Non-selfadjoint matrix Sturm-Liouville operators with eigenvalue-dependent boundary conditions

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Abstract

In this paper we investigate discrete spectrum of the non-selfadjoint matrix Sturm-Liouville operator $L$ generated in $L^2(\mathbb{R}_+, S)$ by the differential expression

$$\ell(y) = -y'' + Q(x)y, \quad x \in \mathbb{R}_+ : [0, \infty),$$

and the boundary condition $y'(0) - (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2) y(0) = 0$ where $Q$ is a non-selfadjoint matrix valued function. Also using the uniqueness theorem of analytic functions we prove that $L$ has a finite number of eigenvalues and spectral singularities with finite multiplicities.

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1. Introduction

The study of the spectral analysis of non self-adjoint Sturm-Liouville operators was begun by Naimark [23] in 1954. He studied the spectral analysis of non-selfadjoint differential operators with continuous and discrete spectrum. Also he investigated the existence of spectral singularities in the continuous spectrum of the non-selfadjoint differential operator. Spectral singularities are poles of the resolvent’s kernel which are in the continuous spectrum and are not eigen-values [26]. General notion of the sets of spectral singularities for closed linear operators on a Banach space was given by Nagy in [22]. Let $L_0$ denote the operator generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$(1.1) \quad \ell_0(y) = -y'' + v(x)y, \quad x \in \mathbb{R}_+$$

and the boundary condition

$$y'(0) - hy(0) = 0$$

where $v$ is a complex valued function and $h \in \mathbb{C}$.

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In [23] it is shown that if
\[ \int_0^\infty \exp(\varepsilon x) |v(x)| \, dx < \infty, \]
for some \( \varepsilon > 0 \), then \( L_0 \) has a finite number of eigenvalues and spectral singularities with a finite multiplicities. Pavlov [25] established the dependence of the structure of the spectral singularities of \( L_0 \) on the behavior of the potential function at infinity. The spectral analysis of the non-selfadjoint operator, generated in \( L^2(\mathbb{R}_+) \) by (1.1) and the integral boundary condition
\[ \int_0^\infty B(x) y(x) \, dx + \alpha y'(0) - \beta y(0) = 0 \]
where \( B \in L^2(\mathbb{R}_+) \) is a complex-valued function, and \( \alpha, \beta \in \mathbb{C} \), was investigated in detail by Krall [15],[16].

Some problems of spectral theory of differential and some other types of operators with spectral singularities were also studied in [1],[3]-[7],[17],[18]. The spectral analysis of the non self-adjoint operator, generated in \( L^2(\mathbb{R}_+) \) by (1.1) and the boundary condition
\[ y'(0) y(0) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 \]
where \( \alpha_i \in \mathbb{C}, i = 0, 1, 2 \) with \( \alpha_2 \neq 0 \) was investigated by Bairamov et al. [8].

The all above mentioned papers related with differential and difference operators are of scalar coefficients. Spectral analysis of the selfadjoint differential and difference operators with matrix coefficients are studied in [2],[9]-[11],[14].

Let \( S \) be a \( n \)-dimensional \( (n < \infty) \) Euclidian space. We denote by \( L^2(\mathbb{R}_+, S) \) the Hilbert space of vector-valued functions with values in \( S \) and the norm
\[ \|f\|_{L^2(\mathbb{R}_+, S)}^2 = \int_0^\infty \|f(x)\|_S^2 \, dx. \]

Let \( L \) denote the operator generated in \( L^2(\mathbb{R}_+, S) \) by the matrix differential expression
\[ \ell(y) = -y'' + Q(x) y, \quad x \in \mathbb{R}_+ \]
and the boundary condition \( y(0) = 0 \), where \( Q \) is a non-selfadjoint matrix-valued function (i.e. \( Q \neq Q^* \)). In [24], [12] discrete spectrum of the non-selfadjoint matrix Sturm-Liouville operator was investigated. Let us consider the BVP in \( L^2(\mathbb{R}_+, S) \)

\[ -y'' + Q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+, \]

\[ y'(0) - (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2) y(0) = 0 \]

where \( Q \) is a non self-adjoint matrix-valued function and \( \beta_0, \beta_1, \beta_2 \) are non self-adjoint matrices with \( \det \beta_2 \neq 0 \).

