Statistical Inference of Stress-Strength Reliability for the Exponential Power (EP) Distribution Based on Progressive Type-II Censored Samples

Neriman AKDAM∗, Ismail KINACI† and Bugra SARACOGLU ‡

Abstract
Suppose that \( X \) represents the stress which is applied to a component and \( Y \) is strength of this component. Let \( X \) and \( Y \) have Exponential Power (EP) distribution with \( (\alpha_1, \beta_1) \) and \( (\alpha_2, \beta_2) \) parameters, respectively. In this case, stress-strength reliability (SSR) is shown by \( P = P(X < Y) \). In this study, the SSR for EP distribution are obtained with numerical methods. Also maximum likelihood estimate (MLE) and approximate bayes estimates by using Lindley approximation method under squared-error loss function for SSR under progressive type-II censoring are obtained. Moreover, performances of these estimators are compared in terms of MSEs by using Monte Carlo simulation. Furthermore coverage probabilities of parametric bootstrap estimates are computed. Finally, real data analysis is presented.

Keywords: Maximum likelihood estimation, Bayes estimation, Exponential Power distribution, Lindley’s approximation, Monte Carlo simulation, Bootstrap estimation.

2000 AMS Classification: 62F15, 62F40

1. Introduction
In this article, it is considered estimation of \( P = P(X < Y) \), when \( X \) and \( Y \) are independent exponential power random variables with different parameters. Stress-Strength model identifies the life of a component which is exposed to \( X \) stress with \( Y \) strength. The probability of \( P = P(X < Y) \) is called as stress-strength reliability (SSR). Stress-Strength models can be used in variety of areas such as engineering, medicine, military. The reliability of a carbon fiber, a bridge, an elevator and so forth can be given as examples to SSR. When the failure time of all components in system may not always observe in reliability analysis and life test. In this case, censored data is obtained.

There are different types of censoring schemes such as type-I, type-II and progressive type-II right (PTR-II) censoring schemes in lifetesting experiments. In type-I censoring scheme, the experiment is stopped at a pre-fixed time. In type-II...
censoring scheme, the experiment is stopped whenever a fixed number of failures has been observed. One of the most widely used censoring schemes is progressive type-II right (PTR-II) censoring scheme. This scheme is described as follows. Assume that \( n \) identical units are placed on a test and \( m \) failures are going to be observed. At the time of the first failure, \( R_1 \) items are chosen at random and removed. Similarly, at the time of the second failure, \( R_2 \) of the remaining items are chosen at random and removed, and so on. Finally, at the time of the \( m^{th} \) failure, all the surviving items are censored. PTR-II censoring scheme is shown with \( R = (R_1, R_2, \ldots, R_m) \). In PTR-II censoring, Type-II censoring is obtained for \( R = (0, 0, \ldots, n - m) \). In this lifetime process, \( X^R = (X^{R_1}_{1:m:n}, X^{R_2}_{2:m:n}, \ldots, X^{R_m}_{m:m:n}) \) with \( X^{R_1}_{1:m:n} < X^{R_2}_{2:m:n} < \cdots < X^{R_m}_{m:m:n} \) is called as PTR-II censored sample with \( R = (R_1, R_2, \ldots, R_m) \) scheme. The joint pdf of this censored sample is given by;

\[
 f_{X^{R_1}_{1:m:n}, X^{R_2}_{2:m:n}, \ldots, X^{R_m}_{m:m:n}}(x_1, x_2, \ldots, x_m) = c \prod_{i=1}^{m} f(x_i) [1 - F(x_i)]^{R_i}, \\
 -\infty < x_1 < x_2 < \cdots < x_m < \infty
\]

where \( c = n \left( n - R_1 - 1 \right) \times \cdots \times \left( n - R_1 - R_2 - \cdots - R_{m-1} - m + 1 \right) \). (Balakrishnan (2007), Balakrishnan and Aggarwala (2000)).

Estimation problem of SSR under PTR-II censoring which reduces the cost and time of an experiment is fairly important. There are many studies about estimation of SSR for some distributions under PTR-II censoring. Among them are Saracoğlu et al. (2012), Lio and Tsai (2012), Valiollahi et al. (2013), Lin and Ke (2013), Rezaei et al. (2015), Basirat et al. (2015).

The exponential power (EP) distribution introduced by Smith & Bain (1975) has been studied by many authors such as Leemis (1986), Rajarshi&Rajarshi (1988) and Chen (1999). In this study, it is assumed that \( X \) and \( Y \) are independent Exponential Power (EP) random variables with \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) parameters, where \( \beta_1, \beta_2 > 0 \) are shape parameters and \( \alpha_1, \alpha_2 > 0 \) are scale parameters, respectively. In this case, the probability density functions (pdfs) and cumulative distribution functions (cdfs) of \( X \) and \( Y \) are as follows;

\[
 f(x; \alpha_1, \beta_1) = \frac{\beta_1}{\alpha_1} \left( \frac{x}{\alpha_1} \right)^{\beta_1-1} \exp \left( \frac{x}{\alpha_1} \right)^{\beta_1} \exp \left[ 1 - \exp \left( \frac{x}{\alpha_1} \right)^{\beta_1} \right], \ x \geq 0
\]

\[
 F(x; \alpha_1, \beta_1) = 1 - \exp \left( 1 - \exp \left( \frac{x}{\alpha_1} \right)^{\beta_1} \right)
\]

\[
 f(y; \alpha_2, \beta_2) = \frac{\beta_2}{\alpha_2} \left( \frac{y}{\alpha_2} \right)^{\beta_2-1} \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \exp \left[ 1 - \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \right], \ y \geq 0
\]

\[
 F(y; \alpha_2, \beta_2) = 1 - \exp \left( 1 - \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \right)
\]
The main objective of this manuscript is to obtained the approximate Bayes estimators for SSR of EP distribution based on PTR-II censored samples under squared error loss function and compare them with maximum likelihood estimators (MLEs) of SSR. The rest of the manuscript is organized as follows. In Section 2, SSR for EP distribution is given. In section 3, the MLEs and bootstrap confidence intervals for SSR are derived. In section 4, the approximate Bayes estimators under squared error loss function are derived by using Lindley’s approximations. Using Monte Carlo simulation, The approximate Bayes estimation are compared with the maximum likelihood estimation in terms of Mean square error (MSE) and results are tabulated in section 5. In section 6, real data analysis is presented. Finally, conclusions are given in Section 7.

