ON $\pi$-MORPHIC MODULES

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Received 02 : 07 : 2012 : Accepted 27 : 03 : 2013

Abstract
Let $R$ be an arbitrary ring with identity and $M$ be a right $R$-module with $S = \text{End}(M_R)$. Let $f \in S$. $f$ is called $\pi$-morphic if $M/f^n(M) \cong r_M(f^n)$ for some positive integer $n$. A module $M$ is called $\pi$-morphic if every $f \in S$ is $\pi$-morphic. It is proved that $M$ is $\pi$-morphic and image-projective if and only if $S$ is right $\pi$-morphic and $M$ generates its kernel. $S$ is unit-$\pi$-regular if and only if $M$ is $\pi$-morphic and $\pi$-Rickart if and only if $M$ is $\pi$-morphic and dual $\pi$-Rickart. $M$ is $\pi$-morphic and image-injective if and only if $S$ is left $\pi$-morphic and $M$ cogenerates its cokernel.

Keywords: Endomorphism rings; $\pi$-morphic rings; $\pi$-morphic modules; unit $\pi$-regular rings.


1. Introduction
Throughout this paper all rings have an identity, all modules considered are unital right modules and all ring homomorphisms are unital (unless explicitly stated otherwise).

A ring $R$ is said to be strongly $\pi$-regular ($\pi$-regular, right weakly $\pi$-regular) if for every element $x \in R$ there exists an integer $n > 0$ such that $x^n \in x^{n+1}R$ (respectively $x^n \in x^nRx^n$, $x^n \in x^nRx^nR$). It is called unit-$\pi$-regular if for every $a \in R$, there exist a unit element $x \in R$ and a positive integer $n$ such that $a^n = a^nxa^n$. In the case of $n = 1$ there exists a unit $x$ such that $a = axa$ for all $a \in R$, then $R$ is unit regular. Clearly, a strongly $\pi$-regular ring is a $\pi$-regular ring.

We say also that the ring $R$ is (von Neumann) regular if for each $a \in R$ there exists $x \in R$ such that $a = axa$ for some element $x$ in $R$, that is, $a$ is regular.

A module $M$ is said to satisfy Fitting’s lemma if, for all $f \in S$, there exists an integer $n \geq 1$, depending on $f$, such that $M = f^nM \oplus \text{Ker}(f^n)$. Hence a module satisfies

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Fitting’s lemma if and only if its endomorphism ring is strongly \( \pi \)-regular (see [4]).

Let \( M \) be a module. It is a well-known theorem of Erlich [2] that a map \( \alpha \in S \) is unit regular if and only if it is regular and \( M/\alpha(M) \cong \ker(\alpha) \). We say that the ring \( R \) is left morphic if every element \( a \) satisfies \( aR = \{0\} \).

In what follows, by \( \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_n \) and \( \mathbb{Z}/n\mathbb{Z} \) we denote, respectively, integers, rational numbers, the ring of integers modulo \( n \) and the \( \mathbb{Z} \)-module of integers modulo \( n \).

We also denote \( r_M(I) = \{m \in M \mid Im = 0\} \) where \( I \) is any subset of \( S \); \( r_R(N) = \{r \in R \mid Nr = 0\} \) and \( l_S(N) = \{f \in S \mid fN = 0\} \) where \( N \) is any subset of \( M \). The maps between modules are assumed to be homomorphisms unless otherwise stated in the context.

### 2. Morphic Modules and \( \pi \)-Morphic Modules

Let \( M \) be a module with \( S = \text{End}(M_R) \), the ring of endomorphisms of the right \( R \)-module \( M \) and \( 1 \) be the identity endomorphism of \( M \). Let \( f \in S \) is called morphic if \( M/f(M) \cong r_M(f) \). The module \( M \) is called morphic if every \( f \in S \) is morphic. Morphic modules are studied in [5]. An endomorphism \( f \in S \) is called \( \pi \)-morphistic if \( M/f^n(M) \cong r_M(f^n) \) for some positive integer \( n \). The module \( M \) is called \( \pi \)-morphistic if every \( f \in S \) is \( \pi \)-morphistic. In the sequel \( S \) will stand for \( \text{End}(M_R) \) for the right \( R \)-module \( M \) is considered.