In this paper using the uniqueness theorem of analytic functions we investigate the eigenvalues and the spectral singularities of \( L \). In particular we prove that \( L \) has a finite number of eigenvalues and spectral singularities with finite multiplicities, if the condition
\[ \lim_{x \to \infty} Q(x) = 0, \quad \int_0^\infty e^{\varepsilon x} \|Q'(x)\| \, dx < \infty, \quad \varepsilon > 0, \]
holds, where \( \| \cdot \| \) denote norm in \( S \). We also show that the analogue of the Pavlov condition for \( L \) is the form
\[
\lim_{x \to \infty} Q(x) = 0, \quad \int_0^\infty e^{\sqrt{x}} \| Q'(x) \| \, dx < \infty, \quad \epsilon > 0.
\]

2. Jost Solution

Let us consider the matrix Sturm-Liouville equation
\[
(2.1) \quad -y'' + Q(x) y = \lambda^2 y, \quad x \in \mathbb{R}_+
\]
where \( Q \) is a non-selfadjoint matrix-valued function and
\[
(2.2) \quad \int_0^\infty x \| Q(x) \| \, dx < \infty
\]
holds. The bounded matrix solution of (2.1) satisfying the condition
\[
\lim_{x \to \infty} y(x,\lambda) e^{-i\lambda x} = I, \quad \lambda \in \mathbb{C}_+ := \{ \lambda : \lambda \in \mathbb{C}, \text{ Im } \lambda \geq 0 \}
\]
will be denoted by \( F(x,\lambda) \). The solution \( F(x,\lambda) \) is called Jost solution of (2.1). It has been shown that, under the condition (2.2), the Jost solution has the representation
\[
(2.3) \quad F(x,\lambda) = e^{i\lambda x} I + \int_0^x K(x,t) e^{i\lambda t} \, dt
\]
where \( I \) denotes the identity matrix in \( S \) and the matrix function \( K(x,t) \) satisfies
\[
(2.4) \quad K(x,t) = \frac{1}{2} \int_{x+t/2}^{x+t} Q(s) ds + \frac{1}{2} \int_{x+t/2}^{x+t} Q(s) K(s,v) dv ds + \frac{1}{2} \int_{x+t/2}^{x+t} Q(s) K(s,v) dv ds.
\]
\( K(x,t) \) is continuously differentiable with respect to their arguments and
\[
(2.5) \quad \| K(x,t) \| \leq c \alpha \left( \frac{x+t}{2} \right)
\]
\[
(2.6) \quad \| K_x(x,t) \| \leq \frac{1}{4} \left\| Q\left( \frac{x+t}{2} \right) \right\| + c \alpha \left( \frac{x+t}{2} \right)
\]
\[
(2.7) \quad \| K_t(x,t) \| \leq \frac{1}{4} \left\| Q\left( \frac{x+t}{2} \right) \right\| + c \alpha \left( \frac{x+t}{2} \right)
\]
where \( \alpha(x) = \int \| Q(s) \| \, ds \) and \( c > 0 \) is a constant. Therefore, \( F(x,\lambda) \) is analytic with respect to \( \lambda \) in \( \mathbb{C}_+ := \{ \lambda : \lambda \in \mathbb{C}, \text{ Im } \lambda > 0 \} \) and continuous on the real axis (\([2], [17], [19]\)).

We will denote the matrix solution of (2.1) satisfying the initial conditions
\[
G(0,\lambda) = I, \quad G'(0,\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2
\]
by \( G(x,\lambda) \). Let us define the following functions:
\[
(2.8) \quad A_\pm(\lambda) = \frac{F_x(0,\pm\lambda) - (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2) F(0,\pm\lambda)}{\lambda e^{\bar{C}_{\pm}}},
\]
where \( \bar{\mathbb{C}}_\pm = \{ \lambda : \lambda \in \mathbb{C}, \pm \text{Im} \lambda \geq 0 \} \). It is obvious that the functions \( A_+(\lambda) \) and \( A_- (\lambda) \) are analytic in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively and continuous on the real axis. It is clear that the resolvent of \( L \) defined by the following

\[
(2.9) \quad R_\lambda (L) \varphi = \int_0^\infty R(x, \xi; \lambda) \varphi (\xi) \, d\xi, \quad \varphi \in L^2(\mathbb{R}_+, S)
\]

