AN APPROACH TO NUMERICAL SEMIGROUPS

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Abstract

In this paper, we give some results on principal ideals of a numerical semigroup $S = \langle s_1, s_2, \ldots, s_p \rangle$ for $p \geq 2$, $p \in \mathbb{N}$. We also describe some relations between Apery subsets and ideals of $S$.

Keywords: Numerical semigroup, Apery set, Ideal, Gap.


1. Introduction

A numerical semigroup $S$ is a subset of $\mathbb{N}$ (the set of nonnegative integers) closed under addition, satisfying $0 \in S$ and for which $\mathbb{N} \setminus S$ has finitely many elements. For a numerical semigroup $S$, $A = \{s_1, s_2, \ldots, s_p\} \subseteq S$ is a generating set of $S$ provided that $S = \langle s_1, s_2, \ldots, s_p \rangle = \{s_1k_1 + s_2k_2 + \cdots + s:pk_p : k_i \in \mathbb{N}, 1 \leq i \leq p\}$. The set $A = \{s_1, s_2, \ldots, s_p\}$ is called a minimal generating set of if no proper subset is a generating set of $S$. It was observed in [1] that the set $\mathbb{N} \setminus S$ is finite if and only if $\gcd\{s_1, s_2, \ldots, s_p\} = 1$ (gcd stands for the greatest common divisor).

Another important invariant of $S$ is the largest integer not belonging to $S$, known as the Frobenius number of $S$ and denoted by $g(S)$, that is $g(S) = \max\{x \in \mathbb{Z} : x \notin S\}$ (see [6, 1]). We define

$$n(S) = \sharp\{(0,1, \ldots, g(S)) \cap S\}$$

where $\sharp(A)$ denotes the cardinality of $A$. It is also well-known that

$$S = \{0, s_1, s_2, \ldots, s_{n-1}, s_n = g(S) + 1, \rightarrow \}$$

where $\rightarrow$ means that every integer greater than $g(S) + 1$ belongs to $S$, $n = n(S)$ and $s_i < s_{i+1}$ for $i = 1, 2, \ldots, n$.

For $m \in S \setminus \{0\}$, the Apery set of $m$ in $S$ is the set $Ap(S, m) = \{s \in S : s - m \notin S\}$. It can easily be proved that $Ap(S, m)$ is formed by the smallest elements of $S$ belonging to the different congruence classes mod $m$. According to this, we have $\sharp(Ap(S, m)) = m$

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and \( g(S) = \max(\text{Ap}(S, m)) - m \). Various aspects and properties of Apery sets are given in [2, 3].

The elements of \( \mathbb{N} \setminus S \), denoted by \( H(S) \), are called the gaps of \( S \) (see [6]). A subset \( I \) of \( S \) is an ideal if \( I + S \subseteq I \) (that is, for all \( x \in I \) and \( s \in S \), the element \( x + s \) is in \( I \)). An ideal \( I \) of \( S \) is generated by \( A \subseteq S \) if \( I = A + S \). We also say that the ideal \( I \) is finitely generated if there exists a finite \( A \subseteq S \) such that \( I = A + S \).

We say that \( I \) is principal if it can be generated by a single element. That is, there exists \( x_0 \in S \) such that \( I = \{x_0\} + S = \{x_0 + s : s \in S\} \). We usually write \( I = \{x_0\} \) instead of \( I = \{x_0\} + S \) (see [5]). The elements of \( H(I) = S \setminus I \) are called the gaps of \( I \). If \( I \) and \( J \) are ideals of \( S \), we define their ideal sum by \( I + J = \{i + j : i \in I, j \in J\} \) (see [1]).

The contents of this study are organized as follows. In section 2, we give some results concerning the sum, union and intersection of principal ideals of \( S \). In particular, the main goal of this section is to prove Theorem 2.5. Furthermore, the aim of Section 3 is to give some relations between the Apery subsets and the principal ideals of \( S \).