2. Stress-Strength Reliability

Let $X$ and $Y$ independent random variables have EP ditribution with $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ parameters, respectively. Suppose that $Y$ represent the strength of a component exposed to $X$ stress, then the SSR of this component is obtained as follows,

$$P = P(X < Y) = \int_{y=0}^{\infty} f_Y(y) F_X(y) \, dy$$

$$= \int_{y=0}^{\infty} \frac{\beta_2}{\alpha_2} \left( \frac{y}{\alpha_2} \right)^{\beta_2 - 1} \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \exp \left[ 1 - \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \right] \times \left[ 1 - \exp \left( \frac{y}{\alpha_1} \right)^{\beta_1} \right] \, dy$$

$$= 1 - \int_{y=0}^{\infty} \frac{\beta_2}{\alpha_2} \left( \frac{y}{\alpha_2} \right)^{\beta_2 - 1} \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \exp \left[ 1 - \exp \left( \frac{y}{\alpha_2} \right)^{\beta_2} \right] \times \exp \left[ 1 - \exp \left( \frac{y}{\alpha_1} \right)^{\beta_1} \right] \, dy$$

This probability can be solved by using numerical methods.

3. Maximum Likelihood Estimation (MLE)

Let $X^R = \left( X^{R_1}_{1:m_1:n_1}, X^{R_2}_{2:m_1:n_1}, \ldots, X^{R_{m_1}}_{m_1:m_1:n_1} \right)$ denote PTR-II censored sample with $R$ censoring scheme taken from $EP(\alpha_1, \beta_1)$ distribution having pdf and cdf defined in Eq.(1.2) and Eq.(1.3) and $Y^S = \left( Y^{S_1}_{1:m_2:n_2}, Y^{S_2}_{2:m_2:n_2}, \ldots, Y^{S_{m_2}}_{m_2:m_2:n_2} \right)$ denote PTR-II censored sample with $S$ censoring scheme taken from $EP(\alpha_2, \beta_2)$
distribution having pdf and cdf defined in Eq.(1.4) and Eq.(1.5). Then, the likelihood function is obtained as follows,

\[
L(\alpha_1, \beta_1, \alpha_2, \beta_2 | x^R, y^S) = \left[ c_1 \prod_{i=1}^{m_1} f(x_{i;m_1:n_1}) [1 - F(x_{i;m_1:n_1})]^{R_i} \right] \\
\times \left[ c_2 \prod_{j=1}^{m_2} f(y_{j;m_2:n_2}) [1 - F(y_{j;m_2:n_2})]^{S_j} \right]
\]

(3.1) \(L(\alpha_1, \beta_1, \alpha_2, \beta_2 | x^R, y^S) = c_1 c_2 \left( \frac{\beta_1}{\alpha_1} \right)^{m_1} \left( \frac{\beta_2}{\alpha_2} \right)^{m_2}
\times \prod_{i=1}^{m_1} \left( \frac{x_{i;m_1:n_1}}{\alpha_1} \right)^{\beta_1-1} \prod_{j=1}^{m_2} \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2-1}
\times \exp \left( \sum_{i=1}^{m_1} \frac{x_{i;m_1:n_1}}{\alpha_1} \right)^{\beta_1} \exp \left( \sum_{i=1}^{m_1} (1 + R_i) \left( 1 - \exp \left( \frac{x_{i;m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right) \right)
\times \exp \left( \sum_{j=1}^{m_2} \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \exp \left( \sum_{j=1}^{m_2} (1 + S_j) \left( 1 - \exp \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right) \right)
\]

where
\[
c_1 = n_1(n_1 - 1 - R_1) \times \cdots \times (n_1 - m_1 + 1 - R_1 - \cdots - R_{m_1-1}),
\]
\[
c_2 = n_2(n_2 - 1 - S_1) \times \cdots \times (n_2 - m_2 + 1 - S_1 - \cdots - S_{m_2-1}).
\]

Then, the log-likelihood function is given by,

\[
\ell \left( \alpha_1, \beta_1, \alpha_2, \beta_2 | x^R, y^S \right) = m_1 \ln \beta_1 - m_1 \ln \alpha_1 + m_2 \ln \beta_2 - m_2 \ln \alpha_2 + \ln c_1 + \ln c_2
\]

(3.2)
\[
+ (\beta_1 - 1) \sum_{i=1}^{m_1} \ln \left( \frac{x_{i;m_1:n_1}}{\alpha_1} \right) + (\beta_2 - 1) \sum_{j=1}^{m_2} \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)
\]
\[
+ \sum_{i=1}^{m_1} \left( \frac{x_{i;m_1:n_1}}{\alpha_1} \right)^{\beta_1} \left[ 1 - \exp \left( \frac{x_{i;m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right]
\]
\[
+ \sum_{j=1}^{m_2} \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \left[ 1 - \exp \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right]
\]

By differentiating partially the log-likelihood function \(\ell \left( \alpha_1, \beta_1, \alpha_2, \beta_2 | x^R, y^S \right)\) with respect to \(\alpha_1, \beta_1, \alpha_2\) and \(\beta_2\) parameters and then equalizing them to zero we get,
\[
\frac{\partial \ell}{\partial \alpha_1} = -\frac{m_1}{\alpha_1} + \sum_{i=1}^{m_1} \left[ \frac{\beta_1}{\alpha_1} \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right]
+ \sum_{i=1}^{m_1} (1 + R_i) \left[ \frac{\beta_1}{\alpha_1} \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right)^{\beta_1} \exp \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right] - \frac{(\beta_1 - 1) m_1}{\alpha_1} = 0
\]

\[
\frac{\partial \ell}{\partial \beta_1} = \frac{m_1}{\beta_1} + \sum_{i=1}^{m_1} \left[ \frac{(x_{i,m_1:n_1})^{\beta_1}}{\alpha_1} \ln \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right) \right] + \sum_{i=1}^{m_1} \ln \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right)
- \sum_{i=1}^{m_1} (1 + R_i) \left[ \frac{(x_{i,m_1:n_1})^{\beta_1}}{\alpha_1} \ln \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right) \exp \left( \frac{x_{i,m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right] = 0
\]

\[
\frac{\partial \ell}{\partial \alpha_2} = -\frac{m_2}{\alpha_2} + \sum_{j=1}^{m_2} \left[ -\frac{\beta_2}{\alpha_2} \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right]
+ \sum_{j=1}^{m_2} (1 + S_j) \left[ \frac{\beta_2}{\alpha_2} \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right)^{\beta_2} \exp \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] - \frac{(\beta_2 - 1) m_2}{\alpha_2} = 0
\]

\[
\frac{\partial \ell}{\partial \beta_2} = \frac{m_2}{\beta_2} + \sum_{j=1}^{m_2} \left[ \frac{(y_{j,m_2:n_2})^{\beta_2}}{\alpha_2} \ln \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right) \right] + \sum_{j=1}^{m_2} \ln \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right)
- \sum_{j=1}^{m_2} (1 + S_j) \left[ \frac{(y_{j,m_2:n_2})^{\beta_2}}{\alpha_2} \ln \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right) \exp \left( \frac{y_{j,m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] = 0
\]