where

\[
R(x, \xi; \lambda) = \begin{cases} R_+(x, \xi; \lambda), & \lambda \in \mathbb{C}_+ \\ R_-(x, \xi; \lambda), & \lambda \in \mathbb{C}_- \end{cases}
\]

\[
(2.10) \quad R_\pm (x, \xi; \lambda) = \begin{cases} -F(x, \pm \lambda) A_\pm^{-1} (\lambda) G^\prime (\xi, \lambda), & 0 \leq \xi \leq x \\ -G(x, \lambda) [A_\pm' (\lambda)]^{-1} F(\xi, \pm \lambda), & x \leq \xi < \infty, \end{cases}
\]

and \( G^\prime (\xi, \lambda) \) and \( A_\pm' (\lambda) \) denote the transpose of the matrix function \( G(\xi, \lambda) \) and \( A_\pm(\lambda) \) respectively.

In the following we will denote the class of non self-adjoint matrix-valued valued absolutely continuous functions in \( \mathbb{R}_+ \) by \( AC(\mathbb{R}_+) \).

**2.1. Lemma.** If

\[
(2.11) \quad Q \in AC(\mathbb{R}_+), \quad \lim_{x \to \infty} Q(x) = 0, \quad \int_0^\infty \| Q'(x) \| < \infty
\]

then \( K_{tt}(x, t) \) exist and

\[
(2.12) \quad K_{tt}(x, t) = -\frac{1}{8} Q'(t^2) + \frac{1}{2} \int_0^\infty Q(s) K_t(s, t+s) \, ds
\]

\[
-\frac{1}{4} Q(t^2) K(t^2, t^2)
\]

\[
-\frac{1}{2} \int_0^t Q(s) [K_t(s, t-s) + K_t(t-x+s)] \, ds.
\]

**Proof.** The proof of lemma direct consequently of (2.4). \( \blacksquare \)

From (2.5)-(2.7) and (2.12) we obtain that

\[
(2.13) \quad \| K_{tt}(0, t) \| \leq c \left\{ \left\| Q'(t^2) \right\| + t \left\| Q(t^2) \right\| + t \alpha(t^2) + \alpha_1(t^2) \right\}
\]

holds, where \( \alpha_1(t) = \int_t^\infty \alpha(s) \, ds \) and \( c > 0 \) is a constant.

**2.2. Lemma.** Under the condition (2.11), \( A_+ \) and \( A_- \) have the representations

\[
(2.14) \quad A_+(\lambda) = -\beta_2 \lambda^2 + A \lambda + B + \int_0^\infty F^+(t)e^{i\lambda t} \, dt, \quad \lambda \in \mathbb{C}_+,
\]

\[
(2.15) \quad A_- (\lambda) = -\beta_2 \lambda^2 + C \lambda + D + \int_0^\infty F^-(t)e^{-i\lambda t} \, dt, \quad \lambda \in \mathbb{C}_-,
\]

where \( A, B, C, D \) are non self-adjoint matrices in \( S \), and \( F^\pm \in L_1(\mathbb{R}_+) \).
Proof. Using (2.3), (2.4) and (2.8) we get (2.14), where
\[ A = i - \beta_1 - i\beta_2 K(0,0), \]
\[ B = -K(0,0) - \beta_0 - i\beta_1 K(0,0) + \beta_2 K_1(0,0), \]
\[ F^+(t) = K_2(0,t) - \beta_0 K(0,t) - i\beta_1 K_1(0,t) + \beta_2 K_1(0,0). \]
From (2.5) to (2.7) and (2.13), \( F^+ \in L_1(\mathbb{R}_+) \). By similar way we obtain (2.15) and \( F^- \in L_1(\mathbb{R}_+) \).

