ON SEMI-E-CONVEX AND QUASI-SEMI-E-CONVEX FUNCTIONS

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Abstract

In this paper we give some necessary and sufficient conditions under which a lower semi-continuous function defined on a real normed space is a semi-E-convex or quasi-semi-E-convex function.

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1. Introduction

The concepts of E-convex set and semi-E-convex function were introduced in [2] and [5]. These concepts are generalizations of convex function and quasi-convex function. Let us recall some definitions and related results. Let $X$ be a topological vector space. Then

(1) A set $U \subset X$ is said to be E-convex if and only if there is a map $E : X \rightarrow X$ such that $\lambda E(x) + (1 - \lambda)E(y) \in U$, for each $x, y \in U$ and $0 \leq \lambda \leq 1$.

(2) A function $f : X \rightarrow \mathbb{R}$ is said to be semi-E-convex on a set $U \subseteq X$ if and only if there is a map $E : X \rightarrow X$ such that $U$ is an E-convex set and $f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(x) + (1 - \lambda)f(y)$, for each $x, y \in U$ and $0 \leq \lambda \leq 1$.

(3) The mapping $f : X \rightarrow \mathbb{R}$ is said to be quasi-semi-E-convex on a set $U \subseteq X$ if $f(\lambda E(x) + (1 - \lambda)E(y)) \leq \max\{f(x), f(y)\}$, for each $x, y \in U$, $\lambda \in [0, 1]$ such that $\lambda E(x) + (1 - \lambda)E(y) \in U$.

Let $E : X \rightarrow X$ be a map and define $E \times I : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ by $(E \times I)(x, t) = (E(x), t)$. It is easy to show that $U \subset X$ is E-convex if and only if $U \times \mathbb{R}$ is $E \times I$-convex. For a function $f : X \rightarrow \mathbb{R}$ we denote by $\text{epi}(f)$ the epigraph of $f$; i.e.

$\text{epi}(f) = \{(x, \alpha) : x \in U, \; \alpha \in \mathbb{R}, \; f(x) \leq \alpha\}$.

Also, let us recall from [1] the following two results that will be used in the sequel.

1.1. Proposition. Let $X$ be a topological vector space and $U \subseteq X$ convex. Then

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Since for each \( \tilde{x} \) notice that \((2.1)\) \( \text{epi}(f) \) is \( E \times I \)-convex on \( X \times \mathbb{R} \).

(b) A function \( f \) is quasi-semi-E-convex on \( U \) if and only if the level set \( K_{\alpha} = \{ x : x \in U, f(x) \leq \alpha \} \) is \( E \)-convex for each \( \alpha \in \mathbb{R} \).

In this paper, for a lower semi continuous function defined on a real normed space we present some statements equivalent to semi-E-convexity and quasi-semi-E-convexity.

2. Results

First, we recall that if \( X \) is a normed space and \( S \subset X \), a function \( f : S \to [-\infty, +\infty] \) is lower semi-continuous if and only if for every real number \( \lambda \) the set \( \{ x \in S : f(x) \leq \lambda \} \) is closed and this is true if and only if its epigraph \( \text{epi}(f) = \{(x, \lambda) \in S \times \mathbb{R} : f(x) \leq \lambda \} \) is closed (as a subset of \( X \times \mathbb{R} \)). See for example [3] and [4].

Let \((x, s), (y, t) \in X \times \mathbb{R} \), with \( x, y \in X \) and \( s, t \in \mathbb{R} \). The line segment \([((x, s), (y, t)]\) (endpoint \((x, s)\) and \((y, t)\)) is the segment \( \{ \gamma(x, s) + (1 - \gamma)(y, t) : 0 \leq \gamma \leq 1 \} \). If \((x, s) \neq (y, t)\), the interior \((x, s), (y, t)\) of \([((x, s), (y, t)]\) is the segment \( \{ \gamma(x, s) + (1 - \gamma)(y, t) : 0 < \gamma < 1 \} \). In a similar way, we can define \([((x, s), (y, t)]\) and \((x, s), (y, t)\).