It is clear that the normal equations do not have explicit forms. MLEs of \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) parameters are obtained by using Newton Raphson method. By using Eq.(2.3) and invariant property of MLE, MLE for \( P \) is obtained as follows.

\[
\hat{P}_{MLE} = 1 - \int_{y=0}^{\infty} \hat{\beta}_2 \left( \frac{y}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} \exp \left( \frac{y}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \times
\]

\[
\times \exp \left[ 1 - \exp \left( \frac{y}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \right] \exp \left[ 1 - \exp \left( \frac{y}{\hat{\alpha}_1} \right)^{\hat{\beta}_1} \right] dy
\]

Moreover, Bootstrap method is used to obtain confidence intervals of \( P \).
3.1. Bootstrap Confidence Intervals. The bootstrap method that is widely used in practise is the percentile bootstrap (Boot-p) proposed by Efron (1982). It is illustrated shortly in following steps how to estimate parametric bootstrap confidence intervals of $P$ using Boot-p method.

Step 1. Generate PTR-II censored samples $(x_{R_1}^{1:m_1:n_1}, x_{R_2}^{2:m_2:n_2}, ..., x_{R_m}^{m_1:n_1})$ and $(y_{S_1}^{1:m_2:n_2}, y_{S_2}^{2:m_2:n_2}, ..., y_{S_m}^{m_2:n_2})$ taken from EP distributions with $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ parameters, respectively.

Step 2. Estimate $P$, say $\hat{P}_{MLE}$.

Step 3. Generate a bootstrap samples $(x_{1:m_1:n_1}^*, x_{2:m_1:n_1}^*, ..., x_{m_1:n_1}^*)$ and $(y_{1:m_2:n_2}^*, y_{2:m_2:n_2}^*, ..., y_{m_2:n_2}^*)$, using $\hat{P}_{MLE}, R_1, R_2, ..., R_m, S_1, S_2, ..., S_m$. Obtain the bootstrap estimate of $P$, say $\hat{P}_{MLE}^*$.

Step 4. Repeat Step 3 NBOOT times.

Step 5. Let $F^*(x) = P(\hat{P}_{MLE} \leq x)$, be the cumulative distribution function of $\hat{P}_{MLE}$. Define $\hat{P}_{MLEboot-p}(x) = F^{*-1}(x)$ for a given $x$. The approximate 100$(1-\gamma)$% confidence interval for $P$ is given by

$$\left(\hat{P}_{MLEboot-p}(\gamma/2), \hat{P}_{MLEboot-p}(1-\gamma/2)\right)$$

4. Bayes Estimation

Let $(X_{R_1}^{1:m_1:n_1}, X_{R_2}^{2:m_2:n_2}, ..., X_{R_m}^{m_1:n_1})$ and $(Y_{S_1}^{1:m_2:n_2}, Y_{S_2}^{2:m_2:n_2}, ..., Y_{S_m}^{m_2:n_2})$ are PTR-II censored samples taken from $EP(\alpha_1, \beta_1)$ and $EP(\alpha_2, \beta_2)$, respectively. For Bayesian estimation of SSR under PTR-II censored sample, we assume that the $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$ parameters have independent prior $Gamma(a_1, b_1)$, $Gamma(a_2, b_2)$, $Gamma(a_3, b_3)$ and $Gamma(a_4, b_4)$ distributions, respectively. These prior density functions are given as follows.

$$\pi_1(\alpha_1) = a_1^{\alpha_1-1} \exp(-b_1 \alpha_1) \frac{1}{\Gamma(\alpha_1)} a_1, b_1, \alpha_1 > 0$$

$$\pi_2(\beta_1) = b_1^{\beta_1-1} \exp(-b_2 \beta_1) \frac{1}{\Gamma(\beta_1)} a_2, b_2, \beta_1 > 0$$

$$\pi_3(\alpha_2) = a_2^{\alpha_2-1} \exp(-b_3 \alpha_2) \frac{1}{\Gamma(\alpha_2)} a_3, b_3, \alpha_2 > 0$$

$$\pi_4(\beta_2) = b_2^{\beta_2-1} \exp(-b_4 \beta_2) \frac{1}{\Gamma(\beta_2)} a_4, b_4, \beta_2 > 0$$

Then, the joint prior density function of $\alpha_1, \beta_1$, $\alpha_2$ and $\beta_2$ can be written as follows.

$$\pi(\alpha_1, \beta_1, \alpha_2, \beta_2) = a_1^{\alpha_1-1} \frac{1}{\Gamma(\alpha_1)} a_2^{\alpha_2-1} \frac{1}{\Gamma(\alpha_2)} a_3^{\alpha_3-1} \frac{1}{\Gamma(\alpha_3)} a_4^{\alpha_4-1} \frac{1}{\Gamma(\alpha_4)} \times$$

$$\times \exp(-b_1 \alpha_1) \exp(-b_2 \beta_1) \exp(-b_3 \alpha_2) \exp(-b_4 \beta_2) a_i, b_i, \alpha_i, \beta_i > 0, i = 1, 2, 3, 4$$
From (4.5), the log of joint prior density function is given by

\[
G(\alpha_1, \beta_1, \alpha_2, \beta_2) = \ln \pi (\alpha_1, \beta_1, \alpha_2, \beta_2) \propto (a_1 - 1) \ln \alpha_1 + (a_2 - 1) \ln \beta_1 + (a_3 - 1) \ln \alpha_2 + (a_4 - 1) \ln \beta_2 - b_1 \alpha_1 - b_2 \beta_1 - b_3 \alpha_2 - b_4 \beta_2
\]