It is clear that every morphic module is \( \pi \)-morphistic.

#### 2.1. Example

There exists a \( \pi \)-morphistic module which is not morphic.

Let \( e_{ij} \) denote \( 3 \times 3 \) matrix units. Consider the ring \( R = \{(e_{11} + e_{22} + e_{13})a + e_{12}b + e_{13}c + e_{23}d \mid a, b, c, d \in \mathbb{Z}_2\} \) and the right \( R \)-module \( M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) where right \( R \)-module operation is given by

\[
(x, y, z)((e_{11} + e_{22} + e_{33})a + e_{12}b + e_{13}c + e_{23}d) = (xa, xb + ya, xc + yd + za)
\]

where \( (x, y, z) \in M, (e_{11} + e_{22} + e_{33})a + e_{12}b + e_{13}c + e_{23}d \in R \). Let \( f \in S = \text{End}(M) \).

It is a routine check that there exist \( x, z \in \mathbb{Z}_2 \) such that \( f(1,0,0) = (x,0,z) \), \( f(0,1,0) = (0,x,0) \), \( f(0,0,1) = (0,0,x) \). For any \((a,b,c) \in M \), \( f(a,b,c) = (xa, ya + xb, za + xc) \).

(i) Let \( x = 0, y = 0, z = 1 \). Then \( f_1(a,b,c) = (0,0,a) \) implies \( f_1^2 = 0 \) which gives \( r_M(f_1^2) = M \). Hence \( M/f_1^2(M) \cong r_M(f_1^2) \).

(ii) Let \( x = 1, y = 0, z = 1 \). Then \( f_2(a,b,c) = (a,b,a + c) \) implies \( r_M(f_2) = 0 \) and \( f_2(M) = M \). Hence \( M/f_2(M) \cong r_M(f_2) \).

(iii) Let \( x = 1, y = 0, z = 0 \). Then \( f_3(a,b,c) = (a,b,c) \) and \( f_3 \) is the identity endomorphism of \( M \).

(iv) Let \( x = 0, y = 1, z = 0 \). Then \( f_4(a,b,c) = (0,0,0) \) and \( f_4^2 = 0 \).

(v) Let \( x = 0, y = 1, z = 1 \). Then \( f_5(a,b,c) = (0,a,a) \) and so \( f_5^2 = 0 \).

(vi) Let \( x = 1, y = 1, z = 0 \). Then \( f_6(a,b,c) = (a,a,b,c) \). Hence \( f_6 \) is an isomorphism.

(vii) Let \( x = 1, y = 1, z = 1 \). Then \( f_7(a,b,c) = (a,a,b,a + c) \). Hence \( f_7 \) is an isomorphism.

(viii) The last one \( f_8 \) is the zero endomorphism.

It follows that \( M \) is \( \pi \)-morphistic. However \( r_M(f_1) = (0) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( f_1(M) = (0) \times (0) \times \mathbb{Z}_2 \) shows that \( M \) is not morphic since, otherwise, \( M/f_1(M) \cong r_M(f_1) \), contrary to the fact that \( e_{12} + e_{13} \in R \) would annihilate \( r_M(f_1) \) from the right but not \( M/(0) \times (0) \times \mathbb{Z}_2 \).

#### 2.2. Lemma

Let \( f \in S \). If \( M/f^n(M) \cong r_M(f^n) \), there exists \( g \in S \) such that \( f^nM = r_M(g) \) and \( g(M) = r_M(f^n) \).
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Proof. Assume that $M/f^n M \cong r_M(f^n)$. Let $M \xrightarrow{\pi} M/f^n M \xrightarrow{h} r_M(f^n)$ where $\pi$ is the coset map and $h$ is the isomorphism. Set $g = h\pi$. Then $g(M) = r_M(f^n)$ and $r_M(g) = f^n(M)$.

2.3. Proposition. Let $M$ be a module, and let $f \in S$ be π-morphic. Then the following conditions are equivalent:

(1) $r_M(f) = 0$.
(2) $f$ is an automorphism.