In the following theorems, we assume that \( X \) is a normed linear space and \( E : X \to X \) a map. Let \( f : X \to [-\infty, +\infty] \) be lower semi-continuous and \( f(E(x)) \leq f(x) \), for all \( x \in X \).

2.1. Theorem. Let \( f : X \to [-\infty, +\infty] \) be lower semi-continuous and suppose that there exists \( \alpha \in (0, 1) \) such that for all \( x, y \in X \) and \( u, v \in \mathbb{R} \), \( f(x) < u \), \( f(y) < v \), and

\[
\frac{f(x) + (1 - \alpha)f(y)}{\alpha u + (1 - \alpha)v} < 1.
\]

Then \( f : X \to [-\infty, +\infty] \) is semi-E-convex.

Proof. It is sufficient to show that \( \text{epi}(f) \) is \( E \times I \)-convex on \( X \times \mathbb{R} \). Suppose on the contrary that there exist \((x_1, \lambda_1), (y, \lambda_2) \in \text{epi}(f) \) (with \( x, y \in X \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \)) and \( \alpha_0 \in (0, 1) \) such that

\[
(\alpha_0 E(x) + (1 - \alpha_0)E(y), \alpha_0 \lambda_1 + (1 - \alpha_0)\lambda_2) \notin \text{epi}(f).
\]

Let us put \( x_0 = \alpha_0 E(x) + (1 - \alpha_0)E(y) \) and \( \lambda_0 = \alpha_0 \lambda_1 + (1 - \alpha_0)\lambda_2 \), then \( (x_0, \lambda_0) \notin \text{epi}(f) \), and

\[
A = \text{epi}(f) \cap [(x_1, \lambda_1), (x_0, \lambda_0)] \quad \text{and} \quad B = \text{epi}(f) \cap [(x_0, \lambda_0), (y, \lambda_2)].
\]

Since \( f \) is lower semi-continuous, \( \text{epi}(f) \) is a closed subset of \( X \times \mathbb{R} \), Consequently \( A \) and \( B \) are bounded and closed subsets of \( X \times \mathbb{R} \), \((x_0, \lambda_0) \notin A \), \((x_0, \lambda_0) \notin B \). Thus, there exist \((\bar{x}, s) \in A \) and \((\bar{y}, t) \in B \) with \( \bar{x}, \bar{y} \in X \) and \( s, t \in \mathbb{R} \) such that

\[
\min_{\alpha \in A} \| \alpha (x_0, \lambda_0) \| = \| (\bar{x}, s) - (x_0, \lambda_0) \|, \quad \text{and} \quad \min_{\beta \in B} \| \beta (x_0, \lambda_0) \| = \| (\bar{y}, t) - (x_0, \lambda_0) \|.
\]

Hence we have

\[
(2.1) \quad \text{epi}(f) \cap ((\bar{x}, s), (\bar{y}, t)) = \emptyset.
\]

Notice that \( \bar{x} \neq \bar{y} \) and \( s \neq t \) and \( ((\bar{x}, s), (\bar{y}, t)) \neq \emptyset \).

On the other hand, since \( (\bar{x}, s), (\bar{y}, t) \in X \times \mathbb{R} \), we have \( f(E(\bar{x})) < s + \epsilon \), \( f(E(\bar{y})) < t + \epsilon \) for each \( \epsilon > 0 \). Since \( \alpha(s + \epsilon) + (1 - \alpha)(t + \epsilon) = \alpha s + (1 - \alpha)t + \epsilon \), by hypothesis, we have

\[
f(\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y})) < \alpha s + (1 - \alpha)t + \epsilon.
\]

Since \( \epsilon \) is an arbitrary positive real number, it follows that

\[
f(\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y})) \leq \alpha s + (1 - \alpha)t.
\]
Hence,
\[ \alpha(\bar{x}, s) + (1 - \alpha)(\bar{y}, t) \in \text{epi}(f), \]
which contradicts (2.1). Thus we conclude that \( \text{epi}(f) \) is \( E \times I \)-convex. This completes the proof.

