1}(0) \leq x_1 \cdots \leq x_n \leq F^{-1}(1)\).

The model of \(gos\) contains as special cases, order statistics, record values, sequential order statistics.

Choosing the parameters appropriately (Cramer, 2002), we get:

**Table 1.1: Variants of the generalized order statistics**

<table>
<thead>
<tr>
<th></th>
<th>(\gamma_n = k)</th>
<th>(\gamma_r)</th>
<th>(m_r)</th>
</tr>
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<tbody>
<tr>
<td>i)</td>
<td>Sequential order statistics</td>
<td>(\alpha_n)</td>
<td>((n-r+1)\alpha_r)</td>
</tr>
<tr>
<td>ii)</td>
<td>Ordinary order statistics</td>
<td>1</td>
<td>(n-r+1)</td>
</tr>
<tr>
<td>iii)</td>
<td>Record values</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>iv)</td>
<td>Progressively type II censored order statistics</td>
<td>(R_n + 1)</td>
<td>(n-r+1 + \sum_{j=r}^{n} R_j)</td>
</tr>
<tr>
<td>v)</td>
<td>Pfeifer’s record values</td>
<td>(\beta_n)</td>
<td>(\beta_r)</td>
</tr>
</tbody>
</table>

For simplicity we shall assume \(m_1 = m_2 = \cdots = m_{n-1} = m\).

The \(pdf\) of the \(r\)th \(gos\), \(X(r, n, m, k), 1 \leq r \leq n\), is

\[
f_{X(r, n, m, k)}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))
\] (1.5)

and the joint \(pdf\) of \(X(r, n, m, k)\) and \(X(s, n, m, k), 1 \leq r < s \leq n\), is

\[
f_{X(r, n, m, k) \times X(s, n, m, k)}(x, y) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x < y.
\] (1.6)
where
\[ \bar{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n - i)(m + 1), \]
\[ h_m(x) = \begin{cases} \frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\ln(1-x), & m = -1 \end{cases} \]
and
\[ g_m(x) = h_m(x) - h_m(0), \quad x \in [0,1). \]


Aggarawala and Balakrishnan (1996) established recurrence relations for single and product moments of progressive type II right censored order statistics from exponential and truncated exponential distributions. Balasooriya and Saw (1998) develop reliability sampling plans for the two parameter exponential distribution under progressive censoring. Balakrishnan et al. (2001) obtained bounds for the mean and variance of progressive type II censored order statistics. Ordinary via truncated distributions and censoring schemes and particularly progressive type II censored order statistics have been discuss by Kamps (1995) and Balakrishnan and Aggarwala (2000), among others.

Kamps (1995) investigated recurrence relations for moments of \( \text{gos} \) based on non-identically distributed random variables, which contains order statistics and record values as special cases. Cramer and Kamps (2000) derived relations for expectations of functions of \( \text{gos} \) within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases. Various developments on \( \text{gos} \) and related topics have been studied by Kamps and Gather (1997), Ahsanullah (2000), Pawlas and Szynal (2001), Kamps and Cramer (2001), Ahmad and Fawzy (2003), Ahmad (2007), Kumar (2010, 2011 and 2013) among others. Characterizations based on \( \text{gos} \) have been studied by some authors, Keseling (1999) characterized some continuous distributions based on conditional distributions of \( \text{gos} \). Bieniek and Szynal (2003) characterized some distributions via linearity of regression of \( \text{gos} \). Cramer et al. (2004) gave a unifying approach on characterization via linear regression of ordered random variables. Khan et al. (2006) characterized some continuous distributions through conditional expectation of functions of \( \text{gos} \).

The aim of the present study is to give some explicit expressions and recurrence relations for single and product moments of \( \text{gos} \) from type II exponentiated log-logistic distribution. In Section 2 we give the explicit expressions and recurrence relations for single moments of type II exponentiated log-logistic distribution and some inverse moments of \( \text{gos} \) are also worked out. Then we show that results for
order statistics and record values are deduced as special cases. In Section 3 we present the explicit expressions and recurrence relations for product moments of type II exponentiated log-logistic distribution and we show that results for order statistics and record values are deduced as special cases and ratio moments of $gos$ are also established. Section 4 provides a characterization result on type II exponentiated log-logistic distribution based on conditional moment of $gos$. Two applications are performed in Section 5. Some concluding remarks are given in Section 6.

2. Relations for Single Moments

In this Section, the explicit expressions, recurrence relations for single moments of $gos$ and inverse moments of $gos$ are considered. First we need the basic result to prove the main Theorem.

**Lemma 2.1:** For type II exponentiated log-logistic distribution as given in (1.2) and any non-negative and finite integers $a$ and $b$ with $m \neq -1$

$$J_j(a,0) = \alpha \sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p (j/\beta)_{(p)}}{\alpha(a+1) + p - (j/\beta)}, \quad \beta > j \quad \text{and} \quad j = 0, 1, \ldots, \quad (2.1)$$

where

$$a_{(i)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+i-1), & i > 0 \\ 1, & i = 0 \end{cases}$$

and

$$J_j(a,b) = \int_0^{\infty} x^j [\overline{F}(x)]^a f(x) g_m^b(F(x)) \, dx. \quad (2.2)$$

**Proof:** From (2.2), we have

$$J_j(a,0) = \int_0^{\infty} x^j [\overline{F}(x)]^a f(x) \, dx. \quad (2.3)$$

By making the substitution $z = [\overline{F}(x)]^{1/\alpha}$ in (2.3), we get

$$J_j(a,0) = \alpha \sigma^j \int_0^1 (1-z)^{j/\beta} z^{a(a+1)-(j/\beta)-1} \, dz$$

$$= \alpha \sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p (j/\beta)_{(p)}}{\alpha(a+1) - (j/\beta) + p - 1} \, dz$$

and hence the result given in (2.1).

