A Link between Topology and Soft Topology

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Abstract
Muhammad Shabir and Munazza Naz have shown that every soft topology gives a parametrized family of topologies on a set $X$. In this paper such a link between topology and soft topology is further discussed.

Keywords: Soft Sets, Soft Topology, Soft Open, Soft Closed, Parameterized family of topologies.

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1 Introduction

The theory of soft sets gives a vital mathematical tool for handling uncertainties and vague concepts. In the year 1999, Molodtsov[9] initiated the study of soft sets. Soft set theory has been applied in several directions. Following this Maji, Biswas, and Roy[7,8] discussed soft set theoretical operations and gave an application of soft set theory to a decision making problem. Recently Muhammad Shabir and Munazza Naz introduced the notion of soft topology[10] and established that every soft topology induces a collection of topologies called the parametrized family of topologies induced by the soft topology. Several mathematicians published papers on applications of soft sets and soft topology[1,2,6,11,12,18]. Soft sets and soft topology have applications to data mining, image processing, decision making problems, spatial modeling and neural patterns[3,4,5,7,13,14,15,16,17]. The purpose of this paper is to study a link between a soft topology and the parametrized family of topologies induced by the soft topology. In particular, we give conditions on a given parameterized family of topologies which ensure there exists a soft topology whose induced family of topologies is the given family.

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2 Preliminaries

Throughout this paper $X$ denotes the universal set and $E$ denotes the parameter space.

**Definition 2.1**[9] A pair $(F, E)$ is called a soft set over $X$, where $F : E \to 2^X$ is a mapping. We denote $(F, E)$ by $\tilde{F}$ and we write $\tilde{F} = \{(e, F(e)) : e \in E\}$.

According to Muhammad Shabir and Munazza Naz[10], for each subset $A$ of $E$ $(F_A, E)$ is a soft set over the universal set $X$, where $F_A : A \to 2^X$ is a mapping. However $F_A : A \to 2^X$ can be extended to $E$ by setting $F_A(e) = \phi$ for all $e \in E - A$. This motivates us to fix the parameter space.

In this paper, the definitions and results of Mohammad Shabir and Munazza Naz[10] are taken and the subset $A$ of $E$ is replaced by the fixed parameter space $E$. Accordingly the following definitions and results are due to Mohammad Shabir and Munazza Naz[10].

**Definition 2.2** For any two soft sets $\tilde{F}$ and $\tilde{G}$ over a common universe $X$, $\tilde{F}$ is a soft subset of $\tilde{G}$ if $F(e) \subseteq G(e)$ for all $e \in E$. If $\tilde{F}$ is a soft subset of $\tilde{G}$ then we write $\tilde{F} \subseteq \tilde{G}$.

Two soft sets $\tilde{F}$ and $\tilde{G}$ over a common universe $X$ are soft equal if $\tilde{F} \subseteq \tilde{G}$ and $\tilde{G} \subseteq \tilde{F}$. That is $\tilde{F} = \tilde{G}$ if and only if $F(e) = G(e)$ for all $e \in E$.

**Definition 2.3** A soft set $\tilde{\Phi}$ over $X$ is said to be the NULL soft set if $\tilde{\Phi} = \{(e, \phi) : e \in E\}$.

**Definition 2.4** A soft set $\tilde{X}$ over $X$ is said to be the absolute soft set if $\tilde{X} = \{(e, X) : e \in E\}$.

**Definition 2.5** The union of two soft sets $\tilde{F}$ and $\tilde{G}$ over $X$ is defined as $\tilde{F} \cup \tilde{G} = (F \cup G, E)$ where $(F \cup G)(e) = F(e) \cup G(e)$ for all $e \in E$.

**Definition 2.6** The intersection of two soft sets $\tilde{F}$ and $\tilde{G}$ over $X$ is defined as $\tilde{F} \cap \tilde{G} = (F \cap G, E)$ where $(F \cap G)(e) = F(e) \cap G(e)$ for all $e \in E$.