With the help of equation (3.1) and (4.5), the joint posterior density function of \(\alpha_1, \beta_1, \alpha_2, \beta_2\) when data are given can be written as follows:

\[
P(\alpha_1, \beta_1, \alpha_2, \beta_2 | X^R, Y^S) =
\]

\[
\frac{\pi (\alpha_1, \beta_1, \alpha_2, \beta_2) k_1 (x_{i:m_1}; \alpha_1, \beta_1) k_2 (y_{j:m_2}; \alpha_2, \beta_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \pi (\alpha_1, \beta_1, \alpha_2, \beta_2) k_1 (x_{i:m_1}; \alpha_1, \beta_1) k_2 (y_{j:m_2}; \alpha_2, \beta_2) d\alpha d\beta}
\]

where

\[
d\alpha d\beta = d\alpha_1 d\beta_1 d\alpha_2 d\beta_2
\]

and

\[
k_1 (x_{i:m_1}; \alpha_1, \beta_1) = \left( \frac{\beta_1}{\alpha_1} \right)^{m_1} \exp \left( \sum_{i=1}^{m_1} \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right) \times
\]

\[
\times \exp \left[ \sum_{i=1}^{m_1} (1 + R_i) \left( 1 - \exp \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right) \right] \prod_{i=1}^{m_1} \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1-1}
\]

and

\[
k_2 (y_{j:m_2}; \alpha_2, \beta_2) = \left( \frac{\beta_2}{\alpha_2} \right)^{m_2} \exp \left( \sum_{j=1}^{m_2} \left( \frac{y_{j:m_2}}{\alpha_2} \right)^{\beta_2} \right) \times
\]

\[
\times \exp \left[ \sum_{j=1}^{m_2} (1 + S_j) \left( 1 - \exp \left( \frac{y_{j:m_2}}{\alpha_2} \right)^{\beta_2} \right) \right] \prod_{j=1}^{m_2} \left( \frac{y_{j:m_2}}{\alpha_2} \right)^{\beta_2-1}
\]

Let any function of \(\alpha_1, \beta_1, \alpha_2\) and \(\beta_2\) is \(u(\alpha_1, \beta_1, \alpha_2, \beta_2)\). The bayesian estimation of this function under squared error loss function can be written as follows,

\[
\hat{u}_B (\alpha_1, \beta_1, \alpha_2, \beta_2) = E \left( u(\alpha_1, \beta_1, \alpha_2, \beta_2) | X^R, Y^S \right)
\]

\[
= \frac{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty u(\alpha_1, \beta_1, \alpha_2, \beta_2) \exp \left[ \ell (\alpha_1, \beta_1, \alpha_2, \beta_2 | x^R, y^S) + G(\alpha_1, \beta_1, \alpha_2, \beta_2) \right] d\alpha d\beta}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \exp \left[ \ell (\alpha_1, \beta_1, \alpha_2, \beta_2 | x^R, y^S) + G(\alpha_1, \beta_1, \alpha_2, \beta_2) \right] d\alpha d\beta}
\]

In Eq.(4.8), if \(u(\alpha_1, \beta_1, \alpha_2, \beta_2)\) is taken as \(P\), Bayes estimator of SSR can be obtained. But, it is not possible to compute Eq.(4.8) in closed-form. Because of this reason, we have obtained the Bayes Estimators of SSR by using Lindley’s approximation under the squared error loss function.
4.1. Lindley’s Approximation. Lindley’s approximation suggested by Lindley (1980) is an approximate method used to compute the ratio of two integrals as in Eq. (4.8) that cannot be solved analytically. The Lindley’s approximation formulas for case with four-parameters by using \((\theta_1, \theta_2, \theta_3, \theta_4)\) notation instead of \((\alpha_1, \beta_1, \alpha_2, \beta_2)\) can be written as follows.

\[
(4.9) \quad u (\hat{\theta})_{Bayes} = E [u(\theta)|X] 
\approx \left\{ u(\hat{\theta_1}, \hat{\theta_2}, \hat{\theta_3}, \hat{\theta_4}) + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} (u_{ij} + 2u_i g_j) \sigma_{ij} \right. \\
+ \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \sum_{l=1}^{4} \ell_{ijkl} \sigma_{ij} \sigma_{kl} u_l \right\}
\]

where \(\hat{\theta_1}, \hat{\theta_2}, \hat{\theta_3}\) and \(\hat{\theta_4}\) are the MLEs of \(\theta_1, \theta_2, \theta_3\) and \(\theta_4\), respectively and other formulas are as follows.

\[
u_i = \frac{\partial u (\theta_1, \theta_2, \theta_3, \theta_4)}{\partial \theta_i}, \quad i = 1, 2, 3, 4
\]

\[
u_{ij} = \frac{\partial^2 u (\theta_1, \theta_2, \theta_3, \theta_4)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2, 3, 4
\]

\[
\ell_{ij} = \frac{\partial^2 \ell (\theta_1, \theta_2, \theta_3, \theta_4)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2, 3, 4
\]

\[
\ell_{ijk} = \frac{\partial^3 \ell (\theta_1, \theta_2, \theta_3, \theta_4)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \quad i, j, k = 1, 2, 3, 4
\]

\([-\ell_{ij}]^{-1} = [\sigma_{ij}], \quad i, j = 1, 2, 3, 4\]

and \(\sigma_{ij}\) is the \((i, j)\)-th element of the matrix \([\sigma_{ij}]\). From (4.6), we have

\[
g_1 = \frac{\partial G(\alpha_1, \beta_1, \alpha_2, \beta_2)}{\partial \alpha_1} = \frac{a_1 - 1}{\alpha_1} - b_1, \quad g_2 = \frac{\partial G(\alpha_1, \beta_1, \alpha_2, \beta_2)}{\partial \beta_1} = \frac{a_2 - 1}{\beta_1} - b_2
\]

\[
g_3 = \frac{\partial G(\alpha_1, \beta_1, \alpha_2, \beta_2)}{\partial \alpha_2} = \frac{a_3 - 1}{\alpha_2} - b_3, \quad g_4 = \frac{\partial G(\alpha_1, \beta_1, \alpha_2, \beta_2)}{\partial \beta_2} = \frac{a_4 - 1}{\beta_2} - b_4
\]

and then, we have the following values of \(\ell_{ij}\) for \(i, j = 1, 2, 3, 4\) and \(\ell_{ijk}\) for \(i, j, k = 1, 2, 3, 4\).

\[
\ell_{11} = \frac{m_1}{\alpha_1^2} + \left( \frac{\beta_1 - 1}{\alpha_1} \right) m_1 + \sum_{i=1}^{m_1} (1 + \beta_1) g_1 (x_{i:m_1:n_1} ; \alpha_1, \beta_1) +
\]

\[
+ \sum_{i=1}^{m_1} (1 + R_i) h_1 (x_{i:m_1:n_1} ; \alpha_1, \beta_1) \left( -\beta_1 - 1 - \beta_1 \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right)
\]

\[
\ell_{12} = \sum_{i=1}^{m_1} g_1 (x_{i:m_1:n_1} ; \alpha_1, \beta_1) \left( -\ln \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right) \alpha_1 - \frac{\alpha_1}{\beta_1} \right) +
\]

\[
+ \sum_{i=1}^{m} h_1 (x_{i:m_1:n_1} ; \alpha_1, \beta_1) (1 + R_i) \left[ \ln \left( \frac{x_{i:m_1:n}}{\alpha_1} \right) \left( 1 + \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right) + \left( \frac{1}{\beta_1} \right) \right] - \frac{m_1}{\alpha_1}
\]
$\ell_{13} = \ell_{31} = 0, \quad \ell_{14} = \ell_{41} = 0, \quad \ell_{21} = \ell_{12}$