**Lemma 2.2:** For the type II exponentiated log-logistic distribution as given in (1.2) and any non-negative finite integers $a$ and $b$

$$J_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} J_j(a+u(m+1),0) \quad (2.4)$$

$$= \frac{\alpha \sigma^j}{(m+1)^b} \sum_{p=0}^{b} \sum_{u=0}^{p+u} (-1)^{p+u} \binom{b}{u}$$

$$\times \frac{(j/\beta)_{(p)}}{\alpha(a+u(m+1)+1) + p - (j/\beta)}, \quad m \neq -1 \quad (2.5)$$

$$= b! a^{b+1} \sigma^j \sum_{p=0}^{\infty} \frac{(j/\beta)_{(p)}}{\alpha(a+1) + p - (j/\beta)}^{b+1}, \quad m = -1, \quad (2.6)$$

where $J_j(a,b)$ is as given in (2.2).
Proof: On expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} \{1 - (\bar{F}(x))^{m+1}\}\right]^b \) binomially in (2.2), we get when \( m \neq -1 \)

\[
J_j(a, b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_0^\infty x^j \left(\bar{F}(x)\right)^{a+u(m+1)} f(x) \, dx
\]

\[
= \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} J_j(a + u(m+1), 0) .
\]

Making use of Lemma 2.1, we establish the result given in (2.5) and when \( m = -1 \) that

\[
J_j(a, b) = \frac{0}{0}, \quad \text{as} \quad \sum_{u=0}^{b} (-1)^u \binom{b}{u} = 0 .
\]

Since (2.5) is of the form \( \frac{0}{0} \) at \( m = -1 \), therefore, we have

\[
J_j(a, b) = A \frac{\sum_{u=0}^{b} (-1)^u \binom{b}{u}}{\alpha^{a+u(m+1)} + p - (j/\beta)^{b+1}}
\]

(2.7)

where

\[
A = \alpha \sigma^j \sum_{p=0}^{\infty} (-1)^p (j/\beta)_{(p)} .
\]

Differentiating numerator and denominator of (2.7) \( b \) times with respect to \( m \), we get

\[
J_j(a, b) = A \alpha^b \frac{b \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{u^b}{\alpha^{a+u(m+1)} + p - (j/\beta)^{b+1}}}{u^b}
\]

On applying the L’ Hospital rule, we have

\[
\lim_{m \to -1} J_j(a, b) = A \alpha^b \frac{b \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{u^b}{\alpha(a+1) + p - (j/\beta)^{b+1}}}{u^b}
\]

(2.8)

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz (1996)

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x - i)^n = n! .
\]

(2.9)

Therefore,

\[
\sum_{u=0}^{b} (-1)^u \binom{b}{u} u^b = b! .
\]

(2.10)

Now on substituting (2.10) in (2.8), we have the result given in (2.6).
Theorem 2.1: For type II exponentiated log-logistic distribution as given in (1.2) and $1 \leq r \leq n$, $k = 1, 2, \ldots, m \neq -1$

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} J_j(\gamma_r - 1, r - 1)$$

$$= \frac{\alpha \sigma^j C_{r-1}}{(r-1)! (m+1)r^{-1}} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{p+u} \binom{r-1}{u} \frac{(j/\beta)_p}{(\alpha\gamma_{r-u} + p - (j/\beta))},$$

$$\beta > j \text{ and } j = 0, 1, 2, \ldots$$

(2.11)

where $J_j(\gamma_r - 1, r - 1)$ is as defined in (2.2).

Proof: From (1.5) and (2.2), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} J_j(\gamma_r - 1, r - 1).$$

Making use of Lemma 2.2, we establish the relation given in (2.12).

Identity 2.1: For $\gamma_r \geq 1$, $k \geq 1$, $1 \leq r \leq n$ and $m \neq -1$

$$\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_r - u} \prod_{i=1}^{r} \gamma_i = \frac{(r-1)! (m+1)^{r-1}}{(r-1)! \gamma_r^{r-1}}.$$  

(2.13)

Proof: (2.13) can be proved by setting $j = 0$ in (2.12).

Special cases

i) Putting $m = 0$, $k = 1$ in (2.12), the explicit formula for the single moments of order statistics of the type II exponentiated log-logistic distribution can be obtained as

$$E[X^j(r,n,0,1,k)] = \alpha \sigma^j C_{r,n} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{p+u} \binom{r-1}{u} \frac{(j/\beta)_p}{(\alpha(n+r+1+u) + p - (j/\beta))},$$

where

$$C_{r,n} = \frac{n!}{(r-1)! (n-r)!}.$$  

ii) Putting $m = -1$ in (2.12), we deduce the explicit expression for the single moments of upper $k$ record values for type II exponentiated log-logistic distribution in view of (2.11) and (2.6) in the form

$$E[X^j(r,n,-1,k)] = E[(Z_r^{(k)})^j]$$

$$= (\alpha k)^j \sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p (j/\beta)_p}{[\alpha k + p - (j/\beta)]^r}$$

and hence for upper records

$$E[(Z_r^{(k)})^j] = E[X^j(r,n,-1,k)]$$

$$= (\alpha k)^j \sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p (j/\beta)_p}{[\alpha + p - (j/\beta)]^r}.$$
Recurrence relations for single moments of gos from (1.5) can be obtained in the following theorem.

**Theorem 2.2** For the distribution given in (1.2) and for $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$,
\[
\left(1 - \frac{\sigma}{\alpha \beta \gamma_r}\right)E[X^j (r, n, m, k)] = E[X^j (r-1, n, m, k)]
+ \frac{j \sigma^{\beta+1}}{\alpha \beta \gamma_r} E[X^{j-\beta} (r, n, m, k)].
\] (2.14)

**Proof** From (1.5), we have
\[
E[X^j (r, n, m, k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) dx.
\] (2.15)

Integrating by parts treating $[\bar{F}(x)]^{\gamma_r-1} f(x)$ for integration and the rest of the integrand for differentiation, we get
\[
E[X^j (r, n, m, k)] = E[X^j (r-1, n, m, k)] + \frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1} (F(x)) dx
\]
the constant of integration vanishes since the integral considered in (2.15) is a definite integral. On using (1.3), we obtain
\[
E[X^j (r, n, m, k)] - E[X^j (r-1, n, m, k)]
= \frac{\sigma}{\alpha \beta \gamma_r (r-1)!} \int_0^\infty x^{j} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1} (F(x)) dx
+ \frac{\sigma^{\beta+1} j C_{r-1}}{\alpha \beta \gamma_r (r-1)!} \int_0^\infty x^{j-\beta} [\bar{F}(x)]^{\gamma_r} g_m^{r-1} (F(x)) dx
\]
and hence the result given in (2.14).

