On the Fibonacci and Lucas numbers, their sums and permanents of one type of Hessenberg matrices

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Abstract
At this paper, we derive some relationships between permanents of one type of lower-Hessenberg matrix family and the Fibonacci and Lucas numbers and their sums.

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1. Introduction
The well-known Fibonacci and Lucas sequences are recursively defined by

\[ F_{n+1} = F_n + F_{n-1}, \quad n \geq 1 \]
\[ L_{n+1} = L_n + L_{n-1}, \quad n \geq 1 \]

with initial conditions \( F_0 = 0, \ F_1 = 1 \) and \( L_0 = 2, \ L_1 = 1 \). The first few values of the sequences are given below:

\[
\begin{array}{c|cccccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  \hline
  F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 \\
  L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & 76 \\
\end{array}
\]

The permanent of a matrix is similar to the determinant but all of the signs used in the Laplace expansion of minors are positive. The permanent of an \( n \)-square matrix is defined by

\[ \text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \]

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where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$ [1].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call $A$ is contractible on column $k$, if column $k$ contains exactly two non zero elements. Suppose that $A$ is contractible on column $k$ with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m - 1) \times (n - 1)$ matrix $A_{ij,k}$ obtained from $A$ replacing row $i$ with $a_{ik}r_i + a_{jk}r_j$ and deleting row $j$ and column $k$ is called the contraction of $A$ on column $k$ relative to rows $i$ and $j$. If $A$ is contractible on row $k$ with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij,k}]^T$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$. We know that if $B$ is a contraction of $A[6]$, then

\begin{equation}
\text{per}A = \text{per}B.
\end{equation}

It is known that there are a lot of relationships between determinants or permanents of matrices and well-known number sequences. For example, the authors [2] investigate relationships between permanents of one type of Hessenberg matrix and the Pell and Perrin numbers.

In [3], Lee defined a $(0 - 1)$-matrix whose permanents are Lucas numbers.

In [4], the author investigate general tridiagonal matrix determinants and permanents. Also he showed that the permanent of the tridiagonal matrix based on $\{a_i\}, \{b_i\}, \{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}, \{b_i\}, \{c_i\}$.

In [5], the authors give $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an $n \times n$ $(-1, 1)$ matrix $S$, such that $\text{per}A = \det(A \circ S)$, where $A \circ S$ denotes Hadamard product of $A$ and $S$.

In the present paper, we consider a particular case of lower Hessenberg matrices. We show that the permanents of this type of matrices are related with Fibonacci and Lucas numbers and their sums.

### 2. Determinantal representation of Fibonacci and Lucas numbers and their sums

Let $H_n = [h_{ij}]_{n \times n}$ be an $n$-square lower Hessenberg matrix as below:

\begin{equation}
H_n = [h_{ij}]_{n \times n} = \begin{cases} 
2, & \text{if } i = j, \text{ for } i, j = 1, 2, \ldots, n - 1 \\
1, & \text{if } j = i - 2 \text{ and } i = j = n \\
(-1)^i, & \text{if } j = i + 1 \\
0, & \text{otherwise}
\end{cases}
\end{equation}

Then we have the following theorem.

**2.1. Theorem.** Let $H_n$ be as in (2.1), then

\[
\text{per}H_n = \text{per}H_n^{(n-2)} = F_{n+1}
\]

where $F_n$ is the $n$th Fibonacci number.
Proof. By definition of the matrix $H_n$, it can be contracted on column $n$. Let $H_n^{(r)}$ be the $r$th contraction of $H_n$. If $r = 1$, then

$$H_n^{(1)} = \begin{pmatrix} 2 & -1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 & -1 \\ 1 & 0 & 2 & 1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & (-1)^{n-2} & 2 \\ 1 & (-1)^{n-1} & 2 \end{pmatrix}. $$

Since $H_n^{(1)}$ also can be contracted according to the last column,

$$H_n^{(2)} = \begin{pmatrix} 2 & -1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 & -1 \\ 1 & 0 & 2 & 1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & (-1)^{n-3} & 3 \\ 2 & (-1)^{n-2} & 3 \end{pmatrix}. $$