$\ell_{22} = -\frac{m_1}{\beta_1^2} + \sum_{i=1}^{m_1} \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \ln \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^2 -$

$- \sum_{i=1}^{m_1} h_1(x_{i:m_1:n_1};\alpha_1,\beta_1) (1 + R_i) \frac{\alpha_1^2}{\beta_1} \ln \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^2 \left[ 1 + \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right]$  

$\ell_{23} = \ell_{32} = 0, \quad \ell_{24} = \ell_{42} = 0$

$\ell_{33} = \frac{m_2}{\alpha_2^2} + \frac{(\beta_2 - 1) m_2}{\alpha_2^2} + \sum_{j=1}^{m_2} (1 + \beta_2) g_2(y_{j:m_2:n_2};\alpha_2,\beta_2) +$

$+ \sum_{j=1}^{m_2} (1 + S_j) h_2(y_{j:m_2:n_2};\alpha_2,\beta_2) \left( -\beta_2 - 1 - \beta_2 \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right)$

$\ell_{34} = \sum_{j=1}^{m_2} g_1(y_{j:m_2:n_2};\alpha_2,\beta_2) \left( -\ln \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right) \alpha_2 - \frac{\alpha_2}{\beta_2} \right) +$

$+ \sum_{j=1}^{m_2} h_2(y_{j:m_2:n_2};\alpha_2,\beta_2) \alpha_2 (1 + S_j) \left[ \ln \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right) \left( 1 + \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right) + \left( \frac{1}{\beta_2} \right) \right] - \frac{m_2}{\alpha_2}$

$\ell_{43} = \ell_{34}$

$\ell_{44} = -\frac{m_2}{\beta_2^2} + \sum_{j=1}^{m_2} \left( \frac{y_{j:m_2:n_2}}{\alpha_1} \right)^{\beta_2} \ln \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right)^2 -$

$- \sum_{j=1}^{m_2} h_2(y_{j:m_2:n_2};\alpha_2,\beta_2) (1 + S_j) \frac{\alpha_2^2}{\beta_2} \ln \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right)^2 \left[ 1 + \left( \frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right]$  

$\ell_{111} = -\frac{2m_1}{\alpha_1^3} - \sum_{i=1}^{m_1} g_1(x_{i:m_1:n_1};\alpha_1,\beta_1) \left[ \beta_1^2 + 3\beta_1 + 2 \right] + \frac{2(\beta_1 - 1) m_1}{\alpha_1^3} +$

$+ \sum_{i=1}^{m_1} h_1(x_{i:m_1:n_1};\alpha_1,\beta_1) (1 + R_i) \left[ \beta_1^2 \left( 1 + 3 \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} + \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{2\beta_1} \right) \right]$

$+ 3\beta_1 \left( 1 + \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right) + 2$  

$\ell_{112} = \frac{m_1}{\alpha_1^2} + \sum_{i=1}^{m_1} g_1(x_{i:m_1:n_1};\alpha_1,\beta_1) \left[ \ln \left( \frac{x_{i:m_1:n_1}}{\alpha_1} \right) (1 + \beta_1) + \left( 2 + \frac{1}{\beta_1} \right) \right] -$
\[-\sum_{i=1}^{m_1} h_1 (x_{i:m_1}; \alpha_1, \beta_1) (1 + R_i) \left[ \beta_1 \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right) \left( 1 + 3 \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right) \right] - \]

\[-\sum_{i=1}^{m_1} h_1 (x_{i:m_1}; \alpha_1, \beta_1) (1 + R_i) \left[ 2 \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right) + \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right] - \]

\[-\sum_{i=1}^{m_1} \frac{h_1 (x_{i:m_1}; \alpha_1, \beta_1)}{\beta_1} (1 + R_i) \]

\[\ell_{122} = \ell_{121} = \ell_{211}, \quad \ell_{122} = \ell_{221} = \ell_{212} \]

\[\ell_{222} = \ell_{232} = \ell_{322} = 0, \quad \ell_{224} = \ell_{242} = \ell_{422} = 0 \]

\[\ell_{233} = \ell_{332} = \ell_{323} = 0, \quad \ell_{133} = \ell_{313} = \ell_{331} = 0 \]

\[\ell_{114} = \ell_{141} = \ell_{411} = 0, \quad \ell_{144} = \ell_{441} = \ell_{414} = 0 \]

\[\ell_{244} = \ell_{442} = \ell_{424} = 0, \quad \ell_{133} = \ell_{313} = \ell_{331} = 0 \]

\[\ell_{122} = \sum_{i=1}^{m_1} g_1 (x_{i:m_1}; \alpha_1, \beta_1) \alpha_1 \beta_1 \left[ -\beta_1 \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right)^2 - 2 \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right) \right] + \]

\[+ \sum_{i=1}^{m_1} h_1 (x_{i:m_1}; \alpha_1, \beta_1) \alpha_1 \beta_1 \left( 1 + R_i \right) \left[ \beta_1 \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right)^2 \left( 1 + 3 \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right) \right] \]

\[+ 2 \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right) \left( 1 + \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right) \]

\[\ell_{222} = \frac{2m_1}{\beta_1^2} + \sum_{i=1}^{m_1} \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{3} - \]

\[-\sum_{i=1}^{m_1} \frac{h_1 (x_{i:m_1}; \alpha_1, \beta_1)}{\beta_1} (1 + R_i) \ln \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{3} \left[ 1 + 3 \left( \frac{x_{i:m_1}}{\alpha_1} \right)^{\beta_1} \right] \]

\[\ell_{123} = \ell_{132} = \ell_{213} = \ell_{231} = \ell_{312} = \ell_{321} = 0 \]

\[\ell_{124} = \ell_{142} = \ell_{214} = \ell_{241} = \ell_{412} = \ell_{421} = 0 \]

\[\ell_{134} = \ell_{143} = \ell_{314} = \ell_{341} = \ell_{413} = \ell_{431} = 0 \]