**Remark 2.2** Setting $m = 0$, $k = 1$ in (2.14), we obtain a recurrence relation for single moments of order statistics for type II exponentiated log-logistic distribution in the form
\[
\left(1 - \frac{\sigma}{\alpha \beta (n-r+1)}\right)E[X^j_{r,n}] = E[X^j_{r-1,n}] + \frac{j \sigma^{\beta+1}}{\alpha \beta (n-r+1)} E[X^{j-\beta}_{r,n}].
\]

**Remark 2.3** Putting $m = -1$, in Theorem 2.2, we get a recurrence relation for single moments of upper $k$ record values from type II exponentiated log-logistic distribution in the form
\[
\left(1 - \frac{\sigma}{\alpha \beta k}\right)E[(X^{(k)}_{U(r)})^j] = E[(X^{(k)}_{U(r-1)})^j] + \frac{j \sigma^{\beta+1}}{\alpha \beta k} E[(X^{(k)}_{U(r)})^{j-\beta}].
\]
Inverse moments of $gos$ from type II exponentiated log-logistic distribution can be obtained by the following Theorem.

**Theorem 2.3:** For type II exponentiated log-logistic distribution as given in (1.2) and $1 \leq r \leq n$, $k = 1, 2, \ldots$,

$$E[X^{1-\beta}(r, n, m, k)] = \sigma^{1-\beta} \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma \left( \beta \right)}{p! \Gamma \left( \beta - p \right)} \frac{1}{\prod_{i=1}^{r} \left( 1 + \frac{p + 1 - (j/\beta)}{\alpha \gamma_i} \right)} , \; \beta > j . \quad (2.16)$$

**Proof:** From (1.5), we have

$$E[X^{1-\beta}(r, n, m, k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}$$

$$\times \int_0^\infty x^{1-\beta} [\bar{F}(x)]^{r-u-1} f(x) \, dx . \quad (2.17)$$

Now letting $t = [\bar{F}(x)]^{1/\alpha}$ in (2.17), we get

$$E[X^{1-\beta}(r, n, m, k)] = \frac{\sigma^{1-\beta} C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{p=0}^{\infty} (-1)^u \binom{r-1}{u} \frac{\Gamma \left( \beta \right)}{p! \Gamma \left( \beta - p \right)}$$

$$\times B \left( \frac{k}{m+1} + n - r + u + \frac{p + 1 - (j/\beta)}{\alpha(m+1)} , 1 \right) .$$

Since

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} B(a + k , \; c) = B(k , \; c + b) \quad (2.18)$$

where $B(a, b)$ is the complete beta function.

Therefore,

$$E[X^{1-\beta}(r, n, m, k)] = \frac{\sigma^{1-\beta} C_{r-1}}{(m+1)^r} \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma \left( \beta \right)}{p! \Gamma \left( \beta - p \right)}$$

$$\times B \left( \frac{\alpha(k + (n-r)(m+1)) + p + 1 - (j/\beta)}{\alpha(m+1)} , 1 \right) . \quad (2.19)$$

and hence the result given in (2.16).
Special cases

iii) Putting \( m = 0, \ k = 1 \) in (2.19), we get inverse moments of order statistics from type II exponentiated log-logistic distribution as:

\[
E[X_{j}^{j-\beta}] = \sigma^{j-\beta} \sum_{p=0}^{\infty} (\frac{-1}{p!}) \Gamma\left(\frac{j}{\beta}\right) \frac{1}{\Gamma\left(\alpha(n-r+1)+p+1-(j/\beta)\right)} \frac{\Gamma\left(\alpha(n+1)+p+1-(j/\beta)\right)}{\Gamma\left(\alpha(n-r+1)+p+1-(j/\beta)\right)}.
\]

iv) Putting \( m = -1 \) in (2.16), to get inverse moments of \( k \) record values from type II exponentiated log-logistic distribution as:

\[
E(X_{U(r)}^{j-\beta}) = \sigma^{j-\beta} \sum_{p=0}^{\infty} (\frac{-1}{p!}) \Gamma\left(\frac{j}{\beta}\right) \frac{1}{\Gamma\left(\alpha(n-r+1)+p+1-(j/\beta)\right)} \frac{\Gamma\left(\alpha(n+1)+p+1-(j/\beta)\right)}{\Gamma\left(\alpha(n-r+1)+p+1-(j/\beta)\right)}.
\]

Recurrence relations for inverse moments of \( gos \) from (1.2) can be obtained in the following theorem.

**Theorem 2.4:** For type II exponentiated log-logistic distribution and for \( 2 \leq r \leq n, \ n \geq 2 \) and \( k = 1, 2, \ldots, \)

\[
\left(1 - \frac{\sigma(j-\beta)}{\alpha \beta \gamma_r}\right) E[X_{j}^{j-\beta}(r,n,m,k)] = E[X_{j}^{j-\beta}(r-1,n,m,k)]
\]

\[
+ \frac{(j-\beta)\sigma^{j+1}}{\alpha \beta \gamma_r} E[X_{j}^{j-2\beta}(r,n,m,k)], \quad \beta > j.
\]

**Proof:** The proof is easy.

**Remark 2.4** Setting \( m = 0, \ k = 1 \) in (2.20), we obtain a recurrence relation for Inverse moments of order statistics for type II exponentiated log-logistic distribution in the form

\[
\left(1 - \frac{\sigma(j-\beta)}{\alpha \beta (n-r+1)}\right) E[X_{j}^{j-\beta}] = E[X_{j-1}^{j-\beta}] + \frac{(j-\beta)\sigma^{j+1}}{\alpha \beta (n-r+1)} E[X_{j}^{j-2\beta}].
\]

**Remark 2.5** Putting \( m = -1, \) in Theorem 2.4, we get a recurrence relation for inverse moments of upper \( k \) record values from type II exponentiated log-logistic distribution in the form

\[
\left(1 - \frac{\sigma(j-\beta)}{\alpha \beta k}\right) E[(X_{U(r)}^{k})^{j-\beta}] = E[(X_{U(r-1)}^{k})^{j-\beta}] + \frac{(j-\beta)\sigma^{j+1}}{\alpha \beta k} E[(X_{U(r)}^{k})^{j-2\beta}].
\]
3. Relations for product moments

In this Section the explicit expressions, recurrence relations for single moments $gos$ and ratio moments of $gos$ are considered. First we need the following Lemmas to prove the main result.