Proof. Assume that $f$ in $S$ is π-morphic. Then there exists a positive integer $n$ such that $M/f^n(M) \cong r_M(f^n)$. By Lemma 2.2 there exists $g \in S$ such that $f^n M = r_M(g)$ and $g(M) = r_M(f^n)$. Assume (1) holds. Then $r_M(f) = 0$ and so $r_M(f^n) = 0$. This shows that $f^n(M) = M$. Hence $f(M) = M$ and $f$ is an automorphism and (2) holds. (2) ⇒ (1) always holds.

2.4. Theorem. Let $M$ be a π-morphic module. Then the following holds.

(1) For any $f \in S$, if $r_M(f) = 0$ then $f^n$ is an automorphism of $M$ for some positive integer $n$.
(2) For any $f \in S$, if $f(M) = M$ then $f^n$ is an automorphism of $M$ for some positive integer $n$.

Proof. (1) Let $f \in S$ with $r_M(f) = 0$. By hypothesis there exists a positive integer $n$ such that $M/f^n M \cong r_M(f^n)$ and $r_M(f) = 0$ implies $r_M(f^n) = 0$. So $M = f^n M$. Hence $f^n$ is an automorphism.

(2) Assume that $f(M) = M$. Then $f^i(M) = M$ for all $i \geq 1$. By hypothesis there exists a positive integer $n$ such that $M/f^n M \cong r_M(f^n)$. Then $r_M(f^n) = 0$. Hence $f^n$ is an automorphism.

Recall that the ring $R$ is called directly finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$. A module $M$ is called directly finite if its endomorphism ring is directly finite, equivalently for any endomorphisms $f$ and $g$ of $M$, $fg = 1$ implies $gf = 1$ where 1 is the identity endomorphism of $M$.

2.5. Corollary. Let $M$ be a π-morphic module. Then it is directly finite.

Proof. Let $f, g \in S$ with $fg = 1$. By Proposition 2.3, $g$ is an automorphism. Hence $gf = 1$.

2.6. Lemma. Let $f$ be a π-morphic element. If $h : M \rightarrow M$ is an automorphism, then there exists a positive integer $n$ such that $f^n h$ and $h f^n$ are both morphic. In particular, every π-unit regular endomorphism is morphic.

Proof. By Lemma 2.2, there exist $g \in S$ and a positive integer $n$ such that $g(M) = r_M(f^n)$ and $r_M(g) = f^n(M)$. Then $(f^n h)(M) = f^n(h(M)) = f^n(M) = r_M(g) = r_M(h^{-1} g)$. Next we show $r_M(f^n h) = (h^{-1} g)(M)$. For if $m \in r_M(f^n h)$, then $(f^n h)(m) = 0$ or $h(m) \in r_M(f^n)$. Hence $m \in (h^{-1} g)(M)$ since $r_M(f^n) = g(M)$. So $r_M(f^n h) \leq (h^{-1} g)(M)$. For the converse inclusion, let $m \in (h^{-1} g)(M)$. Then $h(m) \in g(M)$. So $h(m) \in r_M(f^n)$ since $r_M(f^n) = g(M)$. Hence $(f^n h)(m) = 0$ or $m \in r_M(f^n h)$. Thus $(h^{-1} g)(M) \leq r_M(f^n h)$. It follows that $r_M(f^n h) = (h^{-1} g)(M)$, and so $f^n h$ is morphic. Similarly $h f^n$ is morphic.

2.7. Examples. (1) Every strongly π-regular ring is π-morphic as a right module over itself.
(2) Every module satisfying Fitting’s lemma is π-morphic.
(3) Let $R$ be an Artinian ring. Then every finitely generated $R$ module is π-morphic.
Proof. (1) and (2) are clear. (3) Let \( R \) be an Artinian ring and \( M \) be a finitely generated module. Then \( M \) is both Artinian and Noetherian. By Proposition 11.7 in [1], \( M \) satisfies Fitting’s lemma. Therefore \( M \) is \( \pi \)-morphic.