The next theorem gives a characterization of semi-E-convexity.

**2.2. Theorem.** Let \( f : X \rightarrow (-\infty, +\infty] \) be lower semi-continuous. Then \( f \) is semi-E-convex if and only if for all \( x, y \in X \), there exists \( \alpha \in (0, 1) \) (\( \alpha \) depends \( x, y \)) such that
\[
(2.2) \quad f(\alpha E(x) + (1 - \alpha)E(y)) \leq \alpha f(x) + (1 - \alpha)f(y).
\]

**Proof.** Let \( f : X \rightarrow (-\infty, +\infty] \) be semi-E-convex. It is easy to see that for all \( \alpha \in (0, 1) \)
(2.2) holds. For the converse, it is sufficient to show that \( \text{epi}(f) \) is \( E \times I \)-convex set, as a subset of \( X \times \mathbb{R} \). By contradiction suppose that there exist \( (x, \lambda_1), (y, \lambda_2) \in \text{epi}(f) \) (with \( x, y \in X \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \)) and \( \alpha_0 \in (0, 1) \) such that
\[
(\alpha_0 E(x) + (1 - \alpha_0)E(y), \alpha_0 \lambda_1 + (1 - \alpha_0)\lambda_2) \notin \text{epi}(f).
\]
Let \( x_0 = \alpha_0 E(x) + (1 - \alpha_0)E(y) \) and \( \lambda_0 = \alpha_0 \lambda_1 + (1 - \alpha_0)\lambda_2 \). Then \((x_0, \lambda_0) \notin \text{epi}(f)\).
By following the proof of Theorem 2.1, by defining \( A, B, (\bar{x}, s), \) and \((\bar{y}, t)\), we find that
\[
(2.3) \quad \text{epi}(f) \cap ((\bar{x}, s), (\bar{y}, t)) = \emptyset.
\]
Notice that \((\bar{x}, s), (\bar{y}, t)\) \( \neq \emptyset \).

On the other hand, by the hypothesis of the theorem, for \( \bar{x}, \bar{y} \in X \) there exists \( \alpha \in (0, 1) \) such that
\[
(2.4) \quad f(\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y})) \leq \alpha f(\bar{x}) + (1 - \alpha)f(\bar{y}).
\]
Since \((\bar{x}, s), (\bar{y}, t) \in \text{epi}(f)\), we have
\[
(2.5) \quad f(\bar{x}) \leq s \text{ and } f(\bar{y}) \leq t.
\]
Combining (2.4) and (2.5) we obtain
\[
f(\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y})) \leq s + (1 - \alpha)t.
\]
So, \( \alpha(\bar{x}, s) + (1 - \alpha)(\bar{y}, t) \in \text{epi}(f) \), which contradicts with (2.3). Thus, we conclude that \( \text{epi}(f) \) is \( E \times I \)-convex. Now the result follows.

**2.3. Corollary.** Let \( f : X \rightarrow (-\infty, +\infty] \) be lower semi-continuous. Then \( f \) is semi-E-convex if and only if, for all \( x, y \in X \),
\[
f\left(\frac{1}{2} E(x) + \frac{1}{2} E(y)\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y).
\]

**2.4. Example.** Let us define \( E : [0, \infty) \rightarrow [0, \infty) \) by
\[
E(x) = \begin{cases} 
\frac{1}{n} & \text{if } x = \frac{m}{n}, (m, n) = 1, \\
0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0,
\end{cases}
\]
and the function \( f : [0, \infty) \rightarrow [0, \infty) \) by
\[
f(x) = \begin{cases} 
2x & \text{if } 0 \leq x < 1, \\
x^2 & \text{if } x \geq 1.
\end{cases}
\]
It is clear that \( [0, \infty) \) is E-convex set, and \( f \) is a lower semi-continuous function on \( [0, \infty) \).
Also for each \( x, y \geq 0 \), there exists \( \lambda \in [0, 1] \) such that
\[
f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(x) + (1 - \lambda)f(y),
\]
but it is not semi-E-convex on $[0, \infty)$, because $f\left(\frac{3}{10}\right) = \frac{2}{7}$, $f\left(E\left(\frac{3}{10}\right)\right) = \frac{1}{5}$ or equivalently $f(E(x)) \not\geq f(x)$ and therefore $f(\frac{1}{2}E\left(\frac{1}{2}\right) + \frac{1}{2}E(1)) = \frac{3}{5}$ and $\frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) = 1$.