Lemma 3.1: For type II exponentiated log-logistic distribution as given in (1.2) and non-negative integers $a$, $b$, $c$ with $m \neq -1$

$$J_{i,j}(a,0,c) = \alpha^2 \sigma^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} \frac{(j/\beta)(p)(i/\beta)(q)}{[\alpha(c+1)+p-(j/\beta)]} \times \frac{1}{\alpha(a+c+2)+p+q-(i+j)/(\beta)},$$  \hspace{1cm} (3.1)

where

$$J_{i,j}(a,b,c) = \int_0^\infty \int_0^\infty x^i y^j [\bar{F}(x)]^a f(x)[h_m(F(y)) - h_m(F(x))]^b [\bar{F}(y)]^c f(y)dydx.$$  \hspace{1cm} (3.2)

Proof: From (3.2), we have

$$J_{i,j}(a,0,c) = \int_0^\infty x^i [\bar{F}(x)]^a f(x)G(x)dx,$$  \hspace{1cm} (3.3)

where

$$G(x) = \int_x^\infty y^j [\bar{F}(y)]^c f(y)dy.$$  \hspace{1cm} (3.4)

By setting $z = [\bar{F}(y)]^{1/\alpha}$ in (3.4), we find that

$$G(x) = \alpha \sigma^j \sum_{p=0}^{\infty} (-1)^p \frac{(j/\beta)(p)}{[\alpha(c+1)+p-(j/\beta)]} \frac{[\bar{F}(x)]^{c+1+(p-j/\beta)/\alpha}}{[\alpha(c+1)+p-(j/\beta)]}.$$  \hspace{1cm} (3.6)

On substituting the above expression of $G(x)$ in (3.3), we get

$$J_{i,j}(a,0,c) = \alpha \sigma^j \sum_{p=0}^{\infty} (-1)^p \frac{(j/\beta)(p)}{[\alpha(c+1)+p-(j/\beta)]} \times \int_0^\infty x^i [\bar{F}(x)]^{a+c+1+(p-j/\beta)/\alpha} f(x)dx.$$  \hspace{1cm} (3.5)

Again by setting $t = [\bar{F}(x)]^{1/\alpha}$ in (3.5) and simplifying the resulting expression, we derive the relation given in (3.1).

Lemma 3.2: For the distribution as given in (1.2) and any non-negative integers $a$, $b$, $c$

$$J_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} J_{i,j}(a+(b-v)(m+1),0,c+v(m+1))$$  \hspace{1cm} (3.6)

$$= \frac{\alpha^2 \sigma^{i+j}}{(m+1)^b} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q+v} \binom{b}{v} \frac{(j/\beta)(p)}{[\alpha(c+1+v(m+1))]+p-(j/\beta)]} \times \frac{(i/\beta)(q)}{[\alpha(a+c+2+b(m+1))+p+q-(i+j)/(\beta)]}, \hspace{1cm} m \neq -1$$  \hspace{1cm} (3.7)
\[ = b! \alpha^{b+2} \sigma^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{b+q} (j / \beta)_{(p)}}{[\alpha(c+1) + p - (j / \beta)]^{b+1}} \times \frac{(i / \beta)_{(q)}}{[\alpha(a+c+2) + p + q - (i + j) / \beta]} , \quad m = -1 , \]  

(3.8)

where \( J_{i,j}(a,b,c) \) is as given in (3.2).

**Proof:** When \( m \neq -1 \), we have

\[ [h_m(F(y)) - h_m(F(x))]^b = \frac{1}{(m+1)^b}[(\hat{F}(x))^{m+1} - (\hat{F}(y))^{m+1}]^b \]

\[ = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} \hat{F}(y)^v (m+1) [\hat{F}(x)]^{(b-v)(m+1)}. \]

Now substituting for \([h_m(F(y)) - h_m(F(x))]^b \) in (3.2), we get

\[ J_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} J_{i,j}(a+(b-v)(m+1),0,c+v(m+1)). \]

Making use of the Lemma 3.1, we derive the relation given in (3.7).

When \( m = -1 \), we have

\[ J_{i,j}(a,b,c) = 0 , \quad as \quad \sum_{v=0}^{b} (-1)^v \binom{b}{v} = 0 . \]

On applying L’Hospital rule, (3.8) can be proved on the lines of (2.6).

**Theorem 3.1:** For type II exponentiated log-logistic distribution as given in (1.2) and \( 1 \leq r < s \leq n \), \( k = 1,2,\ldots, m \neq -1 \)

\[ E[X^r (r,n,m,k)X^j (s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1),s-r-1,\gamma_s-1) \]

(3.9)

\[ = \frac{\alpha^2 \sigma^{i+j} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} (-1)^p q+u+v \binom{r-1}{u} \binom{s-r-1}{v} \times \frac{(j / \beta)_{(p)} (i / \beta)_{(q)}}{[\alpha \gamma_{s-v} + p - (j / \beta)][\alpha \gamma_{r-u} + p + q - (i + j) / \beta]}, \]

\[ \beta > \max(i,j) \quad and \quad i,j = 0,1,2,\ldots. \]  

(3.10)

**Proof:** From (1.6), we have

\[ E[X^r (r,n,m,k)X^j (s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times \int_{0}^{\infty} \int_{0}^{\infty} x^r y^j [\hat{F}(x)]^m f(x) g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} \times [\hat{F}(y)]^{\gamma_s-1} f(y) dy dx . \]  

(3.11)
On expanding $g_m^{-1}(F(x))$ binomially in (3.11), we get

$$E[X^i \{r, n, m, k\} X^j \{s, n, m, k\}] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}$$

$$\times \sum_{u=0}^{s-r-1} (-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1), s-r-1, \gamma_s - 1).$$

Making use of the Lemma 3.2, we derive the relation in (3.10).