2.8. Theorem. Every direct summand of a \( \pi \)-morphic module is \( \pi \)-morphic.

Proof. Let \( M = N \oplus K \) and \( S_N = \text{End}_R(N) \) and \( f \in S_N \). Define \( M \xrightarrow{g} M \) by \( g(m) = f(m) + k \) where \( m = n + k \) and \( n \in N, k \in K \). Clearly \( g \in S \) and \( g(M) = f(N) \oplus K \) and \( r_M(g) = r_N(f) \). By hypothesis there exists a positive integer \( n \) such that \( M/g^n(M) \cong r_M(g^n) \). It is apparent that \( g^n(M) = f^n(N) \oplus K \). Hence \( N/f^n(N) \cong (N \oplus K)/(f^n(N) \oplus K) = M/g^n(M) \cong r_M(g^n) = r_N(f^n) \).

2.9. Remark. One may suspect that for \( \pi \)-morphic modules \( M_1 \) and \( M_2 \), \( M_1 \oplus M_2 \) is \( \pi \)-morphic module provided \( \text{Hom}(M_i, M_j) = 0 \) for \( 1 \leq i \neq j \leq 2 \). But we cannot prove it.

Example 2.10 reveals that direct sum of \( \pi \)-morphic modules need not depend on the condition \( \text{Hom}(M_i, M_j) = 0 \).

2.10. Example. Consider the ring \( R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \right\} \) and the right \( R \)-module \( M = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\} \), and the submodules \( N = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\} \) and \( K = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid c \in \mathbb{Z}_2 \right\} \).

Then \( M = N \oplus K \). Clearly \( N \) and \( K \) are \( \pi \)-morphic right \( R \)-modules. Let \( e_{ij} \) denote the \( 3 \times 3 \) matrix units in \( M \) and for \( e_{23}c \in K \) define \( K \xrightarrow{h} N \) by \( h(e_{23}c) = e_{12}c \in N \). Then \( 0 \neq h \in \text{Hom}(K, N) \). For any \( f \in S \), there exist \( a, b, c, u, v \in \mathbb{Z}_2 \) such that \( f \) is given by \( f \left( \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & ax + by + cz \\ 0 & 0 & ax + vz \\ 0 & 0 & 0 \end{pmatrix} \right) \). It is easily checked that all \( f \)'s are morphic endomorphisms.

2.11. Proposition. Let \( M = K \oplus N \) be a \( \pi \)-morphic module and \( K \xrightarrow{i} N \) be a homomorphism. Then \( K \) is isomorphic to a direct summand of \( N \).

Proof. For \( k+n \in M \) where \( k \in K, n \in N \), define \( g(k+n) = f(k) + n \). Then \( g \) is a right \( R \)-module homomorphism of \( M \) and \( g^2 = g \). So \( M = g(M) \oplus (1-g)(M) = (f(K) + N) \oplus \{k-f(k) \mid k \in K \} \). Clearly \( r_M(g) = (1-g)(M) = \{k-f(k) \mid k \in K \} \) is a direct summand of \( N \). By hypothesis there exists a positive integer \( n \) such that \( M/g^n(M) \cong r_M(g^n) \). Since \( g^2 = g \), so \( K \cong K \oplus (N/f(K) + N) \cong (K \oplus N)/(f(K) + N) \cong M/g(M) \cong r_M(g) \) is a direct summand of \( N \).
A module $M$ is called image-projective if, whenever $gM \leq fM$ where $f$, $g \in S$, then $g \in fS$, that is $g = fh$ for some $h \in S$.

2.12. Lemma. Let $M$ be a module with $S = \text{End}_R(M)$.
(1) If $M$ is $\pi$-morphic, then $M$ is left GP-injective $S$-module.
(2) If $M$ is $\pi$-morphic and image-projective, then $S$ is right $\pi$-morphic.
(3) If $S$ is right $\pi$-morphic and $M$ generates its kernel, then $M$ is $\pi$-morphic.