2.3. Theorem. \( A_+(\lambda) \) and \( A_-(\lambda) \) have the asymptotic behavior:
\[ A_{\pm}(\lambda) = -\beta_2 \lambda^2 + A \lambda + B + o(1) \quad \lambda \in \mathbb{C}_\pm, \ |\lambda| \to \infty. \]
Proof. The proof is obvious from (2.5) to (2.7) and (2.13)

We will denote the continuous spectrum of \( L \) by \( \sigma_c \). From Theorem 2 ([22], page 303) we get that
\[ \sigma_c = \mathbb{R}. \]

3. Eigenvalues and Spectral Singularities of \( L \)

Let us suppose that
\[ f_{\pm}(\lambda) := \det A_{\pm}(\lambda). \]
We denote the set of eigenvalues and spectral singularities of \( L \) by \( \sigma_d(L) \) and \( \sigma_{ss}(L) \), respectively. By the definition of eigenvalues and spectral singularities of differential operators we can write
\[ \sigma_d(L) = \{ \lambda: \lambda \in \mathbb{C}_+, \ f_+(\lambda) = 0 \} \cup \{ \lambda: \lambda \in \mathbb{C}_-, \ f_-(\lambda) = 0 \} \]
\[ \sigma_{ss}(L) = \{ \lambda: \lambda \in \mathbb{R} \setminus \{0\}, \ f_+(\lambda) = 0 \} \cup \{ \lambda: \lambda \in \mathbb{R} \setminus \{0\}, \ f_-(\lambda) = 0 \} \]
\[ \left[22], [23], [26]. \right. \] It is clear that \( \sigma_{ss}(L) \subset \mathbb{R}. \)

3.1. Definition. The multiplicity of a zero of \( f_+ \) in \( \mathbb{C}_+ \) (or \( f_- \) in \( \mathbb{C}_- \)) is defined as the multiplicity of the corresponding eigenvalue and spectral singularity of \( L \).

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of \( L \), we need to discuss the quantitative properties of the zeros of \( f_+ \) and \( f_- \) in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively. Assume that
\[ M_1^+ = \{ \lambda: \lambda \in \mathbb{C}_+, f_+(\lambda) = 0 \} \]
and
\[ M_2^+ = \{ \lambda: \lambda \in \mathbb{R}, f_+(\lambda) = 0 \}. \]
From (3.3) and (3.4), we get
\[ \sigma_d(L) = M_1^+ \cup M_2^-, \]
and
\[ \sigma_{ss}(L) = M_1^+ \cup M_2^- - \{0\}. \]

3.2. Theorem. Under the condition (2.11)

i) The set \( \sigma_d(L) \) is bounded and has at most countable number of elements and its limit points can lie only in a bounded subinterval of the real axis.
ii) The set \( \sigma_{ss}(L) \) is bounded and \( \mu(\sigma_{ss}(L)) = 0 \), where \( \mu(\sigma_{ss}(L)) \) denotes the linear Lebesque measure of \( \sigma_{ss}(L) \).
Proof. Using (2.5) and (3.1) we get that the function \( f_\pm \) is analytic in \( \mathbb{C}_+ \) continuous on the real axis and

\begin{equation}
(3.6) \quad f_\pm (\lambda) = -\lambda^2 \det \beta_2 + O (\lambda), \quad \lambda \in \overline{\mathbb{C}_+}, |\lambda| \to \infty,
\end{equation}

Equation (3.6) shows the boundedness of the sets \( \sigma_d (L) \) and \( \sigma_{ss} (L) \). From the analyticity of the function \( f_\pm \) in \( \mathbb{C}_+ \) we obtain that \( \sigma_d (L) \) has at most countable number of elements and its limit points can lie only in a bounded subinterval of the real axis. By the boundary value uniqueness theorem of analytic functions, we find that \( \mu (\sigma_{ss} (L)) = 0 \), [13].

We will denote the sets of limit points of \( M^+_s \) and \( M^+_s \) by \( M^+_s \) and \( M^+_s \), and the set of all zeros of \( A_+ \) with infinite multiplicity in \( \mathbb{C}_+ \) by \( M^+_s \). Analogously define the sets \( M^+_s \), \( M^+_s \) and \( M^+_s \).

It is explicit from the boundary uniqueness theorem of analytic functions that \[13\]

\begin{equation}
(3.7) \quad M^+_s \cap M^+_s = \emptyset, \quad M^+_s \subset M^+_s, \quad M^+_s \subset M^+_s,
\end{equation}

\[ M^+_s \subset M^+_s, \quad M^+_s \subset M^+_s, \quad M^+_s \subset M^+_s \]

and \( \mu (M^+_s) = \mu (M^+_s) = \mu (M^+_s) = 0 \).