**Identity 3.1:** For $\gamma_r, \gamma_s \geq 1$, $k \geq 1$, $1 \leq r < s \leq n$ and $m \neq -1$

$$\sum_{v=0}^{s-r-1} (-1)^v \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}} = \frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{i=r+1}^s \gamma_i}. \quad (3.12)$$

**Proof:** At $i = j = 0$ in (3.10), we have

$$1 = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{s-r-1} \sum_{v=0}^{s-2} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \frac{1}{\gamma_{s-v} \gamma_{s-r-u}}.$$

Now on using (2.13), we get the result given in (3.12). At $r = 0$, (3.12) reduces to (2.13).

**Special cases**

i) Putting $m = 0$, $k = 1$ in (3.10), the explicit formula for the product moments of order statistics of the type II exponentiated log-logistic distribution can be obtained as

$$E[X_{r,n}^i X_{s,n}^j] = \alpha^2 \sigma^{i+j} C_{r,s,n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} (-1)^{p+q+u+v} \binom{r-1}{u} \binom{s-r-1}{v} \frac{(j/\beta)_{(p)}(i/\beta)_{(q)}}{[\alpha(n-s+1+v) + p - (j/\beta)][\alpha(n-r+1+u) + p + q - (i+j)/\beta}] \cdot$$

where

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$
\[
E[X^i_{U(r)} X^j_{U(s)}] = \alpha^s \sigma^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{p+q} (j/l \beta)(p)(i/l \beta)(q)}{(\alpha + p - (j/l \beta))^{s-r}(\alpha + p + q - (i + j/l \beta))^r}.
\]

**Remark 3.1:** At \( j = 0 \) in (3.10), we have
\[
E[X^i (r, n, m, k)] = \frac{\alpha \sigma^i C_{s-1}}{(r-1)!(s-r-1)!} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \frac{(-1)^{q+u} v^{r-1}}{u^{v+1}} \left( \frac{r-1}{\gamma_{s-v}} \left[ \beta \gamma_{r-u} + q - (i/l \beta) \right] \right).
\]

Making use of (3.12) in (3.13) and simplifying the resulting expression, we get
\[
E[X^i (r, n, m, k)] = \frac{\alpha \sigma^i C_{r-1}}{(r-1)!} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \frac{(-1)^{q+u} v^{r-1}}{u^{v+1}} \left( \frac{r-1}{\beta \gamma_{r-u} + q - (i/l \beta)} \right)
\]
as obtained in (2.12).

Making use of (1.6), we can derive recurrence relations for product moments of \( gos \) from (1.2).

**Theorem 3.2** For the given type II exponentiated log-logistic distribution and \( n \in \mathbb{N} \), \( m \in \mathbb{R} \), \( 1 < r < s < n-1 \)
\[
\left(1 - \frac{\sigma j}{\alpha \beta \gamma_s}\right) E[X^i (r, n, m, k) X^j (s, n, m, k)] = E[X^i (r, n, m, k) X^j (s-1, n, m, k)]
\]
\[
+ \frac{j \sigma^{\beta+1}}{\alpha \beta \gamma_s} E[X^i (r, n, m, k) X^{j-\beta} (s, n, m, k)].
\]

**Proof** From (1.6), we have
\[
E[X^i (r, n, m, k) X^j (s, n, m, k)]
\]
\[
= \frac{C_{s-1}}{(r-1)!} \int_0^\infty x^i \left[ \bar{F}(x) \right]^m f(x) g_{m-1}^{-1}(F(x)) I(x) dx,
\]
where
\[
I(x) = \int_0^\infty y^j \left[ \bar{F}(y) \right]^{p-1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} f(y) dy.
\]
Solving the integral in \( I(x) \) by parts and substituting the resulting expression in (3.15), we get
\[
E[X^i (r, n, m, k) X^j (s, n, m, k)] - E[X^i (r, n, m, k) X^j (s-1, n, m, k)]
\]
\[
= \frac{j C_{s-1}}{\gamma_s (r-1)!} \int_0^\infty x^i y^{j-1} \left[ \bar{F}(x) \right]^m f(x) g_{m-1}^{-1}(F(x))
\]
\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} \left[ \bar{F}(y) \right] dy dx.
\]
the constant of integration vanishes since the integral in \( I(x) \) is a definite integral. On using the relation (1.3), we obtain

\[
E[X^i (r,n,m,k) X^j (s,n,m,k)] - E[X^i (r,n,m,k) X^j (s-1,n,m,k)]
\]

\[
= \frac{j \sigma C_{s-1}}{\alpha \beta \gamma_s (r-1)! (s-r-1)!} \int_x^\infty x^y \left[ \tilde{F}(x) \right]^m f(x) g_m^{r-1} (F(x))
\]

\[
\times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} [\tilde{F}(y)]^{y_r-1} f(y) dy dx
\]

\[
+ \frac{j \sigma^{\beta+1} C_{s-1}}{\alpha \beta \gamma_s (r-1)! (s-r-1)!} \int_x^\infty x^y \left[ \tilde{F}(x) \right]^m f(x) g_m^{r-1} (F(x))
\]

\[
\times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} [\tilde{F}(y)]^{y_s-1} f(y) dy dx
\]

and hence the result given in (3.14).

**Remark 3.1** Setting \( m = 0 \), \( k = 1 \) in (3.14), we obtain recurrence relations for product moments of order statistics of the type II exponentiated log-logistic distribution in the form

\[
\left( 1 - \frac{\sigma j}{\alpha \beta (n-s+1)} \right) E(X^{(i,j)}_{r,s:n}) = E(X^{(i,j)}_{r,s-1:n}) + \frac{j \sigma^{\beta+1}}{\alpha \beta (n-s+1)} E(X^{i,j-\beta}_{r,s:n}).
\]

**Remark 3.2:** Putting \( m = -1 \), \( k \geq 1 \) in (3.5), we get the recurrence relations for product moments of upper \( k \) records of the type II exponentiated log-logistic distribution in the form

\[
\left( 1 - \frac{\sigma j}{\alpha \beta k} \right) E[(X^{(k)}_{U(r)})^i \Lambda (X^{(k)}_{U(s)})^j] = E[(X^{(k)}_{U(r)})^i \Lambda (X^{(k)}_{U(s-1)})^j]
\]

\[
+ \frac{j \sigma^{\beta+1}}{\alpha \beta k} E[(X^{(k)}_{U(r)})^i \Lambda (X^{(k)}_{U(s)})^{j-\beta}].
\]

Ratio moments ofgos from type II exponentiated log-logistic distribution can be obtain by the following Theorem.