Proof. (1) Let $f \in S$. By hypothesis there exist a positive integer $n$ and $g \in S$ such that $f^nM = r_M(g)$ and $r_M(f^n) = gM$. Since $l_S(f^n) = l_S(f^nM)$, $r_M l_S(f^n) = r_M l_S(f^nM) = r_M l_S(g) = r_M(g) = f^nM$.

(2) Let $f \in S$. By hypothesis there exist $g \in S$ and a positive integer $n$ such that $f^n = r_M(g)$ and $r_M(f^n) = gM$. Then $g f^n = 0$. Hence $f^n \subseteq r_M(g)$ and so $f^nS \subseteq r_M(g)$. Let $h \in r_M(g)$. Then $g h(M) = 0$ and $h(M) \subseteq r_M(g) = f^n(M)$. By image-projectivity of $M$ there exists $h' \in S$ such that $f^n h' = h f^nS$ or $r_M(g) \subseteq f^nS$. Thus $r_M(g) = f^n S$. Next we prove $r_M(f^n) = g S$. If $h \in r_M(f^n)$, then $f^n h = 0$ and $f^n h(M) = 0$ and $h(M) \subseteq r_M(f^n) = g (M)$. By image-projectivity of $M$ there exists an $h' \in S$ such that $h = gh'$. $r_M(f^n) = g (M)$ implies $f^n g = 0$. Hence $g S \subseteq r_M(f^n)$ and so $g S = r_M(f^n)$.

(3) Let $f \in S$. There exist $g \in S$ and a positive integer $n$ such that $f^nS = r_M(g)$ and $r_M(f^n) = gS$. We prove $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. $f^nS = r_M(g)$ implies $g f^n = 0$ and so $f^n(M) \subseteq r_M(g)$. Let $h \in S$ such that $h(M) \subseteq r_M(g)$. So $g h = 0$ and $h \in f^n S$. There exists $h' \in S$ such that $h = f^n h'$. Hence $h(M) \subseteq f^n h'(M) \subseteq f^n(M)$. Since $M$ generates $r_M(g)$, $r_M(g) \subseteq f^n(M)$, $r_M(g) = f^n(M)$. Next we prove $r_M(f^n) = g(M)$. $r_M(f^n) = gS$ implies $f^n g = 0$. Then $g(M) \subseteq r_M(f^n)$. Let $h(M) \subseteq r_M(f^n)$. Then $f^n h(M) = 0$ and so $f^n h = 0$ and $h \in r_M(f^n) = g S$. There exists $h' \in S$ such that $h = gh'$. Hence $h(M) \subseteq gh'(M) \subseteq g(M)$ and $r_M(f^n) \subseteq g(M)$ since $M$ generates $r_M(f^n)$. Thus $r_M(f^n) = g(M)$. 

The following theorem generalizes Theorem 32 in [5] to $\pi$-morphic modules.

2.13. Theorem. Let $M$ be a module. Then the following are equivalent:
(1) $M$ is $\pi$-morphic and image-projective.
(2) $S$ is right $\pi$-morphic and $M$ generates its kernel.


Let $M$ be a module. In [7], the module $M$ is called $\pi$-Rickart if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer $n$ such that $r_M(f^n) = eM$, while in [3], $M$ is said to be Rickart if for any $f \in S$, there exists $e^2 = e \in S$ such that $r_M(f) = eM$. Dickart module is named as kernel-direct in [5]. In [8], $M$ is called dual $\pi$-Rickart if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer $n$ such that $f^n(M) = eM$, while in [3], $M$ is said to be dual Rickart if for any $f \in S$, there exists $e^2 = e \in S$ such that $f(M) = eM$. Dual-Rickart module is named as image-direct in [5]. Erlich [2] proved that a map $f \in S$ is unit-regular if and only if $f$ is regular and morphic. We state and prove this theorem for $\pi$-regular rings.

2.14. Theorem. Let $f \in S$. Then the following are equivalent:
(1) $f$ is unit-$\pi$-regular.
(2) $f$ is $\pi$-regular and morphic.