The arbitrary union and the arbitrary intersection of soft sets are defined as follows.

$\cup \{F_\alpha : \alpha \in \Delta\} = \{(\cup \{F_\alpha : \alpha \in \Delta\}, E)\}$ and $\cap \{F_\alpha : \alpha \in \Delta\} = \{(\cap \{F_\alpha : \alpha \in \Delta\}, E)\}$

where $(\cup \{F_\alpha : \alpha \in \Delta\})(e) = \cup \{F_\alpha(e) : \alpha \in \Delta\}$ and $(\cap \{F_\alpha : \alpha \in \Delta\})(e) = \cap \{F_\alpha(e) : \alpha \in \Delta\}$.

**Definition 2.7** The complement of a soft set $\tilde{F}$ is denoted by $(\tilde{F})' = (F', E)$ where $F' : E \to 2^X$ is the mapping given by $F'(e) = X - F(e)$ for all $e \in E$.

**Definition 2.8** If $\tilde{\tau}$ is a collection of soft sets over $X$, then $\tilde{\tau}$ is said to be a soft topology on $X$ if

(i)$\tilde{\Phi}, \tilde{X}$ belong to $\tilde{\tau}$
(ii) arbitrary union of soft sets in \( \tilde{\tau} \) belongs to \( \tilde{\tau} \),

(iii) the intersection of any two soft sets in \( \tilde{\tau} \) belongs to \( \tilde{\tau} \).

If \( \tilde{\tau} \) is a soft topology over a universal set \( X \) with parameter space \( E \), then \( (X, \tilde{\tau}, E) \) is called a soft topological space and the members of \( \tilde{\tau} \) are called soft open sets over \( (X, E) \).

Muhammad Shabir and Munazza Naz introduced a parametrized family of topologies and established that every soft topology induces the parametrized family of topologies as shown in the following lemma.

**Lemma 2.9** Let \( (X, \tilde{\tau}, E) \) be a soft topological space over \( X \). Then the collection \( \tilde{\tau}_e = \{ F(e) : \tilde{F} \in \tilde{\tau} \} \) for each \( e \in E \), defines a topology on \( X \).

### 3 Link

**Definition 3.1** Let \( (X, \tilde{\tau}, E) \) be a soft topological space over \( X \). Then the collection \( E(\tilde{\tau}) = \{ \tilde{\tau}_e : e \in E \} \) denotes the parameterized family of topologies induced by the soft topology \( \tilde{\tau} \).

**Proposition 3.2** Let \( (X, \tilde{\tau}, E) \) be a soft topology over \( X \) with parameter space \( E \). Then \( |E(\tilde{\tau})| \leq |E| \) and \( |\tilde{\tau}_e| \leq |\tilde{\tau}| \) for every \( e \in E \).

**Proof**

Let \( \tilde{\tau} \) be a soft topological space over \( X \) with parameter space \( E \). Define \( \varphi : E \to E(\tilde{\tau}) \) by \( \varphi(e) = \tilde{\tau}_e \). Clearly \( \varphi \) is onto but it need not be one-to-one.

This proves that \( |E(\tilde{\tau})| \leq |E| \). Now define \( \theta_e : \tilde{\tau} \to \tilde{\tau}_e \) by \( \theta_e(F) = F(e) \). \( \theta_e \) is onto but need not be one-to-one. Therefore \( |\tilde{\tau}_e| \leq |\tilde{\tau}| \)

The above proposition has been illustrated in the following examples.

**Example 3.3** Let \( X = \{ h_1, h_2, h_3 \} \), \( E = \{ e_1, e_2 \} \) and \( \tilde{\tau} = \{ \phi, X, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \tilde{F}_5, \tilde{F}_6, \tilde{F}_7, \tilde{F}_8, \tilde{F}_9 \} \)

where \( \Phi, X, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \tilde{F}_5, \tilde{F}_6, \tilde{F}_7, \tilde{F}_8, \tilde{F}_9 \) are soft sets over \( X \).