Continuing this method, we obtain the $r$th contraction

$$H_n^{(r)} = \begin{pmatrix} 2 & -1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 & -1 \\ 1 & 0 & 2 & 1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & (F_r+1) & 2 \\ F_r+1 & (-1)^{r-1}(F_r+2 - F_{r+1}) & (F_r+2) \end{pmatrix}, \ n \text{ is even}$$

$$H_n^{(r)} = \begin{pmatrix} 2 & -1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 & -1 \\ 1 & 0 & 2 & 1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & (F_r+1) & 2 \\ F_r+1 & (-1)^{r-1}(F_r+2 - F_{r+1}) & (F_r+2) \end{pmatrix}, \ n \text{ is odd}$$

where $2 \leq r \leq n - 4$. Hence

$$H_n^{(n-3)} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ F_{n-2} & (F_{n-2} - F_{n-1}) & F_{n-1} \end{pmatrix}$$

by contraction of $H_n^{(n-3)}$ on column 3,

$$H_n^{(n-2)} = \begin{pmatrix} 2 & -1 \\ F_{n-2} & F_n \end{pmatrix}.$$ 

By (1.1), we have $\per H_n = \per H_n^{(n-2)} = F_{n+1}$. \hfill \square

Let $K_n = [k_{ij}]_{n \times n}$ be an $n$-square lower Hessenberg matrix in which the superdiagonal entries are alternating $-1$s and $1$s starting with 1, except the first one which is $-3$, the
main diagonal entries are 2s, except the last one which is 1, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. Clearly:

\[ K_n = \begin{pmatrix} 2 & -3 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & -1 & 1 \\ & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & (-1)^{n-1} & 1 & 0 & 1 \end{pmatrix} \]

2.2. Theorem. Let \( K_n \) be as in (2.2), then

\[ \text{per} K_n = \text{per} K_n^{(n-2)} = L_{n-2} \]

where \( L_n \) is the \( n \)th Lucas number.

Proof. By definition of the matrix \( K_n \), it can be contracted on column \( n \). By consecutive contraction steps, we can write down,

\[ K_n^{(r)} = \begin{pmatrix} 2 & -3 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & -1 & 1 \\ & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & (-1)^{r-1} & 2 F_{r+1} & (-1)^{r-2} (F_{r+2} - F_{r+1}) & F_{r+2} \\ F_{r+1} & (-1)^{r-1} (F_{r+2} - F_{r+1}) & F_{r+2} & \end{pmatrix} \]

for \( 1 \leq r \leq n-4 \). Hence

\[ K_n^{(n-3)} = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 2 & 1 \\ F_{n-2} & F_{n-2} - F_{n-1} & F_{n-1} \end{pmatrix} \]

by contraction of \( K_n^{(n-3)} \) on column 3, gives

\[ K_n^{(n-2)} = \begin{pmatrix} 2 & -3 \\ F_{n-2} & -2 \end{pmatrix} \]

By applying (1.1), we have \( \text{per} K_n = \text{per} K_n^{(n-2)} = 2F_n - 3F_{n-2} = L_{n-2} \), which is desired. \( \square \)

Let \( M_n = [m_{ij}]_{n \times n} \) be an \( n \)-square lower Hessenberg matrix as below:

\[ M_n = \begin{pmatrix} 2, \text{ if } i = j, \text{ for } i, j = 1, 2, \ldots, n \\ 1, \text{ if } j = i - 2 \\ (-1)^i, \text{ if } j = i + 1 \\ 0, \text{ otherwise} \end{pmatrix} \]
2.3. Theorem. Let $M_n$ be as in (2.3), then

$$\text{per} M_n = \text{per} M_n^{(n-2)} = \sum_{i=0}^{n+1} F_i = F_{n+3} - 1$$

where $F_n$ is the $n$th Fibonacci number.

Proof. By contraction method on column $n$, we have

$$M_n^{(r)} = \begin{pmatrix}
2 & -1 & 0 & 2 & 1 \\
0 & 2 & 1 & 0 & 2 & 1 \\
1 & 0 & 2 & -1 & 2 & 1 \\
1 & 0 & 2 & -1 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 2 & \cdots & \cdots & \cdots \\
& & & & & \sum_{i=0}^{r+1} F_i \sum_{i=0}^{r-1} F_i \sum_{i=0}^{r+2} F_i
\end{pmatrix}, \quad n \text{ is odd}$$