Proof. (1) $\Rightarrow$ (2) Every unit-$\pi$-regular ring is $\pi$-regular. There exist a unit $g$ and a positive integer $n$ such that $f^n = f^n g f^n$. Then $g f^n$ is an idempotent, $r_M(f^n) = (1 - g f^n)M$ and
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\[ M \cong f^n(M) \oplus (1 - g^n)M. \] Hence \( M/f^n(M) \cong r_M(f^n) \).

(2) \( \Rightarrow \) (1) Let \( f^n = f^n g^n \) where \( g \in S \). Then
\[ M = f^n M \oplus (1 - f^n g)M = r_M(f^n) \oplus (g f^n)M. \]
Let \( h : f^n M \rightarrow g f^n(M) \) be defined by \( h f^n(m) = g f^n(m) \) where \( f^n(m) \in f^n(M) \). Then \( h \) and \( f^n \) are homomorphisms and inverse each other. Now
\[ M = f^n(M) \oplus (1 - f^n g)(M) \]
Let \( n = f^n g f^n \) where \( f^n = f_n f^n \) and \( g f^n = f^n g f^n \).
By morphic condition we have \( M/f^n(M) \cong r_M(f^n) \).

2.15. Theorem. Let \( M \) be a module with \( S = \text{End}_R(M) \). The following are equivalent:

1. \( S \) is unit-\( \pi \)-regular.
2. \( M \) is \( \pi \)-morphic and \( \pi \)-Rickart.
3. \( M \) is \( \pi \)-morphic and dual \( \pi \)-Rickart.

Proof. (1) \( \Rightarrow \) (2) Let \( S \) be unit-\( \pi \)-regular and \( f \in S \). There exist a unit \( g \in S \) and a positive integer \( n \) such that \( f^n = f^n g^n \). By virtue of Theorem 2.14, \( M \) is \( \pi \)-morphic.

(2) \( \Rightarrow \) (3) Let \( f \in S \). There exists a positive integer \( n \) such that \( M/(f^n M) \cong r_M(f^n) \).
By Lemma 2.2 there exists a \( g \in S \) such that \( g(M) = r_M(f^n) \) and \( r_M(g) = f^n(M) \).
By (2), \( r_M(g) \) is \( \pi \)-Rickart, therefore \( f^n(M) \) is direct summand.

(3) \( \Rightarrow \) (1) Let \( f \in S \). By (3), there exist a positive integer \( n \) and \( g \in S \) such that \( f^n M = r_M(g) \) and \( r_M(f^n) = g(M) \).
By (3), \( f^n M \) and \( g(M) \) are direct summand and so is \( r_M(f^n) \).
Hence \( S \) is \( \pi \)-regular ring by [9, Corollary 3.2]. By Theorem 2.14, \( S \) is unit-\( \pi \)-regular.

Example 2.16 shows that there exists a \( \pi \)-Rickart module which is not \( \pi \)-morphic.

2.16. Example. Consider \( M = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \) as a \( \mathbb{Z} \)-module. It can be easily determined that \( S = \text{End}_\mathbb{Z}(M) \) is
\[ \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix} \]
For any \( f = \begin{bmatrix} a & 0 \\ \bar{b} & \bar{c} \end{bmatrix} \in S \), we have the following cases.

Case 1. Assume that \( a = 0 \), \( \bar{b} = \bar{0} \), \( \bar{c} = \bar{0} \) or \( a = 0 \), \( \bar{b} = \bar{1} \), \( \bar{c} = \bar{1} \). In both cases \( f \) is an idempotent, and so \( r_M(f) = (1 - f)M \).

Case 2. If \( a \neq 0 \), \( \bar{b} = \bar{0} \), \( \bar{c} = \bar{0} \) or \( a \neq 0 \), \( \bar{b} = \bar{1} \), \( \bar{c} = \bar{0} \), then \( r_M(f) = 0 \).

Case 3. If \( a \neq 0 \), \( \bar{b} = \bar{1} \) or \( a \neq 0 \), \( \bar{b} = \bar{0} \), \( \bar{c} = \bar{0} \), then \( r_M(f) = 0 \). Hence \( r_M(f^n) = M \).