The soft sets are defined as follows

\[
\begin{align*}
\tilde{F}_1 &= \{(e_1, \{h_2\}), (e_2, \{h_1\})\} \\
\tilde{F}_2 &= \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2\})\}, \\
\tilde{F}_3 &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\}, \\
\tilde{F}_4 &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_3\})\}, \\
\tilde{F}_5 &= \{(e_1, X), (e_2, \{h_1, h_2\})\},
\end{align*}
\]
\[\tilde{F}_0 = \{(e_1, \{h_2\}), (e_2, \{h_1, h_2\})\},\]
\[\tilde{F}_7 = \{(e_1, \{h_2, h_3\}), (e_2, X)\},\]
\[\tilde{F}_8 = \{(e_1, \{h_1, h_2\}), (e_2, X)\},\]
\[\tilde{F}_9 = \{(e_1, \{h_2\}), (e_2, X)\}.\]

Then \(\tilde{\tau}\) defines a soft topology on \(X\) and \((X, \tilde{\tau}, E)\) is a soft topological space over \(X\). It can be easily seen that \(\tilde{\tau}_{e_1} = \{\phi, X, \{h_2\}, \{h_2, h_3\}, \{h_1, h_2\}\}\) and \(\tilde{\tau}_{e_2} = \{\phi, X, \{h_1\}, \{h_1, h_3\}, \{h_1, h_2\}\}\) are topologies on \(X\).

Here \(e_1 \neq e_2\) and \(\tilde{\tau}_{e_1} \neq \tilde{\tau}_{e_2}\). Since \(\varphi(e_1) \neq \varphi(e_2)\), \(\varphi\) is one-to-one. Here \(|E(\tilde{\tau})| = 2\), \(|E| = 2\) and \(|E(\tilde{\tau})| = |E|\).

Also \(\tilde{F}_1(e_1) = \tilde{F}_6(e_1)\) but \(\tilde{F}_1 \neq \tilde{F}_6\). Since \(\theta_{e_1}(\tilde{F}_1) = \theta_{e_1}(\tilde{F}_6)\), \(\theta_{e_1}\) is not one-to-one. Here \(|\tilde{\tau}_{e_1}| = 5\) and \(|\tilde{\tau}| = 11\). Therefore \(|\tilde{\tau}_{e_1}| < |\tilde{\tau}|\). Again since \(\theta_{e_2}(\tilde{F}_2) = \theta_{e_2}(\tilde{F}_3) = \{h_1, h_2\}\), \(\theta_{e_2}\) is not one-to-one. Here \(|\tilde{\tau}_{e_2}| = 5 < 11 = |\tilde{\tau}|\).

**Example 3.4** Let \(X = \{h_1, h_2\}, E = \{e_1, e_2\}\) and \(\tilde{\tau} = \{\Phi, X, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \tilde{F}_5, \tilde{F}_6\}\) where \(\Phi, X, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \tilde{F}_5, \tilde{F}_6\) are soft sets over \(X\).

The soft sets are defined as follows
\[\tilde{F}_1 = \{(e_1, \{h_2\}), (e_2, \{h_2\})\},\]
\[\tilde{F}_2 = \{(e_1, X), (e_2, \{h_2, h_3\})\},\]
\[\tilde{F}_3 = \{(e_1, \{h_2\}), (e_2, X)\},\]
\[\tilde{F}_4 = \{(e_1, \{h_2\}), (e_2, \{h_2, h_3\})\},\]
\[\tilde{F}_5 = \{(e_1, \{h_2, h_3\}), (e_2, X)\},\]
\[\tilde{F}_6 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_2, h_3\})\}.

Then \(\tilde{\tau}\) defines a soft topology on \(X\) and hence \((X, \tilde{\tau}, E)\) is a soft topological space over \(X\).