$$M_n^{(r)} = \begin{pmatrix}
2 & -1 & 0 & 2 & 1 \\
0 & 2 & 1 & 0 & 2 & 1 \\
1 & 0 & 2 & -1 & 2 & 1 \\
1 & 0 & 2 & -1 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 2 & \cdots & \cdots & \cdots \\
& & & & & \sum_{i=0}^{r+1} F_i \sum_{i=0}^{r-2} F_i \sum_{i=0}^{r+2} F_i
\end{pmatrix}, \quad n \text{ is even}$$

for $1 \leq r \leq n - 4$. Hence

$$M_n^{(n-3)} = \begin{pmatrix}
2 & -1 & 0 \\
0 & 2 & 1 \\
\sum_{i=0}^{n-2} F_i & - \sum_{i=0}^{n-3} F_i & \sum_{i=0}^{n-1} F_i
\end{pmatrix}$$

by contraction of $M_n^{(n-3)}$ on column 3, gives

$$M_n^{(n-2)} = \begin{pmatrix}
2 & -1 \\
\sum_{i=0}^{n-2} F_i & \sum_{i=0}^{n} F_i
\end{pmatrix}.$$  

By applying (1.1), we have

$$\text{per} M_n = \text{per} M_n^{(n-2)} = \sum_{i=0}^{n+1} F_i = F_{n+3} - 1$$

which is desired. \qed

Let $N_n = [n_{ij}]_{n \times n}$ be an $n$-square lower Hessenberg matrix in which the superdiagonal entries are alternating $-1$s and $1$s starting with $1$, except the first one which is $-2$, the main diagonal entries are $2$s except the first one is 3, the subdiagonal entries are $0$s, the lower-subdiagonal entries are $1$s and otherwise $0$. In this content:
(2.4) \[ N_n = \begin{pmatrix} 3 & -2 & & & \\ 0 & 2 & 1 & & \\ 1 & 0 & 2 & -1 & \\ & 1 & 0 & 2 & 1 \\ & & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & \cdots & (-1)^{n-1} \\ 1 & 0 & 2 & \end{pmatrix} \]

2.4. Theorem. Let \( N_n \) be an \( n \)-square matrix (\( n \geq 2 \)) as in (2.4), then
\[
\text{per} N_n = \text{per} N_n^{(n-2)} = \sum_{i=0}^{n} L_i = L_{n+2} - 1
\]
where \( L_n \) is the \( n \)th Lucas number.

Proof. By contraction method on column \( n \), we have
\[
N_n^{(r)} = \begin{pmatrix} 3 & -2 & & & \\ 0 & 2 & 1 & & \\ 1 & 0 & 2 & -1 & \\ & 1 & 0 & 2 & 1 \\ & & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & \cdots & (-1)^{r} \\ 1 & 0 & 2 & \sum_{i=0}^{r+1} F_i \ (-1)^{r-1} \sum_{i=0}^{r} F_i \sum_{i=0}^{r+2} F_i \end{pmatrix}, \text{ } n \text{ is odd}
\]
\[
N_n^{(r)} = \begin{pmatrix} 3 & -2 & & & \\ 0 & 2 & 1 & & \\ 1 & 0 & 2 & -1 & \\ & 1 & 0 & 2 & 1 \\ & & \ddots & \ddots & \ddots \\ 1 & 0 & 2 & \cdots & (-1)^{r} \\ 1 & 0 & 2 & \sum_{i=0}^{r+1} F_i \ (-1)^{r} \sum_{i=0}^{r} F_i \sum_{i=0}^{r+2} F_i \end{pmatrix}, \text{ } n \text{ is even}
\]
for \( 1 \leq r \leq n-4 \). Hence
\[
N_n^{(n-3)} = \begin{pmatrix} 3 & -2 & 0 & & \\ 0 & 2 & 1 & & \\ & \sum_{i=0}^{n-2} F_i \ - \sum_{i=0}^{n-3} F_i \ \sum_{i=0}^{n-1} F_i \end{pmatrix}
\]
by contraction of \( N_n^{(n-3)} \) on column 3, gives
\[
N_n^{(n-2)} = \begin{pmatrix} 3 & -2 & \sum_{i=0}^{n-2} F_i \ \sum_{i=0}^{n} F_i \end{pmatrix}
\]
By applying (1.1), we have
\[
\text{per} N_n = \text{per} N_n^{(n-2)} = \sum_{i=0}^{n} L_i = L_{n+2} - 1
\]
by the identity \( F_{n-1} + F_{n+1} = L_n \). \( \square \)
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References