\[\ell_{234} = \ell_{243} = \ell_{342} = \ell_{423} = \ell_{432} = 0 \]
\[ \ell_{344} = \frac{m_2}{\alpha_2} + \sum_{j=1}^{m_2} g_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) \left[ \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right) (1 + \beta_2) + \left( 2 + \frac{1}{\beta_2} \right) \right] - \\
- \sum_{j=1}^{m_2} h_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) (1 + S_j) \left[ \beta_2 \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right) \left( 1 + 3 \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right) + \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{2\beta_2} \right] - \\
- \sum_{j=1}^{m_2} h_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) (1 + S_j) \left[ 2 + \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right) + 2 \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] - \\
- \sum_{j=1}^{m_2} h_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) (1 + S_j) \frac{1}{\beta_2} \left( 1 + S_j \right) \] 

\[ \ell_{433} = \ell_{434}, \quad \ell_{344} = \ell_{443} = \ell_{434} \]

\[ \ell_{333} = -\frac{2m_2}{\alpha_2} - \sum_{j=1}^{m_2} g_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) \frac{\beta_2}{\alpha_2} \left[ \beta_2^2 + 3\beta_2 + 2 \right] + \frac{2(\beta_2 - 1)m_2}{\alpha_2^3} + \\
+ \sum_{j=1}^{m_2} h_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) (1 + S_j) \left[ \beta_2^2 \left( 1 + 3 \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right) + \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{2\beta_2} \right] + \\
+ 3\beta_2 \left( 1 + \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right) + 2 \right] \]

\[ \ell_{444} = \frac{2m_2}{\beta_2^3} + \sum_{i=1}^{m_2} \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^3 - \\
- \sum_{j=1}^{m_2} h_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) \frac{\alpha_2^2}{\beta_2} (1 + S_j) \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^3 \left[ 1 + 3 \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] + \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{2\beta_2} \right] \]

\[ \ell_{443} = \sum_{j=1}^{m_2} g_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) \frac{\alpha_2}{\beta_2} \left[ - \beta_2 \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^2 - 2 \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right) \right] + \\
+ \sum_{j=1}^{m_2} h_2 \left( y_{j;m_2:n_2}; \alpha_2, \beta_2 \right) \frac{\alpha_2}{\beta_2} (1 + S_j) \left[ \beta_2 \ln \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^2 \left( 1 + \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{2\beta_2} \right) \right] + \\
+ 3\beta_2 \left( 1 + \left( \frac{y_{j;m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right) + 2 \right] \]
where \( g_1(x_i; m_i; \alpha_1, \beta_1), g_1(x_i; m_i; \alpha_1, \beta_1), g_2(y_j; m_2; \alpha_2, \beta_2) \) and \( h_2(y_j; m_2; \alpha_2, \beta_2) \) are given by

\[
g_1(x_i; m_1; \alpha_1, \beta_1) = \frac{\left( \frac{x_i^{m_1}}{\alpha_1} \right)^{\beta_1}}{\alpha_1^2}
\]

\[
g_2(y_j; m_2; \alpha_2, \beta_2) = \frac{\left( \frac{y_j^{m_2}}{\alpha_2} \right)^{\beta_2}}{\alpha_2^2}
\]

\[
h_1(x_i; m_1; \alpha_1, \beta_1) = \frac{\left( \frac{x_i^{m_1}}{\alpha_1} \right)^{\beta_1} \exp \left( \frac{x_i^{m_1}}{\alpha_1} \right)}{\alpha_1^2}
\]

and

\[
h_2(y_j; m_2; \alpha_2, \beta_2) = \frac{\left( \frac{y_j^{m_2}}{\alpha_2} \right)^{\beta_2} \exp \left( \frac{y_j^{m_2}}{\alpha_2} \right)}{\alpha_2^2}
\]

Finally, the approximate Bayes estimator under the squared error loss function of SSR for EP distribution based on PTR-II censored samples are obtained as follows,

\[
(4.10) \quad \hat{\beta}_{BS} = \hat{\beta}_{MLE} + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{4} (u_{ij} + 2u_i g_j) \sigma_{ij}^2 + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \sum_{l=1}^{4} \ell_{ijkl} \sigma_{ij} \sigma_{kl} u_l
\]

where \( \hat{\beta}_{MLE} \) is defined in Eq.(3.7).

5. Simulation Study

In this section, a Monte Carlo simulation study is given to observe the performances of two estimation methods for different sample sizes and different censoring schemes for SSR based on EP distribution under PTR-II censoring. For different sample sizes and censoring schemes, by using the algorithm presented in Balakrishnan and Sandhu (1995), PTR-II censored samples are generated from EP distribution. This algorithm is as follows:

1. \( W_1, W_2, \ldots, W_{m_1} \) are \( m_1 \)-sized samples generated from Uniform (0,1) distribution.
2. \( V_i = W_i \left( \sum_{j=m_i}^{m_1} R_j \right)^{-1} \) is defined for \( i = 1, 2, \ldots, m_1 \).
3. \( U_{i:m_i} \) is defined for \( i = 1, 2, \ldots, m_1 \).

Thus \( U_{1:m_i,n_1}^R < U_{2:m_i,n_1}^R < \ldots < U_{m_i:m_i,n_1}^R \) are obtained PTR-II censored samples with censoring scheme \( R = (R_1, R_2, \ldots, R_{m_1}) \) taken from Uniform (0,1) distribution. Finally, PTR-II censored \( i^{th} \) order statistics with censoring scheme \( R = (R_1, R_2, \ldots, R_{m_1}) \) taken from EP (\( \alpha_1, \beta_1 \)) is obtained by transformation

\[
X_{i:m_i,n_1}^R = \ln \left( -\ln \left( U_{i:m_i,n_1}^R + 1 \right) + 1 \right) \frac{1}{\alpha_1}, \quad i = 1, 2, \ldots, m_1
\]

and PTR-II censored \( j^{th} \) order statistics with censoring scheme \( S = (S_1, S_2, \ldots, S_{m_2}) \) taken from EP (\( \alpha_2, \beta_2 \)) distribution is obtained by transformation

\[
Y_{j:m_2}^S = \ln \left( -\ln \left( U_{j:m_2}^S + 1 \right) + 1 \right) \frac{1}{\alpha_2}, \quad j = 1, 2, \ldots, m_2
\]
Then, the MSEs of the MLE of SSR are compared with MSEs of the approximate Bayes estimates obtained by Lindley approximation under the squared error loss function based on PTR-II censoring. For comparison purposes, the approximate Bayes estimates under informative priors \( a1 = 1; b1 = 1; a2 = 1; b2 = 1; a3 = 1; b3 = 1; a4 = 1; b4 = 1 \) are computed. We have reported MSE of the ML estimates and approximate bayes estimates over 5000 replications. The results are given in Table 1. Moreover, we have reported length and coverage probabilities based on bootstrap confidence intervals of SSR over 1000 bootstrap sample with 2000 replications. This results are given by Table 2.