It can be easily seen that \(\tilde{\tau}_{e_1} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\}\) and \(\tilde{\tau}_{e_2} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\}\) are topologies on \(X\). Here \(e_1 \neq e_2\) but \(\tilde{\tau}_{e_1} = \tilde{\tau}_{e_2}\). Since \(\varphi(e_1) = \varphi(e_2)\), \(\varphi\) is not one-to-one. Here \(|E(\tilde{\tau})| = 1\), \(|E| = 2\) and \(|E(\tilde{\tau})| < |E|\).

Also \(\tilde{F}_1(e_1) = \tilde{F}_4(e_1)\) but \(\tilde{F}_1 \neq \tilde{F}_4\). Since \(\theta_{e_1}(\tilde{F}_1) = \theta_{e_1}(\tilde{F}_4)\), \(\theta_{e_1}\) is not one-to-one. Here \(|\tilde{\tau}_{e_1}| = 4\) and \(|\tilde{\tau}| = 8\). Therefore \(|\tilde{\tau}_{e_1}| < |\tilde{\tau}|\). Again since \(\theta_{e_2}(\tilde{F}_3) = \theta_{e_2}(\tilde{F}_6) = \{h_2, h_3\}\), \(\theta_{e_2}\) is not one-to-one. Here \(|\tilde{\tau}_{e_2}| = 4\), \(|\tilde{\tau}| = 8\). Therefore \(|\tilde{\tau}_{e_2}| < |\tilde{\tau}| = 8\).

**Example 3.5** Let \(X = \{h_1, h_2\}, E = \{e_1, e_2\}\) and \(\tilde{\tau} = \{\Phi, X, \tilde{F}_1, \tilde{F}_2\}\) where \(\Phi, X, \tilde{F}_1, \tilde{F}_2\) are soft sets over \(X\). The soft sets are defined as follows
\[ \tilde{\tau}_1 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_2, h_3\})\}, \]
\[ \tilde{\tau}_2 = \{(e_1, \{h_2\}), (e_2, \{h_2\})\}, \]

Then \( \tilde{\tau} \) defines a soft topology on \( X \) and hence \((X, \tilde{\tau}, E)\) is a soft topological space over \( X \). It can be easily seen that here \( \tilde{\tau}_{e_1} = \{\phi, X, \{h_2, h_3\}\} \) and \( \tilde{\tau}_{e_2} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\} \) are topologies on \( X \). Here \( e_1 \neq e_2 \) and \( \tilde{\tau}_{e_1} = \tilde{\tau}_{e_2} \). Since \( \varphi(e_1) = \varphi(e_2) \), \( \varphi \) is not one-to-one. Here \( |E(\tilde{\tau})| = 1 \) and \( |E| = 2 \). Therefore \( |E(\tilde{\tau})| < |E| \).

Also \( \tilde{\tau}_1(e_1) \neq \tilde{\tau}_2(e_1) \) and \( \tilde{\tau}_1 \neq \tilde{\tau}_2 \). Since \( \theta_{e_1}(\tilde{\tau}_1) \neq \theta_{e_1}(\tilde{\tau}_2) \), \( \theta_{e_1} \) is one-to-one and onto. Here \( |\tilde{\tau}_{e_1}| = 4 \) and \( |\tilde{\tau}| = 4 \). Therefore \( |\tilde{\tau}_{e_1}| = |\tilde{\tau}| \). Again since \( \theta_{e_2}(\tilde{\tau}_1) \neq \theta_{e_2}(\tilde{\tau}_2) \), \( \theta_{e_2} \) is one-to-one. Here \( |\tilde{\tau}_{e_2}| = 4, |\tilde{\tau}| = 4 \). Therefore \( |\tilde{\tau}_{e_2}| = |\tilde{\tau}| = 4 \).

Muhammad Shabir and Munazza Naz established that every soft topology induces a parameterized family of topologies and further gave an example (Example 2, page 1790 of [10]) to show that the converse is not true. Then the following question will arise.

