BEST SUBORDINANTS OF THE STRONG DIFFERENTIAL SUPERORDINATION

Gheorghe Oros*† and Adela Olimpia Tăut‡

Received 23 : 03 : 2009 : Accepted 29 : 04 : 2009

Abstract


In (Strong differential superordination, Acta Universitatis Apulensis 19, 110–106, 2009), Georgia I. Oros introduces the dual concept of strong differential superordinations. The aim of this paper is to obtain the best subordinants of the strong differential superordinations.

Keywords: Differential subordination, Differential superordination, Strong differential subordination, Strong differential superordination, Best subordinator, Univalent function, Analytic function.

2000 AMS Classification: 30 C 45, 30 A 20, 34 A 30.

*Department of Mathematics, University of Oradea, Str. Universității No. 1, 410087 Oradea, Romania. E-mail: gh_oros@yahoo.com
†Corresponding Author.
‡Faculty of Environmental Protection, University of Oradea, B-dul Gen. Magheru, 26, Oradea, Romania. E-mail adela_taut@yahoo.com
1. Introduction and preliminaries

Let $U$ denote the unit disc of the complex plane:
$$U = \{ z \in \mathbb{C} : |z| < 1 \}$$
and
$$\overline{U} = \{ z \in \mathbb{C} : |z| \leq 1 \}.$$ 
Let $\mathcal{H}(U)$ denote the space of holomorphic functions in $U$ and
$$A_n = \{ f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \cdots, z \in U \}$$
with $A_1 = A$, and
$$S = \{ f \in A; \ f \text{ is univalent in } U \},$$
$$\mathcal{H}[a,n] = \{ f \in \mathcal{H}(U) : \ f(z) = a_n z^n + a_{n+1}z^{n+1} + \cdots, z \in U \}.$$ 
Let $\Omega$ and $\Delta$ be any sets in the complex plane $\mathbb{C}$, let $p$ be analytic in the unit disc $U$ and $\psi: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$.

In a series of articles such as [4, 6, 7, 8] the authors have determined properties of functions $p$ that satisfy the strong differential subordination

(i) $\{ \psi(p(z),zp'(z),z^2p''(z); z,\xi) \mid z \in U, \ \xi \in \overline{U} \} \subset \Omega \Rightarrow p(U) \subset \Delta.$

In [5] the author considers the dual problem of determining properties of functions $p$ that satisfy the strong differential superordination

(ii) $\Omega \subset \{ \psi(p(z),zp'(z),z^2p''(z); z,\xi) \mid z \in U, \ \xi \in \overline{U} \} \Rightarrow \Delta \subset p(U).$

1.1. Definition. [5] Let $H(z, \xi)$ be analytic in $U \times \overline{U}$ and $f(z)$ analytic and univalent in $U$. The function $f(z)$ is called strongly subordinate to $H(z, \xi)$, or $H(z, \xi)$ is said to be strongly superordinate to $f(z)$, written $f(z) \prec \prec H(z, \xi)$, if $f(z)$ is subordinate to $H(z, \xi)$ as a function of $z$, for all $\xi \in \overline{U}$. If $H(z, \xi)$ is univalent in $U$ for all $\xi \in \overline{U}$, then $f(z) \prec \prec H(z, \xi)$ if and only if $f(0) = H(0, \xi)$ for all $\xi \in \overline{U}$ and $f(U) \subset H(U \times \overline{U})$.

If $\Omega$ or $\Delta$ in (ii) is a simply connected domain, then it may be possible to rephrase (ii) in terms of strong differential superordination.

If $p$ is univalent in $U$, and if $\Delta$ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping $q$ of $U$ onto $\Delta$ such that $q(0) = p(0)$. In this case, (ii) can be rewritten as

(iii) $\Omega \subset \{ \psi(p(z),zp'(z),z^2p''(z); z,\xi) \mid z \in U, \ \xi \in \overline{U} \}$ implies $q(z) \prec p(z), z \in U.$

If $\Omega$ is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping $h$ of $U$ onto $\Omega$ such that $h(0) = \psi(p(0),0,0,\xi)$. If, in addition, the function $\psi(p(z),zp'(z),z^2p''(z); z,\xi)$ is univalent in $U$ for all $\xi \in \overline{U}$, then (iii) can be rewritten as

(iv) $h(z) \prec \prec \psi(p(z),zp'(z),z^2p''(z); z,\xi)$ implies $q(z) \prec p(z), z \in U.$

In the implication (iv), the functions $h$ and $q$ can be analytic and not necessarily univalent.

This last result leads us to some of the important definitions that will be used in this article.

1.2. Definition. [5] Let $\varphi: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let $h$ be analytic in $U$. If $p$ and $\varphi(p(z),zp'(z),z^2p''(z); z,\xi)$ are univalent in $U$ for all $\xi \in \overline{U}$ and satisfy the (second-order) strong differential superordination

(j) $h(z) \prec \prec \varphi(p(z),zp'(z),z^2p''(z); z,\xi)$
then $p$ is called a \textit{solution} of the strong differential superordination.