Therefore \( M \) is a \( \pi \)-Rickart module. Now we prove it is not \( \pi \)-morphic. Let
\[ f = \begin{bmatrix} 2 & 0 \\ 0 & \bar{1} \end{bmatrix} \in S. \]
For each positive integer \( n \), \( r_M(f^n) = 0 \) and
\[ f^n(M) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}). \]
Then \( M/f^n(M) \cong (\mathbb{Z}/2\mathbb{Z})^n \). But \( (\mathbb{Z}/2\mathbb{Z})^n \) can not be isomorphic to \( r_M(f^n) = 0 \).

In [5], \( M \) is called an image-injective module if for each \( f \in S \), every \( R \)-module homomorphisms from \( f(M) \) to \( M \) extends to \( M \). By this definition we state and prove dual versions of Lemma 2.12.

2.17. Lemma. Let \( M \) be a module with \( S = \text{End}_R(M) \).

1. If \( S \) is left \( \pi \)-morphic, then \( M \) is image-injective.
2. If \( M \) is \( \pi \)-morphic and image-injective, then \( S \) is left \( \pi \)-morphic.
3. If \( S \) is left \( \pi \)-morphic and \( M \) cogenerates its cokernel, then \( M \) is \( \pi \)-morphic.
Proof. (1) By Lemma 2.12, $S$ is right GP-injective. Let $f, g \in S$. There exists a positive integer $n$ depending on $f$ such that $f^n \neq 0$ and any map $f^n S \rightarrow S$ extends to an endomorphism of $S$. Let $f^n(M) \rightarrow M$ be a right $R$-module homomorphism and set $h = g f^n$. Then $r_S(f^n) \leq r_S(h)$. The map $f^n S \rightarrow hS$ defined by $t(f^n s) = hs$ where $s \in S$ is well defined right $S$-module homomorphism. By the GP-injectivity of $S$, $t$ extends to an endomorphism $g'$ of $S$ so that $g' f^n = h$. Let $m \in M$. $g' f^n(m) = h(m) = g f^n(m)$. Hence $g$ extends to $g' \in S$. Thus $M$ is image-injective.

(2) Let $f \in S$. There exist $g \in S$ and a positive integer $n$ such that $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. We prove $Sf^n = l_S(g)$ and $l_S(f^n) = Sg$. $r_M(f^n) = g(M)$ implies $f^n g = 0$. Then $f^n \in l_S(g)$ and so $Sf^n \leq l_S(g)$. Let $h \in l_S(g)$. Then $hg = 0$ or $f^n(M) = g(M) \leq r_M(h)$. Since $f^n(M) = g(M)$, the map defined $t$ by $f^n(M) \rightarrow h(M)$ extends to an endomorphism $\alpha$ of $M$. Then $\alpha f^n = h \in Sf^n$. Hence $l_S(g) \leq Sf^n$ and so $l_S(g) = Sf^n$.

(3) Let $f \in S$. We prove that there exist $g \in S$ and a positive integer $n$ such that $f^n(M) = r_M(g)$ and $r_M(f^n) = g(M)$. By hypothesis $S$ is left $\pi$-morphic, there exist $g \in S$ and a positive integer $n$ such that $Sf^n = l_S(g)$ and $l_S(f^n) = Sg$. $Sf^n = l_S(g)$ implies $f^n g = 0$ and $g(M) \leq r_M(f^n)$. Let $m \in r_M(f^n) = g(M)$. Then $0 \neq m \in M/g(M)$. By hypothesis, $M$ cogenerated by $M/g(M)$. There exists a map $M/g(M) \rightarrow M$ such that $t(m) = 0$. Now define $M \rightarrow M$ by $\alpha(x) = t(\overline{x})$. Then $t g = 0$ for all $x \in M$. Hence $\alpha g = 0$. So $\alpha \in l_S(g) = Sf^n$. There exists $s \in S$ such that $\alpha = sf^n$. This leads us a contradiction since $0 \neq \alpha(m) = sf^n(m) = 0$. Thus $r_M(f^n) = g(M)$.