Table 1. The MSEs of MLEs and Approximate Bayes Estimates for \( \alpha_1 = 1, \beta_1 = 2, \alpha_2 = 1 \) and \( \beta_2 = 1 \)

<table>
<thead>
<tr>
<th>Censoring scheme</th>
<th>( R_{MLE} )</th>
<th>( \hat{R}_{BAYES} )</th>
<th>MSE</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1, m_1 )</td>
<td>( n_2, m_2 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5,5</td>
<td>5,5</td>
<td>(5 * 0)</td>
<td>(5 * 0)</td>
<td>0.0273</td>
</tr>
<tr>
<td>10,5</td>
<td>10,5</td>
<td>(4 * 0, 5)</td>
<td>(4 * 0, 5)</td>
<td>0.0200</td>
</tr>
<tr>
<td>10,5</td>
<td>10,5</td>
<td>(5, 4 * 0)</td>
<td>(5, 4 * 0)</td>
<td>0.0229</td>
</tr>
<tr>
<td>10,5</td>
<td>10,5</td>
<td>(5 * 1)</td>
<td>(5 * 1)</td>
<td>0.0201</td>
</tr>
<tr>
<td>10,10</td>
<td>10,10</td>
<td>(10 * 0)</td>
<td>(10 * 0)</td>
<td>0.0156</td>
</tr>
<tr>
<td>20,10</td>
<td>20,10</td>
<td>(9 * 0, 10)</td>
<td>(9 * 0, 10)</td>
<td>0.0119</td>
</tr>
<tr>
<td>20,10</td>
<td>20,10</td>
<td>(10, 9 * 0)</td>
<td>(10, 9 * 0)</td>
<td>0.0134</td>
</tr>
<tr>
<td>20,20</td>
<td>20,20</td>
<td>(10 * 1)</td>
<td>(10 * 1)</td>
<td>0.0116</td>
</tr>
<tr>
<td>30,10</td>
<td>30,10</td>
<td>(9 * 0, 20)</td>
<td>(9 * 0, 20)</td>
<td>0.0147</td>
</tr>
<tr>
<td>30,10</td>
<td>30,10</td>
<td>(20, 9 * 0)</td>
<td>(20, 9 * 0)</td>
<td>0.0132</td>
</tr>
<tr>
<td>30,30</td>
<td>30,30</td>
<td>(10 * 2)</td>
<td>(10 * 2)</td>
<td>0.0127</td>
</tr>
<tr>
<td>100,50</td>
<td>100,50</td>
<td>(49 * 0, 50)</td>
<td>(49 * 0, 50)</td>
<td>0.0032</td>
</tr>
<tr>
<td>100,50</td>
<td>100,50</td>
<td>(50, 49 * 0)</td>
<td>(50, 49 * 0)</td>
<td>0.0030</td>
</tr>
<tr>
<td>100,50</td>
<td>100,50</td>
<td>(50 * 1)</td>
<td>(50 * 1)</td>
<td>0.0027</td>
</tr>
<tr>
<td>100,100</td>
<td>100,100</td>
<td>(100 * 0)</td>
<td>(100 * 0)</td>
<td>0.0017</td>
</tr>
</tbody>
</table>

Table 2. Average length and coverage probabilities of bootstrap confidence intervals for \( P \) for \( \alpha_1 = 1, \beta_1 = 2, \alpha_2 = 1 \) and \( \beta_2 = 1 \)

<table>
<thead>
<tr>
<th>Censoring scheme</th>
<th>Average length</th>
<th>Coverage Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1, m_1 )</td>
<td>( n_2, m_2 )</td>
<td></td>
</tr>
<tr>
<td>20,20</td>
<td>20,20</td>
<td>(20 * 0)</td>
</tr>
<tr>
<td>(1) 100,50</td>
<td>100,50</td>
<td>(49 * 0, 50)</td>
</tr>
<tr>
<td>100,50</td>
<td>100,50</td>
<td>(50, 49 * 0)</td>
</tr>
<tr>
<td>100,50</td>
<td>100,50</td>
<td>(50 * 1)</td>
</tr>
<tr>
<td>100,100</td>
<td>100,100</td>
<td>(100 * 0)</td>
</tr>
</tbody>
</table>

6. Data Analysis

\[ Y_j^{(n_1,m_2)} = \ln \left( -\ln \left( U_j^{(n_1,m_2)} + 1 \right) \right)^{\frac{1}{\beta_2}} \alpha_2, \quad j = 1, 2, \ldots, m_2. \]
To compute estimation methods of $P$, it is considered two datasets, originally reported by Bader and Priest (1982), on failure stresses (in GPa) of single carbon fibers of lengths 20 mm and 50 mm, respectively (Ghitany et al. (2014)). The datasets are given in Table 3 and in Table 4.

Table 3. Length 20 mm: ($n_1 = 69$)

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.312</td>
<td>1.314</td>
<td>1.479</td>
<td>1.552</td>
<td>1.700</td>
<td>1.803</td>
<td>1.861</td>
<td>1.865</td>
<td>1.944</td>
<td></td>
</tr>
<tr>
<td>1.958</td>
<td>1.966</td>
<td>1.997</td>
<td>2.006</td>
<td>2.021</td>
<td>2.027</td>
<td>2.055</td>
<td>2.063</td>
<td>2.098</td>
<td></td>
</tr>
<tr>
<td>2.140</td>
<td>2.179</td>
<td>2.224</td>
<td>2.240</td>
<td>2.253</td>
<td>2.270</td>
<td>2.272</td>
<td>2.274</td>
<td>2.301</td>
<td></td>
</tr>
<tr>
<td>2.301</td>
<td>2.359</td>
<td>2.382</td>
<td>2.382</td>
<td>2.426</td>
<td>2.434</td>
<td>2.435</td>
<td>2.478</td>
<td>2.490</td>
<td></td>
</tr>
<tr>
<td>2.511</td>
<td>2.514</td>
<td>2.535</td>
<td>2.554</td>
<td>2.566</td>
<td>2.570</td>
<td>2.586</td>
<td>2.629</td>
<td>2.633</td>
<td></td>
</tr>
<tr>
<td>2.642</td>
<td>2.648</td>
<td>2.684</td>
<td>2.697</td>
<td>2.726</td>
<td>2.770</td>
<td>2.773</td>
<td>2.800</td>
<td>2.809</td>
<td></td>
</tr>
<tr>
<td>2.818</td>
<td>2.821</td>
<td>2.848</td>
<td>2.880</td>
<td>2.954</td>
<td>3.012</td>
<td>3.067</td>
<td>3.084</td>
<td>3.090</td>
<td></td>
</tr>
<tr>
<td>3.096</td>
<td>3.128</td>
<td>3.233</td>
<td>3.433</td>
<td>3.585</td>
<td>3.585</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Length 50 mm: ($n_2 = 65$)