Throughout this paper, we will assume the numerical semigroup \( S \) satisfies

\[
S = \langle s_1, s_2, \ldots, s_p \rangle = \{s_1k_1 + s_2k_2 + \cdots + s_pk_p : k_1, k_2, \ldots, k_p \in \mathbb{N}, 1 \leq i \leq p \},
\]

and that its principal ideals are \( I_i \), for \( i = 1, 2, \ldots, p \) (\( p \geq 2, p \in \mathbb{N} \)), respectively.

### 2. Some results for principal ideals of numerical semigroups

In this section, we give some results concerning the sum, union and intersection of principal ideals of a numerical semigroup \( S \). In particular, we obtain elements belonging to the intersection of the principal ideals of \( S \) which are not in the sum of the principal ideals of \( S \).

#### 2.1. Lemma

\[
\sum_{i=1}^{p} I_i \subseteq I_i, \quad \text{where} \quad \sum_{i=1}^{p} I_i = \left[ \sum_{i=1}^{p} s_i \right] \quad \text{and} \quad s_i \in S.
\]

**Proof.** If \( x \in \sum_{i=1}^{p} I_i = \left[ \sum_{i=1}^{p} s_i \right] \), then there exists \( s \in S \) such that \( x = \sum_{i=1}^{p} s_i + s \). Thus, we find \( x \in S \). Therefore, we get \( \sum_{i=1}^{p} I_i \subseteq S \implies \sum_{i=1}^{p} I_i + I_k \subseteq S + I_k \subseteq I_k, \quad 1 \leq k \leq p. \) \( \square \)

We obtain the following result from Lemma 2.1.

#### 2.2. Corollary

\[
\sum_{i=1}^{p} I_i \subseteq \bigcap_{i=1}^{p} I_i.
\]

#### 2.3. Lemma

Let \( S \) and \( I_i \) be a numerical semigroup and principal ideals of \( S \), respectively. Then \( s_p \notin I_i \) for \( i = 1, 2, \ldots, p - 1 \).

**Proof.** If \( s_p \in I_i \), then there exists \( s \in S \) such that \( s_i + s = s_p \) for \( i = 1, 2, \ldots, p - 1 \). Thus it follows that \( S = \{s_1, s_2, \ldots, s_{p-1}\} \), which is a contradiction since \( A = \{s_1, s_2, \ldots, s_p\} \) is a minimal generating set of \( S \). \( \square \)

#### 2.4. Lemma

Let \( S \) and \( I_i \) be a numerical semigroup and principal ideals of \( S \), respectively. Then \( \bigcup_{i=1}^{p} I_i \subseteq S \setminus \{0, s_p\} \).

**Proof.** If \( x \in \bigcup_{i=1}^{p} I_i \), then it follows that \( x \neq s_p \) and \( x \neq 0 \) from definition of principal ideal of \( S \), and Lemma 2.3. \( \square \)
2.5. **Theorem.** Let $S$ be a numerical semigroup, $I_i$ and $g(S)$ be its principal ideals and Frobenius number, respectively. Then

$$g(S) + \sum_{i=1}^{p} s_i \in \bigcap_{i=1}^{p} I_i \setminus \bigcup_{i=1}^{p} I_i.$$ 

**Proof.** Firstly, we show that $g(S) + \sum_{i=1}^{p} s_i \in \bigcap_{i=1}^{p} I_i$:

$$g(S) + \sum_{i=1}^{p} s_i = s_1 + (g(S) + s_2 + s_3 + \cdots + s_p) \in I_1,$$

since $(g(S) + s_2 + s_3 + \cdots + s_p) \in S$,

$$g(S) + \sum_{i=1}^{p} s_i = s_2 + (g(S) + s_1 + s_3 + \cdots + s_p) \in I_2,$$

since $(g(S) + s_1 + s_3 + \cdots + s_p) \in S$,

$\cdots$ \hfill \hfill $\cdots$ \hfill \hfill $\cdots$

$$g(S) + \sum_{i=1}^{p} s_i = s_p + (g(S) + s_1 + s_2 + \cdots + s_{p-1}) \in I_p,$$

since $(g(S) + s_1 + s_2 + \cdots + s_{p-1}) \in S$.