**Theorem 3.3:** For type II exponentiated log-logistic distribution as given in (1.2)

\[
E[X^i (r,n,m,k) X^{j-\beta} (s,n,m,k)]
\]

\[
= \sigma^{i+j-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} \frac{p! q!}{\Gamma \left( \frac{j}{\beta} - p \right)} \frac{\Gamma \left( \frac{i}{\beta} + 1 \right)}{\Gamma \left( \frac{i}{\beta} + 1 - p \right)} \times \prod_{a=1}^{r} \left( \frac{1}{1 + \frac{p+q-(i+j)/\beta}{\alpha \gamma_a}} \right) \prod_{b=r+1}^{s} \left( \frac{1}{1 + \frac{p+1-(j/\beta)}{\alpha \gamma_b}} \right), \quad \beta > j.
\]

(3.16)
**Proof:** From (1.6), we have

\[
E[X^i (r, n, m, k) X^{j-\beta} (s, n, m, k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-2}}
\]

\[
\times \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \int_{0}^{\infty} x^i [F(x)]^{(s-r+u-v)(m+1)-1} f(x) J(x) dx,
\]

(3.17)

where

\[
J(x) = \int_{x}^{\infty} y^{j-\beta} [F(y)]^{\gamma_{s-v}-1} f(y) dy.
\]

(3.18)

By setting \( z = \frac{1}{[F(y)]^{1/\alpha}} \) in (3.18), we find that

\[
J(x) = \sigma^{j-\beta} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma \left( \frac{j}{\beta} \right)}{p! \Gamma \left( \frac{j}{\beta} - p \right)} \frac{\gamma_{s-v}^{p+1-(j/\beta)}}{[F(x)]^{(p+1-(j/\beta)) \alpha}}.
\]

On substituting the above expression of \( J(x) \) in (3.17), we get

\[
E[X^i (r, n, m, k) X^{j-\beta} (s, n, m, k)] = \frac{\sigma^{j-\beta} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-2}}
\]

\[
\times \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v+p} \binom{r-1}{u} \binom{s-r-1}{v} \frac{\Gamma \left( \frac{j}{\beta} \right)}{p! \Gamma \left( \frac{j}{\beta} - p \right)} \frac{\gamma_{s-v}^{p+1-(j/\beta)}}{[F(x)]^{(p+1-(j/\beta)) \alpha}}
\]

\[
\times \int_{0}^{\infty} x^i [F(x)]^{\gamma_{s-v}^{p+1-(j/\beta)} \alpha^{-1}} f(x) dx.
\]

(3.19)

Again by setting \( t = \frac{1}{[F(x)]^{1/\alpha}} \) in (3.19), we get

\[
E[X^i (r, n, m, k) X^{j-\beta} (s, n, m, k)] = \frac{\sigma^{j+\beta} C_{s-1}}{(m+1)^{s}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q}
\]

\[
\times \Gamma \left( \frac{j}{\beta} \right) \Gamma \left( \frac{i}{\beta} + 1 \right) \Gamma \left( \frac{\alpha(k+(n-r)(m+1)) + p+q-(i+j)/\beta}{\alpha(m+1)} \right)
\]

\[
\times \Gamma \left( \frac{\alpha(k+n(m+1)) + p+q-(i+j)/\beta}{\alpha(m+1)} \right)
\]

\[
\times \Gamma \left( \frac{\alpha(k+(n-r)(m+1)) + p+1-(j/\beta)}{\alpha(m+1)} \right)
\]

\[
\times \Gamma \left( \frac{\alpha(k+n(m+1)) + p+1-(j/\beta)}{\alpha(m+1)} \right)
\]

(3.20)

and hence the result given in (3.16).
Special cases

iii) Putting \( m = 0 \), \( k = 1 \) in (3.20), the explicit formula for the ratio moments of order statistics of the type II exponentiated log-logistic distribution can be obtained as

\[
E[X_{r n X_{s n}}^i X_{j-\beta}^{j-\beta}] = \frac{n! \sigma^{i+j-\beta}}{(n-s)!} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} \frac{\Gamma\left(\frac{j}{\beta}\right) \Gamma\left(\frac{i}{\beta}+1\right)}{p! q! \Gamma\left(\frac{i}{\beta}-p\right) \Gamma\left(\frac{i}{\beta}+1-q\right)} \times \frac{\Gamma[\alpha(n-r+1)+p+q-(i+j)/\beta]}{\Gamma[\alpha(n-s+1)+p+1-(i+j)/\beta]} \frac{\Gamma[\alpha(n+1)+p+q-(i+j)/\beta]}{\Gamma[\alpha(n-r+1)+p+1-(i+j)/\beta]}.
\]

iv) Putting \( m = -1 \) in (3.16), the explicit expression for the ratio moments of upper \( k \) record values for the type II exponentiated log-logistic distribution can be obtained as

\[
E[(X_{U(s)}^{(k)})^i X_{U(s)}^{j-\beta}] = \sigma^{i+j-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} \frac{\Gamma\left(\frac{j}{\beta}\right) \Gamma\left(\frac{i}{\beta}+1\right)}{p! q! \Gamma\left(\frac{i}{\beta}-p\right) \Gamma\left(\frac{i}{\beta}+1-q\right)} \times \frac{1}{\left(1 + \frac{p+q-(i+j)/\beta}{\alpha k}\right)^r \left(1 + \frac{p+1-(i+j)/\beta}{\alpha k}\right)^{s-r}}
\]

Making use of (1.6), we can derive recurrence relations for ratio moments of \( gos \) from (1.2).