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.339</td>
<td>1.434</td>
<td>1.549</td>
<td>1.574</td>
<td>1.589</td>
<td>1.613</td>
<td>1.746</td>
<td>1.753</td>
<td>1.764</td>
<td></td>
</tr>
<tr>
<td>1.807</td>
<td>1.812</td>
<td>1.840</td>
<td>1.852</td>
<td>1.852</td>
<td>1.862</td>
<td>1.864</td>
<td>1.931</td>
<td>1.952</td>
<td></td>
</tr>
<tr>
<td>1.974</td>
<td>2.019</td>
<td>2.051</td>
<td>2.055</td>
<td>2.058</td>
<td>2.088</td>
<td>2.125</td>
<td>2.162</td>
<td>2.171</td>
<td></td>
</tr>
<tr>
<td>2.172</td>
<td>2.180</td>
<td>2.194</td>
<td>2.211</td>
<td>2.270</td>
<td>2.272</td>
<td>2.280</td>
<td>2.299</td>
<td>2.308</td>
<td></td>
</tr>
<tr>
<td>2.335</td>
<td>2.349</td>
<td>2.356</td>
<td>2.386</td>
<td>2.390</td>
<td>2.410</td>
<td>2.430</td>
<td>2.431</td>
<td>2.458</td>
<td></td>
</tr>
<tr>
<td>2.471</td>
<td>2.497</td>
<td>2.514</td>
<td>2.558</td>
<td>2.577</td>
<td>2.593</td>
<td>2.601</td>
<td>2.604</td>
<td>2.620</td>
<td></td>
</tr>
<tr>
<td>2.633</td>
<td>2.670</td>
<td>2.682</td>
<td>2.699</td>
<td>2.705</td>
<td>2.735</td>
<td>2.785</td>
<td>3.020</td>
<td>3.042</td>
<td></td>
</tr>
<tr>
<td>3.116</td>
<td>3.174</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

First, it is checked whether EP distribution can be used or not to analyze these data sets. The Kolmogorov-Smirnov (KS) distances have been used to check the goodness-of-fit. ML estimates, KS-Z and p-values based on above data are shown in Table 5.

Table 5. Kolmogorov-Smirnov Z and the corresponding p-values for MLE

<table>
<thead>
<tr>
<th>Carbon Fiber</th>
<th>MLE</th>
<th>Kolmogorov-Smirnov Z</th>
<th>p-values</th>
</tr>
</thead>
<tbody>
<tr>
<td>X (Length 20 mm)</td>
<td>$\hat{\alpha} = 2.9920$</td>
<td>$\hat{\beta} = 3.7061$</td>
<td>0.8519</td>
</tr>
<tr>
<td>Y (Length 30 mm)</td>
<td>$\hat{\alpha} = 2.6964$</td>
<td>$\hat{\beta} = 4.0975$</td>
<td>0.6509</td>
</tr>
</tbody>
</table>

Progressive censored samples based on X and Y for $m_1 = m_2 = 50$, $n_1 = 69$, $n_2 = 65$ are as follows:

Progressive censored samples for X (Length 20 mm)

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.312</td>
<td>1.314</td>
<td>1.479</td>
<td>1.552</td>
<td>1.700</td>
<td>1.803</td>
<td>1.861</td>
<td>1.865</td>
<td>1.944</td>
<td></td>
</tr>
<tr>
<td>1.958</td>
<td>1.997</td>
<td>2.006</td>
<td>2.021</td>
<td>2.027</td>
<td>2.055</td>
<td>2.063</td>
<td>2.098</td>
<td>2.140</td>
<td></td>
</tr>
<tr>
<td>2.179</td>
<td>2.224</td>
<td>2.240</td>
<td>2.270</td>
<td>2.272</td>
<td>2.382</td>
<td>2.426</td>
<td>2.434</td>
<td>2.435</td>
<td></td>
</tr>
<tr>
<td>2.490</td>
<td>2.511</td>
<td>2.535</td>
<td>2.566</td>
<td>2.629</td>
<td>2.633</td>
<td>2.642</td>
<td>2.648</td>
<td>2.684</td>
<td></td>
</tr>
<tr>
<td>2.697</td>
<td>2.726</td>
<td>2.770</td>
<td>2.773</td>
<td>2.818</td>
<td>2.821</td>
<td>2.848</td>
<td>3.012</td>
<td>3.084</td>
<td></td>
</tr>
<tr>
<td>3.090</td>
<td>3.128</td>
<td>3.233</td>
<td>3.433</td>
<td>3.585</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Progressive censored samples for Y (Length 50 mm)
In this case, ML and approximate Bayes estimates for SSR under progressive type II right Censoring are as follows.

<table>
<thead>
<tr>
<th>Censoring scheme</th>
<th>$\hat{R}_{MLE}$</th>
<th>$\hat{P}_{BAYES}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(n_1, m_1)$</td>
<td>$(n_2, m_2)$</td>
<td></td>
</tr>
<tr>
<td>69,50</td>
<td>65,50</td>
<td></td>
</tr>
<tr>
<td>$(19, 49 \times 0)$</td>
<td>$(15, 49 \times 0)$</td>
<td>0.4158</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4187</td>
</tr>
</tbody>
</table>

Moreover, bootstrap confidence intervals for SSR are (0.2765, 0.4649) and bootstrap estimate of SSR is 0.3706.

7. Corollary

In this study, we have considered the estimation problem of SSR for the EP distribution under PTR-II censored sample. It is observed that the maximum likelihood estimators of SSR can be obtained by using some numerical methods. We have obtained the approximate Bayes estimators of SSR by using the idea of Lindley’s approximation under the squared-error loss function. We have compared the MSEs of the approximate Bayes estimators and the MLEs through Monte Carlo simulations, and it has been observed that the performances of the approximate bayes estimates are more satisfactory than ML estimators. When $m_1/n_1$ and $m_2/n_2$ rate increased, MSEs of the MLE and approximate Bayes estimates tended to decrease. The best estimates have been obtained when $n_1 = m_1$ and $n_2 = m_2$ (complete sample case).

References