Now, we must show that $g(S) + \sum_{i=1}^{p} s_i \notin \bigcup_{i=1}^{p} I_i$. Suppose on the contrary that $g(S) + \sum_{i=1}^{p} s_i \in \bigcup_{i=1}^{p} I_i$. Then, there exists $s \in S$ such that $g(S) + \sum_{i=1}^{p} s_i = s_1 + s_2 + s_3 + \cdots + s_p + s$. Thus, we get $g(S) = s \in S$, which is a contradiction. \hfill $\square$

2.6. **Example.** Let us consider the numerical semigroup $S$ given by $S = \langle 4, 7, 9 \rangle = \{0, 4, 7, 8, 9, 11, \ldots \}$. The Frobenius number of $S$ is $g(S) = 10$. Then the principal ideals of $S$ are described by:

$I = [4] = 4 + S = \{4, 8, 11, 12, 13, 15, \ldots \}$,

$J = [7] = 7 + S = \{7, 11, 14, 15, 16, 18, \ldots \}$, and

$K = [9] = 9 + S = \{9, 13, 16, 17, 18, 20, \ldots \}$.

In this case, we find that

$I + J + K = [20] = \{20, 24, 27, 28, 29, 31, \ldots \} \subset I, J, K$,

$I \cup J = \{4, 7, 8, 11, \ldots \} \subseteq S \setminus \{0, 9\}$,

$I \cap J \cap K = \{16, 18, 20, 21, \ldots \} \supset I + J + K$,

and $g(S) + s_1 + s_2 + s_3 = 10 + 4 + 7 + 9 = 30 \in I \cap J \cap K$ but 30 not in $[4 + 7 + 9]$. 

3. **The relation between principal ideals and Apéry sets**

In this section, we obtain some relations between the principal ideals $I_i = [s_i]$ and the Apéry sets $\text{Ap}(S, s_i) = \{s \in S : s - s_i \notin S\}$ for $1 \leq i \leq p$.

**3.1. Lemma.** $\text{Ap}(S, s_i) \subseteq I_i^c$ for each $i$, $1 \leq i \leq p$.

**Proof.** We must show that $I_i \cap \text{Ap}(S, s_i) = \emptyset$. Suppose that $I_i \cap \text{Ap}(S, s_i) \neq \emptyset$ for some $i \in \{1, 2, \ldots, p\}$. If $x \in [s_i] \cap \text{Ap}(S, s_i)$, then $x = s_i + s$ for some $s \in S$, and $x - s_i \notin S$ for this $i$. But this is contradiction since $x \in S$. \hfill $\square$
The following is a consequence of Lemma 3.1.

3.2. Corollary. For each \( i \in \{1, 2, \ldots, p\} \) the family \( \{ s_i \} \), \( \text{Ap}(S, s_i) \) is a partition of \( S \).

Proof. Take \( i \in \{1, 2, \ldots, p\} \). According to lemma 3.1, it is sufficient to show that \( S = [s_i] \cup \text{Ap}(S, s_i) \). It is clear that \( [s_i] \cup \text{Ap}(S, s_i) \subseteq S \), so take \( x \in S \) with \( x \notin [s_i] \). Then we have \( x - s_i \notin S \), so \( x \in \text{Ap}(S, s_i) \) which gives the required result. \( \square \)

3.3. Lemma. \( \sum_{i=1}^{p} s_i \notin \text{Ap}(S, s_i) \) for each \( i \in \{1, 2, \ldots, p\} \).

Proof. The result follows from the fact that \( \sum_{i=1}^{p} s_i - s_j = s_1 + s_2 + \cdots + s_{j-1} + s_{j+1} \in S \) for each \( i \in \{1, 2, \ldots, p\} \) and \( 2 \leq j \leq p-1 \). \( \square \)

3.4. Lemma. \( \text{Ap}(S, s_i) \subset \text{Ap}(S, \sum_{i=1}^{p} s_i) \) for each \( i \in \{1, 2, \ldots, p\} \).

Proof. For each \( i \in \{1, 2, \ldots, p\} \), if \( x \notin \text{Ap}(S, \sum_{i=1}^{p} s_i) \), then \( x - s_1 - s_2 - \cdots - s_p \in S \). It follows that \( x - s_i \in S \), and hence \( x \notin \text{Ap}(S, s_i) \). \( \square \)