**Theorem 3.4** For the given type II exponentiated log-logistic distribution

\[
\left(1 - \frac{\sigma(j-\beta)}{\alpha \beta \gamma_s}\right) E[X^i (r,n,m,k) X^{j-\beta} (s,n,m,k)] = E[X^i (r,n,m,k) X^{j-\beta} (s-1,n,m,k)] + \frac{(j-\beta) \sigma^{\beta+1}}{\alpha \beta \gamma_s} E[X^i (r,n,m,k) X^{j-2\beta} (s,n,m,k)], \quad \beta > j
\]

**Proof:** The proof is easy.

**Remark 3.4** Setting \( m = 0 \), \( k = 1 \) in (3.21), we obtain a recurrence relation for Ratio moments of order statistics for type II exponentiated log-logistic distribution in the form

\[
\left(1 - \frac{\sigma(j-\beta)}{\alpha \beta (n-s+1)}\right) E[X_{r n X_{s n}}^i X_{j-\beta}^{j-\beta}] = E[X_{r n X_{s n}}^i X_{s-\beta}^{j-\beta} X_{s n}^{j-\beta}] + \frac{(j-\beta) \sigma^{\beta+1}}{\alpha \beta (n-s+1)} E[X_{r n X_{s n}}^i X_{s n}^{j-2\beta}].
\]
Remark 3.5 Putting $m = -1$, in Theorem 3.4, we get a recurrence relation for ratio moments of upper $k$ record values from type II exponentiated log-logistic distribution in the form

$$
 \left( 1 - \frac{\sigma (j - \beta)}{\alpha \beta k} \right) E[(X_U^{(k)})^i (X_U^{(s)})^{j-\beta}] = E[(X_U^{(k)})^i (X_U^{(s-1)})^{j-\beta}]
$$

$$
+ \frac{(j - \beta)\sigma^{\beta+1}}{\alpha \beta k} E[(X_U^{(k)})^i (X_U^{(s)})^{j-2\beta}].
$$

Remark 3.6: At $\gamma_r = n - r + 1 + \sum_{i=r}^j m_i$, $1 \leq r \leq j \leq n$, $m_i \in N$, $k = m_n + 1$ in (3.16) the product moment of progressive type II censored order statistics of type II exponentiated log-logistic distribution can be obtained.

Remark 3.7: The result is more general in the sense that by simply adjusting $j - \beta$ in (3.16), we can get interesting results. For example if $j - \beta = -i$ then

$$
 E \left[ \frac{X(r, n, m, k)}{X(s, n, m, k)} \right]^i
$$

gives the moments of quotient. For $j - \beta > 0$, $E[X^i (r, n, m, k)X^{j-\beta} (s, n, m, k)]$ represent product moments, whereas for $j < \beta$, it is moment of the ratio of two generalized order statistics of different powers.

4. Characterization

This Section contains characterization of type II exponentiated log-logistic distribution by using the conditional expectation of $gos$.

Let $X(r, n, m, k)$, $r = 1, 2, \ldots, n$ be $gos$ from a continuous population with $cdf$ $F(x)$ and $pdf$ $f(x)$, then the conditional $pdf$ of $X(s, n, m, k)$ given $X(r, n, m, k) = x$, $1 \leq r < s \leq n$, is in view of (1.5) and (1.6), is

$$
f_{X(s, n, m, k) \mid X(r, n, m, k)} (y \mid x) = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}} [\bar{F}(x)]^{m-\gamma_r + 1}

\times [(h_m(F(y)) - h_m(F(x))]^{r-1} [\bar{F}(y)]^{\gamma_r - 1} f(y).
$$

Theorem 4.1: Let $X$ be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$
E[ X(s, n, m, k) \mid X(r, n, m, k) = x] = \sigma \sum_{p=0}^{s-r} (1/\beta)_{(p)} [1 + (x/\sigma)^\beta]^p

\times \prod_{j=1}^{s-r} \frac{\gamma_{r+j}}{\gamma_{r+j} - p/\alpha}
$$

if and only if
\[
\bar{F}(x) = \left(1 + \left\{ \frac{x}{\sigma} \right\}^\beta \right)^{-\alpha}, \quad x \geq 0, \quad \sigma > 0, \quad \beta > 1, \quad \alpha > 0.
\]

**Proof:** From (4.1), we have

\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \int_x^\infty y \left[1 - \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{m+1} \right]^{s-r-1} \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_y-1} f(y) \frac{1}{\bar{F}(x)} dy.
\]

(4.3)

By setting \( u = \frac{\bar{F}(y)}{\bar{F}(x)} = \left( \frac{1 + (x/\sigma)^\beta}{1 + (y/\sigma)^\beta} \right) \) from (1.2) in (4.3), we obtain

\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \int_0^1 \left[ \frac{1 + (x/\sigma)^\beta}{1 + (y/\sigma)^\beta} \right]^{\gamma_y-1} (1 - u^{m+1})^{s-r-1} du
\]

\[
= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \sum_{p=0}^\infty \left( \frac{1}{\beta} \right)_p [1 + (x/\sigma)^\beta]^p
\]

\[
\times \int_0^1 t^{\gamma_y-(p/\alpha)-1} (1 - t^{m+1})^{s-r-1} dt
\]

(4.4)

Again by setting \( t = u^{m+1} \) in (4.4), we get

\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \int_0^1 \left( \frac{k - (p/\alpha)}{(m+1)} + n - s \right)^{s-r-1} dt
\]

\[
= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \sum_{p=0}^\infty \left( \frac{1}{\beta} \right)_p [1 + (x/\sigma)^\beta]^p
\]

\[
\times \Gamma \left( \frac{k - (p/\alpha)}{(m+1)} + n - s \right) \Gamma(s-r)
\]

\[
\times \Gamma \left( \frac{k - (p/\alpha)}{(m+1)} + n - r \right)
\]

\[
= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \sum_{p=0}^\infty \left( \frac{1}{\beta} \right)_p [1 + (x/\sigma)^\beta]^p
\]

\[
\times \left( \frac{m+1}{s-r} \right)^{s-r} \Gamma(s-r)
\]

\[
\prod_{j=1}^{s-r} (\gamma_{r+j} - p/\alpha)
\]

and hence the relation in (4.2).
To prove sufficient part, we have from (4.1) and (4.2)

\[
\frac{C_{s-1}}{(s - r - 1)C_{r-1}(m + 1)^{s-r-1}} \int_{x}^{\infty} y[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1} \times [\bar{F}(y)]^{\gamma r-1} f(y) \, dy = [\bar{F}(x)]^{\gamma r+1} H_r(x),
\]

where

\[H_r(x) = \sigma \sum_{p=0}^{\infty} \left(1/\beta\right)_{(p)}[1 + (x/\sigma)^{\beta}]^p \prod_{j=1}^{s-r} \left(\gamma_{r+j} - p/\alpha\right).\]