An analytic function $q$ is called a \textit{subordinate of the solutions of the strong differential superordination}, or more simply a \textit{subordinate}, if $q < p$ for all $p$ satisfying \( (j) \).

A univalent subordinate $\tilde{q}$ that satisfies $q < \tilde{q}$ for all subordinates $q$ of \( (j) \) is said to be the \textit{best subordinate}.

Note that the best subordinate is unique up to a rotation of $U$.

1.3. Definition. [2, Definition 2.2.b, p. 21] We denote by $Q$ the set of functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ f \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of $Q$ for which $f(0) = a$ is denoted by $Q(a)$.

1.4. Definition. [5] Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ with $q(z) \neq 0$. The class of \textit{admissible functions} $\phi_n[\Omega,q]$, consists of those functions $\varphi : \mathbb{C}^3 \times U \times U \to \mathbb{C}$ that satisfy the admissibility condition:

\begin{enumerate}
    \item[(A)] $\varphi(r,s,t,z,\xi) \in \Omega$
\end{enumerate}

whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$ and $\text{Re} \left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \text{Re} \left[ \frac{zq''(z)}{q(z)} + 1 \right]$, where $z \in U$, $z \in \partial U$, $\xi \in U$ and $m \geq n \geq 1$.

When $n = 1$ we write $\phi_1[\Omega,q]$ as $\phi[\Omega,q]$.

In the special case when $h$ is an analytic mapping of $U$ onto $\Omega \neq \mathbb{C}$ we denote this class $\phi_n[h(U),q]$ by $\phi_n[h,q]$.

In order to prove the main results, we need the following lemma.

1.5. Lemma. [5, Theorem 2] Take $q \in \mathcal{H}[a,n]$, let $h$ be analytic in $U$ and $\varphi \in \phi_n[h,q]$. If $p \in Q(a)$ and $\varphi(p(z),zq'(z),z^2p''(z);z,\xi)$ is univalent in $U$ for all $\xi \in \overline{U}$, then

$$h(z) \prec \varphi(p(z),zq'(z),z^2p''(z);z,\xi), \ z \in U, \ \xi \in \overline{U}$$

implies

$$q(z) \prec p(z), \ z \in U.$$

1.6. Remark. The conclusion of Lemma 1.5 can be written in the generalized form:

$$h(w(z)) \prec \varphi(p(w(z)),w(z)p'(w(z))),(w^2(z)p''(w(z)),w(z);\xi)), \ z \in U, \ \xi \in \overline{U},$$

where $w : U \to U$.

2. Main results

Using the following theorem, the result from Lemma 1.5 can be extended to those cases in which the behavior of $q$ on the boundary of $U$ is unknown.

2.1. Theorem. Let $h$ and $q$ be univalent in $U$, with $q(0) = a$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\varphi : \mathbb{C}^3 \times U \times U \to \mathbb{C}$ satisfy one of the following conditions:

\begin{enumerate}
    \item[(i)] $\varphi \in \phi_n[h,q_\rho]$, for some $\rho \in (0,1)$, or
    \item[(ii)] There exists $\rho_0 \in (0,1)$ such that $\varphi \in \phi_n[h_\rho,q_\rho]$, for all $\rho \in (\rho_0,1)$.
\end{enumerate}
If \( p \in \mathcal{H}[a, n] \), \( \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \) is univalent in \( U \) for all \( \xi \in \overline{U} \) and

\[
(2.1) \quad h(z) \ll \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \quad \xi \in \overline{U},
\]
then

\[
q(z) \prec p(z), \quad z \in U.
\]

**Proof.** Case (i). By applying Lemma 1.5 we obtain

\[
q_\rho(z) \prec p(z), \quad z \in U.
\]

Since \( q(z) \prec q_\rho(z) \) we deduce

\[
q(z) \prec p(z), \quad z \in U.
\]

Case (ii). If we let \( p_\rho(z) = p(\rho z) \), then

\[
\varphi(p_\rho(z), zp_\rho'(z), z^2p_\rho''(z); z, \xi) = \varphi(p(\rho z), \rho zp'(\rho z), \rho^2 z^2p''(\rho z); \rho z, \xi) \supset h_\rho(U).
\]

By using Remark 1.6 and Lemma 1.5 with \( w(z) = \rho z \), we obtain

\[
q_\rho(z) \prec p_\rho(z), \quad \text{for } \rho \in (\rho_0, 1).
\]

By letting \( \rho \to 1 \) we obtain

\[
q(z) \prec p(z), \quad z \in U.
\]

\( \square \)

The next two theorems yield best subordinants of the differential superordination (1).

The following theorems provide the existence of best subordinants of (1) for certain \( \varphi \) and also provide a method for finding the best subordinant for the cases \( n = 1 \) and \( n > 1 \).