2.18. Theorem. Let $M$ be a module. Then the following are equivalent:
(1) $M$ is $\pi$-morphic and image injective.
(2) $S$ is left $\pi$-morphic and $M$ cogenerated its cokernel.

Proof. Clear from Lemma 2.17. □

A ring $R$ is said to be right Kasch if every simple right $R$-module embeds in $R$, equivalently, if $l(I) \neq 0$ for every proper (maximal) right ideal $I$ of $R$ (see also [6, page 51]). Let $M$ be a module. $M$ is called Kasch module if any simple module in $\sigma[M]$ embeds in $M$, where $\sigma[M]$ is the category consisting of all $M$-subgenerated right $R$-modules, while $M$ is strongly Kasch if any simple right $R$-module embeds in $M$. It is easy to see that a ring $R$ is right Kasch if and only if the right $R$-module $R$ is Kasch if and only if the right $R$-module $R$ is strongly Kasch since $\sigma[R]$ is just the category of all right $R$-modules for details see [10].

2.19. Proposition. Let $M$ be a $\pi$-morphic module. If every maximal right ideal of $S$ is principal, then $S$ is a right Kasch ring.

Proof. Let $I$ be maximal right ideal of $S$. Then $I = fS$ for some $f \in S$. There exists a positive integer $n$ such that $M/f^n M \cong r_M(f^n)$. Assume that $r_M(f^n) = 0$. Then $f^n M = M = fM$. Hence $f^n$ is an isomorphism. Thus $I = S$. It is a contradiction.
It follows that for any nonzero \( f \neq 0 \in I \) there exists a positive integer \( n \) such that \( M/f^nM \cong r_M(f^n) \neq 0 \). Consider the diagram \( M \xrightarrow{\pi} M/f^nM \xrightarrow{\phi} r_M(f^n) \) where \( \pi \) is coset map and \( \phi \) is the isomorphism. Then \( \phi \pi f^n = 0 \). Hence \( 0 \neq \phi \pi f^{n-1} \in I_S(f) \). \( \square \)

2.20. Corollary. Let \( R \) be a right \( \pi \)-morphic ring and every maximal right ideal be principal. Then \( R \) is right Kasch.

Proof. Clear from Lemma 2.19 by considering \( M = R_R \) and \( S = \text{End}_R(R) \cong R \). \( \square \)

2.21. Proposition. Let \( S \) be a right \( \pi \)-morphic ring. Then the following conditions are equivalent:

1. \( S \) is a right Kasch ring.
2. Every maximal right ideal of \( S \) is an annihilator.
3. Every maximal right ideal of \( S \) is principal.

Proof. Note that every \( \pi \)-morphic ring is directly finite by Corollary 2.5. In [6] it is noted that (1) \( \Rightarrow \) (2) always holds.

(2) \( \Rightarrow \) (3) Let \( I \) be a maximal right ideal of \( S \). Then there exists a nonzero right ideal \( A \) of \( S \) such that \( I = l(A) \). Let \( 0 \neq a \in A \), there exist \( b \in S \) and a positive integer \( n \) such that \( a^nS = r(b) \) and \( r(a^n) = bS \). Hence \( I \subseteq l(a^n) \neq S \). Therefore, \( I = r(a^n) \).

(3) \( \Rightarrow \) (1) To complete the proof we show that \( l(I) \neq 0 \) for every maximal right ideal \( I \) of \( S \). Let \( I \) be a maximal right ideal. By (3), \( I = aS \) for some \( a \in S \). We invoke hypothesis here to find \( b \in S \) and a positive integer \( n \) such that \( a^nS = r(b) \) and \( r(a^n) = bS \). Then \( a^nb = 0 \) and \( ba^n = 0 \). If \( b = 0 \), then \( a^nS = S \). By Corollary 2.5, \( a \) is invertible and so \( I = S \). This contradicts being \( I \) maximal. It follows that \( b \neq 0 \). Let \( t \) be a nonzero positive integer such that \( ba^t = 0 \) and \( ba^{t-1} \neq 0 \). Hence \( ba^t = 0 \) implies \( 0 \neq ba^{t-1} \in l(I) \). So \( S \) is right Kasch. \( \square \)

References