Differentiating (4.5) both sides with respect to \(x\), we get

\[
- \frac{C_{s-1}[\bar{F}(x)]^{m} f(x)}{(s - r - 2)C_{r-1}(m + 1)^{s-r-2}} \int_{x}^{\infty} y[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-2} \times [\bar{F}(y)]^{\gamma r-1} f(y) \, dy = H'_r(x)[\bar{F}(x)]^{\gamma r+1} + \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma r+1} f(x)
\]

or

\[- \gamma_{r+1} H_{r+1}(x)[\bar{F}(x)]^{\gamma r+1} f(x) = H'_r(x)[\bar{F}(x)]^{\gamma r+1} + \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma r+1} f(x).\]

Therefore,

\[
\frac{f(x)}{\bar{F}(x)} = - \frac{H'_r(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} = \frac{\alpha \beta (x/\sigma)^{\beta-1}}{\sigma[1 + (x/\sigma)^{\beta}]}
\]

which proves that

\[\bar{F}(x) = \left(1 + \left\{\frac{x}{\sigma}\right\}^{\beta}\right)^{-\alpha}, \quad x \geq 0, \quad \sigma > 0, \quad \beta > 1, \quad \alpha > 0.\]

**Remark 4.1:** For \(k = 1, m = 0\) and \(k = 1, m = -1\), we obtain the characterization results of the type II exponentiated log-logistic distribution based on order statistics and record values respectively.

5. **Application**

In this Section we suggest some applications based on moments discussed in Section 2. Order statistics, record values and their moments are widely used in statistical inference [see for example Balakrishnan and Sandhu (1996), Sultan and Moshref (2000) and Mahmoud et al. (2003), among several others].

i) **Estimation:** The moments of order statistics and record values given in Section 2 can be used to obtain the best linear unbiased estimate of the parameters of the type II exponentiated log-logistic distribution. Some works of this nature based on gos have
been done by Ahsanullah and habibullah (2000), Malinowska et al. (2006) and Burkchat et al. (2007).

ii) Characterization: The type II exponentiated log-logistic distribution given in (1.2) can be characterized by using recurrence of single moment of generalized order statistics as follows:

Let \( L(a,b) \) stand for the space of all integrable functions on \( (a,b) \). A sequence \( (h_n) \subset L(a,b) \) is called complete on \( L(a,b) \) if for all functions \( g \in L(a,b) \) the condition
\[
\int_a^b g(x) f_n(x) dx = 0, \quad n \in \mathbb{N},
\]
implies \( g(x) = 0 \) a.e. on \( (a,b) \). We start with the following result of Lin (1986).

**Proposition 5.1:** Let \( n_0 \) be any fixed non-negative integer, \(-\infty \leq a < b \leq \infty \) and \( g(x) \geq 0 \) an absolutely continuous function with \( g'(x) \neq 0 \) a.e. on \( (a,b) \). Then the sequence of functions \( \{(g(x))^n e^{-g(x)}, n \geq n_0\} \) is complete in \( L(a,b) \) iff \( g(x) \) is strictly monotone on \( (a,b) \).

Using the above Proposition we get a stronger version of Theorem 2.2.

**Theorem 5.1:** A necessary and sufficient conditions for a random variable \( X \) to be distributed with \( pdf \) given by (1.1) is that
\[
1 - \frac{\alpha^j}{\alpha \beta \gamma_r} E[X^j (r,n,m,k)] = E[X^j (r-1,n,m,k)] + \frac{j \alpha^j \beta + 1}{\alpha \beta \gamma_r} E[X^{j-\beta} (r,n,m,k)]. \tag{5.1}
\]

Proof: The necessary part follows immediately from (2.14) on the other hand if the recurrence relation (5.1) is satisfied then on using (1.5), we have
\[
\frac{C_{r-1}}{r-1} \int_0^\infty x^j [\mathcal{F}(x)]^{\gamma_r-1} f(x) g_m^{-1} (F(x)) dx
\]
\[
= \frac{C_{r-1}}{\gamma_r (r-2)!} \int_0^\infty x^j [\mathcal{F}(x)]^{\gamma_r+1} f(x) g_m^{-2} (F(x)) dx
\]
\[
+ \frac{\alpha^j \beta + 1}{\alpha \beta \gamma_r (r-1)!} \int_0^\infty x^j [\mathcal{F}(x)]^{\gamma_r-1} f(x) g_m^{-1} (F(x)) dx
\]
\[
+ \frac{\alpha^j \beta + 1}{\alpha \beta \gamma_r (r-1)!} \int_0^\infty x^{j-\beta} [\mathcal{F}(x)]^{\gamma_r-1} f(x) g_m^{-1} (F(x)) dx. \tag{5.2}
\]

Integrating the first integral on the right-hand side of the above equation by parts and simplifying the resulting expression, we get
\[
\frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^j [\mathcal{F}(x)]^{\gamma_r-1} g_m^{-1} (F(x)) \left[ \mathcal{F}(x) - \frac{\sigma x}{\alpha \beta} f(x) - \frac{\sigma^j \beta + 1}{\alpha \beta x^{\beta-1}} f(x) \right] dx = 0.
\]
It now follows from Proposition 5.1, we get

\[ \alpha \beta \bar{F}(x) = \sigma [1 + 1/(x/\sigma)^\beta] x f(x), \]

which proves that \( f(x) \) has the form (1.1).

6. Concluding Remarks

In the study presented above, we established some new explicit expressions and recurrence relations between the single and product moments of \( gos \) from the type II exponentiated log-logistic distribution. In addition ratio and inverse moments of type II exponentiated log-logistic distribution are also established. Further, the conditional expectation of \( gos \) is used to characterize the distribution.

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References