Given a collection \( \{\tau_e : e \in E\} \) of topologies on \( X \), are there conditions under which there exist a soft topology \( \tilde{\tau} \) over \( X \) with parameter space \( E \) such that \( \tau_e = \tilde{\tau}_e \), for all \( e \in E \). The following theorem gives an answer to the above question.

**Theorem 3.6:** Let \( X \) be a universal set and \( E \) be a parameter space. Let \( \{\tau_\alpha : \alpha \in E\} \) be a family of topologies on \( X \) satisfying the following conditions.

(i) There is an index set \( J \) such that for each \( \alpha \in E \), \( \tau_\alpha = \{G_{\alpha j} : j \in J\} \)

(ii) If \( \Delta \subseteq J \), then \( \exists r, s \in J \) such that \( \cap \{G_{\alpha j} : j \in \Delta\} = G_{\alpha r} \) for finite \( \Delta \) and \( \cup \{G_{\alpha j} : j \in \Delta\} = G_{\alpha s} \) for each \( \alpha \in E \).

(iii) There exist \( j_0, j_1 \in J \) such that \( G_{\alpha j_0} = \phi \) and \( G_{\alpha j_1} = X \) for all \( \alpha \in E \).

Then \( \tilde{\tau} = \{\tilde{\tau}_j : j \in J\} \) where \( \tilde{\tau}_j(\alpha) = G_{\alpha j} \) for each \( \alpha \in E \) is a soft topology on \( X \) with parameter space \( E \) satisfying \( \tilde{\tau}_\alpha = \tau_\alpha \) for all \( \alpha \in E \).

**Proof:** For each \( j \in J \) define \( \tilde{F}_j : E \to 2^X \) by \( \tilde{F}_j(\alpha) = G_{\alpha j} \) for all \( \alpha \in E \).

Then \( \{\tilde{F}_j : j \in J\} \) is a collection of soft sets over \( X \) with parameter space \( E \).

Claim: \( \tilde{\tau} = \{\tilde{F}_j : j \in J\} \) is a soft topology on \( X \).

Let \( \Delta \) be a non-empty subset of \( J \) and let \( \alpha \in E \).

\[
(\bigcup \{\tilde{F}_j : j \in \Delta\})(\alpha) = \bigcup \{\tilde{F}_j(\alpha) : j \in \Delta\} = \bigcup \{G_{\alpha j} : j \in \Delta\} = G_{\alpha s} \text{ for some } s \in J
\]

Since this is true for every \( \alpha \in E \), \( (\bigcup \{\tilde{F}_j : j \in \Delta\})(\alpha) = \tilde{F}_s(\alpha) \).

Therefore \( \tilde{\tau} \) is closed under arbitrary union.

\[
(\tilde{F}_j \cap \tilde{F}_k)(\alpha) = F_j(\alpha) \cap F_k(\alpha) = G_{\alpha j} \cap G_{\alpha k} = G_{\alpha r}
\]
\[ = F_{r}(\alpha) \]

That is \( \tilde{F}_{j} \cap \tilde{F}_{k} = \tilde{F}_{r} \in \tilde{\tau} \)

To prove that \( \tilde{\tau}_{\alpha} = \tau_{\alpha} \) for all \( \alpha \in E \)

\[ \tilde{\tau}_{\alpha} = \{ \tilde{F}_{j}(\alpha) : \tilde{F}_{j} \in \tilde{\tau} \} \]

\[ = \{ G_{\alpha j} : j \in J \} \]

\[ = \tau_{\alpha} \text{ for all } \alpha \in E. \]

**Remark 3.7:** Obtaining the necessary and sufficient conditions for theorem 3.6 is an open problem for researchers in soft topology.

### 4 Conclusion

In this paper, a link between a soft topology and the parametrized family of topologies induced by the soft topology is identified and characterized.

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### References