**2.2. Theorem.** Let \( h \) be univalent in \( U \) and \( \varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C} \). Suppose that the differential equation

\[
(2.2) \quad \varphi(q(z), zq'(z), z^2q''(z); z) = h(z)
\]
has a solution \( q \in Q(a) \). If \( \varphi \in \phi[h, q] \), \( p \in Q(a) \) and \( \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \) is univalent in \( U \), for all \( \xi \in \overline{U} \) then

\[
(2.3) \quad h(z) \ll \varphi(p(z), zp'(z), z^2p''(z); z, \xi)
\]
implies \( q(z) \prec p(z) \) and \( q \) is the best subordinant.

**Proof.** Since \( \varphi \in \phi[h, q] \), by applying Lemma 1.5 we deduce that \( q \) is a subordinant of (2.3). Since \( q \) also satisfies (2.2), it is also a solution of the strong differential superordination (2.3) and therefore all subordinants of (2.3) will be subordinate to \( q \). Hence, \( q \) will be the best subordinant of (2.3).

\( \square \)

From this theorem we see that the problem of finding the best subordinant of (2.3) essentially reduces to showing that the differential equation (2.2) has a univalent solution and checking that \( \varphi \in \phi[h, q] \).

The conclusion of the theorem can be written in the symmetric form

\[
\varphi(q(z), zq'(z), z^2q''(z); z, \xi) \ll \varphi(p(z), zp'(z), z^2p''(z); z, \xi)
\]
implies

\[
q(z) \prec p(z), \quad z \in U, \quad \xi \in \overline{U}.
\]

This result can be extended to those cases in which the behavior of \( q \) on the boundary of \( U \) is unknown, by the following theorem.
2.3. Theorem. Let $h$ be univalent in $U$ and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation

\[
\varphi(q(z), zq'(z), z^2q''(z); z) = h(z)
\]

has a solution $q$ with $q(0) = a$, and that one of the following conditions is satisfied:

(i) $q \in Q$ and $\varphi \in \phi[h, q]$, or
(ii) $q$ is univalent in $U$ and $\varphi \in \phi[h, q]$, for some $\rho \in (0, 1)$, or
(iii) $q$ is univalent in $U$ and there exists $\rho_0 \in (0, 1)$ such that

$$\varphi \in \phi[h_\rho, q_\rho]$$

for all $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, 1]$ and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in $U$, for all $\xi \in \overline{U}$ and if $p$ satisfies

\[
h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \quad \xi \in \overline{U}
\]

then

$$q(z) \prec p(z), \quad z \in U,$$

and $q$ is the best subordinant.

Proof. By applying Lemma 1.5 and Theorem 2.1 we deduce that $q$ is a subordinant of (2.5). Since $q$ satisfies (2.4), it is a solution of (2.5) and therefore $q$ will be subordinated by all subordinants of (2.5). Hence $q$ will be the best subordinant of (2.5). \qed

2.4. Example. Let $q(z) = 1 + z, h(z) = q(z) + zq'(z) + z^2q''(z) = 1 + 2z, p \in \mathcal{H}[1, n]$ and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$, with

$$\text{Re } \varphi(p(z), zp'(z), z^2p''(z); z, \xi) > 0, \quad z \in U, \quad \xi \in \overline{U}.$$

If

$$1 + 2z \prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \quad \xi \in \overline{U}$$

then from Theorem 2.2 we have

$$1 + z \prec p(z), \quad z \in U,$$

and $q(z) = 1 + z$ is the best subordinant.

2.5. Theorem. Let $h$ be univalent in $U$ and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation

\[
\varphi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^{2n-1}q''(z)) = h(z)
\]

has a solution $q$, with $q(0) = a$, and that one of the following conditions is satisfied:

(i) $q \in Q$ and $\varphi \in \phi[h, q]$, or
(ii) $q$ is univalent in $U$ and $\varphi \in \phi_\alpha[h, q_\alpha]$, for some $\rho \in (0, 1)$, or
(iii) $q$ is univalent in $U$ and there exists $\rho_0 \in (0, 1)$ such that $\varphi \in \phi_\alpha[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, n], \varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in $U$ for all $\xi \in \overline{U}$, and if $p$ satisfies

\[
h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \quad \xi \in \overline{U},
\]

then

$$q(z) \prec p(z),$$

and $q$ is the best subordinant.
Proof. By applying Lemma 1.5 and Theorem 2.1 we deduce that $q$ is a subordinant of (2.7). If we let $p(z) = q(z^n)$, then
\[zp'(z) = nz^nq'(z^n)\]
and
\[z^2p''(z) = n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n).\]
Therefore, from (6) we obtain
\[
\begin{align*}
\varphi(p(z), zp'(z), z^2p''(z); z, \xi) & = \varphi(q(z^n), nz^nq'(z^n), n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n); z, \xi) \\
& = h(z^n) \\
& \prec h(z) \\
\varphi(q(z^n), nz^nq'(z^n), n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n); z, \xi) & \prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi).
\end{align*}
\]
Since $q(U) = p(U)$, we conclude that $q$ is the best subordinant. □

References