3.3. Theorem. If

\begin{equation}
(3.8) \quad Q \in AC (\mathbb{R}_+), \quad \lim_{x \to -\infty} Q(x) = 0, \quad \int_0^\infty e^{\epsilon t} \left\| Q'(x) \right\| \, dx < \infty, \quad \epsilon > 0
\end{equation}

the operator \( L \) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. By (2.5), (2.13), (2.14) and (3.8) we observe that, the function \( A_+ \) has an analytic continuation to the half plane \( \text{Im} \lambda > -\frac{\epsilon}{2} \). So, the limit points of zeros of \( A_+ \) in \( \overline{\mathbb{C}_+} \) can not lie in \( \mathbb{R} \). From analyticity of \( A_+ \) for \( \text{Im} \lambda > -\frac{\epsilon}{2} \), we obtain that all zeros of \( A_+ \) in \( \overline{\mathbb{C}_+} \) have a finite multiplicity. We obtain similar results for \( A_- \). Consequently by (3.4) and (3.5) the sets \( \sigma_d (L) \) and \( \sigma_{ss} (L) \) have a finite number of elements with a finite multiplicity.

Now let us suppose that hold, the conditions which is weaker than (3.8).

3.4. Theorem. If

\begin{equation}
(3.9) \quad Q \in AC (\mathbb{R}_+), \quad \lim_{x \to -\infty} Q(x) = 0, \quad \sup_{x \in \mathbb{R}_+} \left\{ \exp \left( \epsilon \sqrt{T} \right) \left\| Q'(x) \right\| \right\} < \infty, \quad \epsilon > 0
\end{equation}

holds, then \( M^+_s = M^+_s = \emptyset \).

Proof. From (3.1) and (3.9) we have \( f_\pm \) is analytic in \( \mathbb{C}_+ \) and all of its derivatives are continuous on the \( \overline{\mathbb{C}_+} \). For sufficiently large \( P > 0 \) we have

\begin{equation}
(3.10) \quad \left| \frac{d^m}{d\lambda^m} f_\pm (\lambda) \right| \leq T_m, \quad m = 0, 1, 2, ..., \lambda \in \overline{\mathbb{C}_+}, |\lambda| < P
\end{equation}

where

\begin{equation}
(3.11) \quad T_m := 2^m c \int_0^\infty t^m e^{-\epsilon t/2} \sqrt{t} \, dt, \quad m = 0, 1, 2, ...
\end{equation}

where \( c > 0 \) is a constant. Since the function \( f_\pm \) is not equal to zero identically, using Pavlov’s Theorem [25] we get that \( M^+_s \) satisfies

\begin{equation}
(3.12) \quad \int_0^\alpha \ln G(s) \, d\mu \left( M^+_s, s \right) > -\infty
\end{equation}
where \( G(s) = \inf_{m} \frac{T_m s^m}{m!} \), \( \mu(M^+, s) \) is the linear Lebesque measure of \( s \)-neighborhood of \( M^+ \) and \( a > 0 \) is a constant.

We obtain the following estimates for \( T_m \)

\[
T_m \leq B b^m m! \]

where \( B \) and \( b \) are constants depending on \( c \) and \( \varepsilon \). Substituting (3.13) in the definition of \( G(s) \), we arrive at

\[
G(s) = \inf_{m} \frac{T_m s^m}{m!} \leq B \exp \left( -e^{-1} b^{-1} s^{-1} \right).
\]

Now by (3.12), we get

\[
\int_{0}^{a} s^{-1} d\mu(M^+, s) < \infty.
\]

Consequently (3.14) holds for an arbitrary \( s \) if and only if \( \mu(M^+, s) = 0 \) or \( M^+ = \phi \). In a similar way we can show \( M^- = \phi \).

### 3.5. Theorem

Under the condition (3.9) the operator \( L \) has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity.

**Proof.** We have to show that the functions \( f_+ \) and \( f_- \) have a finite number of zeros with a finite multiplicities in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \), respectively. We prove only for \( f_+ \).

It follows from (3.7) and Theorem 3.4 that \( M^+_3 = M^+_4 = \phi \). So the bounded set \( M^+_3 \) and \( M^+_4 \) have no limit points, i.e. the function \( f_+ \) has only finite number of zeros in \( \mathbb{C}^+ \). Since \( M^+_3 = \phi \), these zeros are of finite multiplicity.

### References