3.5. Lemma. \( \text{Ap}(S, s_i) = H(I_i) \) for each \( i \in \{1, 2, \ldots, p\} \).

Proof. The result follows from the following observation: for each \( i \in \{1, 2, \ldots, p\} \),

\[
\begin{align*}
x \in \text{Ap}(S, s_i) & \iff x - s_i \notin S \iff \forall s \in S, s \neq x - s_i \\
& \iff x \neq s_i + s \iff x \notin I_i \iff x \in H(I_i).
\end{align*}
\]

The following result is a consequence of Lemma 3.5.

3.6. Corollary. \( S \setminus \left( \sum_{i=1}^{p} I_i \right) = \text{Ap}(S, \sum_{i=1}^{p} s_i) \).

3.7. Lemma. \( \bigcup_{i=1}^{p} H(I_i) \subseteq H \left( \sum_{i=1}^{p} I_i \right) \).

Proof. From Lemma 2.1 we have \( \sum_{i=1}^{p} I_i \subseteq I_i \), and so \( H(I_i) \subseteq H \left( \sum_{i=1}^{p} I_i \right) \) for each \( i \in \{1, 2, \ldots, p\} \). Thus, we obtain \( \bigcup_{i=1}^{p} H(I_i) \subseteq H \left( \sum_{i=1}^{p} I_i \right) \). \( \square \)

3.8. Example. Let us consider a numerical semigroup \( S \) given by \( S = \langle 5, 7, 9, 11, 13 \rangle = \{0, 5, 7, 9, \ldots \} \). The Frobenius number of \( S \) is \( g(S) = 8 \). The principal ideals \( I_i \) of \( S \) (for \( i = 1, 2, 3, 4, 5 \)) are respectively:

\[
\begin{align*}
I_1 &= [5] = \{5, 10, 12, 14, \ldots \}, \\
I_2 &= [7] = \{7, 12, 14, 16, \ldots \}, \\
I_3 &= [9] = \{9, 14, 16, 18, \ldots \}, \\
I_4 &= [11] = \{11, 16, 18, 20, \ldots \}, \text{ and} \\
I_5 &= [13] = \{13, 18, 20, 22, \ldots \}.
\end{align*}
\]
Now, the subsets $A_p(S, s_i)$ of $S$ (for $i = 1, 2, 3, 4, 5$) are respectively;

$A_p(S, 5) = \{ s \in S : s - 5 \notin S \} = \{ 0, 7, 9, 11, 13 \}$

$= H(I_1)$,

$A_p(S, 7) = \{ s \in S : s - 7 \notin S \} = \{ 0, 5, 9, 10, 11, 13, 15 \}$

$= H(I_2)$,

$A_p(S, 9) = \{ s \in S : s - 9 \notin S \} = \{ 0, 5, 7, 10, 11, 12, 13, 15, 17 \}$

$= H(I_3)$,

$A_p(S, 11) = \{ s \in S : s - 11 \notin S \} = \{ 0, 5, 7, 9, 10, 12, 13, 14, 15, 17, 19 \}$

$= H(I_4)$, and,

$A_p(S, 13) = \{ s \in S : s - 13 \notin S \} = \{ 0, 5, 7, 9, 10, 11, 12, 14, 15, 16, 17, 19, 21 \}$

$= H(I_5)$.

From Corollary 3.2, we can write

$S = [s_i] \cup A_p(S, s_i), \quad [s_i] \cap A_p(S, s_i) = \emptyset, \quad \sum_{i=1}^{5} s_i = 45 \notin A_p(S, s_i),$

and

$A_p(S, s_i) \subset A_p(S, 45), \quad \text{for } i = 1, 2, 3, 4, 5.$

On the other hand, we have $S \setminus \sum_{i=1}^{5} I_i = A_p(S, 45)$ and $\bigcup_{i=1}^{5} H(I_i) \subset H([45]).$

References