Our last result give a necessary and sufficient condition for a real-valued lower semi-continuous to be quasi-semi-E-convex.

2.5. Theorem. Let $f : X \rightarrow \mathbb{R}$ be lower semi-continuous. Then $f$ is quasi-semi-E-convex if and only if, for all $x, y \in X$, there exists an $\alpha \in (0, 1)$ ($\alpha$ depends on $x, y$) such that

$$f(\alpha E(x) + (1 - \alpha)E(y)) \leq \max\{f(x), f(y)\}.$$ 

Proof. By part (b) of Proposition 1.1, it can be easily checked that $f : X \rightarrow \mathbb{R}$ is quasi-semi-E-convex if and only if for every real number $\lambda$, the level set $\{x \in X : f(x) \leq \lambda\}$ is E-convex. Suppose on the contrary that there exists a real number $\lambda^*$ such that the set $F_{\lambda^*} = \{x \in X : f(x) \leq \lambda^*\}$ is not a E-convex set. Thus there exist $x, y \in F_{\lambda^*}$, and $\alpha_0 \in (0, 1)$ such that $\alpha_0 E(x) + (1 - \alpha_0)E(y) \notin F_{\lambda^*}$. Let $x_0 = \alpha_0 E(x) + (1 - \alpha_0)E(y)$, then $x_0 \notin F_{\lambda^*}$. Let

$$A = F_{\lambda^*} \cap [x, x_0] \text{ and } B = F_{\lambda^*} \cap [x_0, y],$$

where $[x, x_0] = \{\gamma x + (1 - \gamma)x_0 : 0 \leq \gamma \leq 1\}$ and $[x_0, y] = \{\gamma x_0 + (1 - \gamma)y : 0 \leq \gamma \leq 1\}$. Notice that $F_{\lambda^*}$ is a closed set [3]. Consequently $A$ and $B$ are bounded and closed subsets of $X$, and $x_0 \notin A, x_0 \notin B$. Thus there exist $\bar{x} \in A$ and $\bar{y} \in B$ such that

$$\min_{a \in A} \|a - x_0\| = \|\bar{x} - x_0\|$$

and

$$\min_{b \in B} \|b - x_0\| = \|\bar{y} - x_0\|,$$

where $\| \cdot \|$ is the norm on $X$. Hence we have

$$F_{\lambda^*} \cap [\bar{x}, x_0] = \emptyset \text{ and } F_{\lambda^*} \cap [x_0, \bar{y}] = \emptyset.$$ 

Therefore,

$$F_{\lambda^*} \cap (\bar{x}, \bar{y}) = \emptyset.$$ 

Notice that $\bar{x} \neq \bar{y}$ and so $(\bar{x}, \bar{y}) \neq \emptyset$.

On the other hand, by the hypothesis of the theorem, for $\bar{x}, \bar{y} \in X$, there exists an $\alpha \in (0, 1)$ such that

$$f(\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y})) \leq \max\{f(\bar{x}), f(\bar{y})\}.$$ 

Since $\bar{x}, \bar{y} \in F_{\lambda^*}$ we have

$$f(\bar{x}) \leq \lambda^* \text{ and } f(\bar{y}) \leq \lambda^*.$$ 

Combining (2.7) and (2.8), we obtain

$$f(\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y})) \leq \lambda^*.$$ 

So $\alpha E(\bar{x}) + (1 - \alpha)E(\bar{y}) \in F_{\lambda^*}$, which contradicts (6). Thus we conclude that $F_{\lambda^*}$ is convex. This completes the proof. \qed

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