THE BASE POINTS OF INDEFINITE QUADRATIC FORMS IN THE CYCLE AND PROPER CYCLE OF AN INDEFINITE QUADRATIC FORM

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Abstract

Let \( F = (a, b, c) \) be an indefinite quadratic form of discriminant \( \Delta > 0 \). In the first section, we give some preliminaries from binary quadratic forms. In the second section, we derive some results concerning the base points of indefinite quadratic forms in the cycle and proper cycle of \( F \) using the transformations \( \tau(F) = (-a, b, -c) \), \( \xi(F) = (c, b, a) \), \( \chi(F) = (-c, b, -a) \), \( \psi(F) = (-a, -b, -c) \), and the right neighbor \( R^i(F) \) of \( F \) for \( i \geq 0 \).

Keywords: Quadratic form, Indefinite form, Cycle, Proper cycle, Right neighbor, Base point.

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1. Introduction.

A real binary quadratic form (or just a form) \( F \) is a polynomial in two variables \( x \) and \( y \) of the type

\[
F = F(x, y) = ax^2 + bxy + cy^2
\]

with real coefficients \( a, b, c \). We denote \( F \) briefly by \( F = (a, b, c) \). The discriminant of \( F \) is defined by the formula \( b^2 - 4ac \) and is denoted by \( \Delta = \Delta(F) \). The form \( F \) is an integral form if and only if \( a, b, c \in \mathbb{Z} \), and is indefinite if and only if \( \Delta(F) > 0 \). An indefinite quadratic form \( F = (a, b, c) \) of discriminant \( \Delta \) is said to be reduced if

\[
(1.1) \quad \left| \sqrt{\Delta} - 2|a| \right| < b < \sqrt{\Delta}.
\]

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Let $GL(2, \mathbb{Z})$ be the modular multiplicative group of $2 \times 2$ matrices $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ such that $r, s, t, u \in \mathbb{Z}$ and $\det g = \pm 1$. Gauss (1777-1855) defined the group action of $GL(2, \mathbb{Z})$ on the set of forms as follows: Let $F = (a, b, c)$ be a form and $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL(2, \mathbb{Z})$.

Then the form $gF$ is defined by

$$gF(x, y) = a(rx + ty)^2 + b(rx + ty)(sx + uy) + c(sx + uy)^2$$

(1.2)

$$= (ar^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy$$

$$+ (at^2 + btu + cuy^2),$$

that is, $gF$ is gotten from $F$ by making the substitution $x \rightarrow rx + tu$, $y \rightarrow sx + uy$.

Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in GL(2, \mathbb{Z})$, that is, the action of $GL(2, \mathbb{Z})$ on forms leaves the discriminant invariant. If $F$ is indefinite or integral, then so is $gF$ for all $g \in GL(2, \mathbb{Z})$.

Let $F$ and $G$ be two forms. If there exists a $g \in GL(2, \mathbb{Z})$ such that $gF = G$, then $F$ and $G$ are called equivalent. If $\det g = 1$, then $F$ and $G$ are called properly equivalent. If $\det g = -1$, then $F$ and $G$ are called improperly equivalent.

Let $\rho(F)$ denotes the normalization of $(c, -b, a)$, see [1, p. 88] for further details. To be more explicit, we set

$$\rho(F) = (c, -b + 2cs, cs^2 - bs + a),$$

where

$$s = s(F) = \left\{ \begin{array}{ll}
\text{sign}(c) \left\lfloor \frac{b}{2|c|} \right\rfloor & \text{for } |c| \geq \sqrt{\Delta} \\
\text{sign}(c) \left\lfloor \frac{b + \sqrt{\Delta}}{2|c|} \right\rfloor & \text{for } |c| < \sqrt{\Delta}.
\end{array} \right.$$ 

Note that, if $F$ is reduced, then $\rho(F)$ is also reduced by (1.1). In fact, $\rho$ is a permutation of the set of all reduced indefinite forms.

Let $F = (a, b, c)$ be a quadratic form. Define

$$\tau(F) = \tau(a, b, c) = (-a, b, -c).$$

Then $\rho(\rho(F)) = \rho(\tau(F)) = (c, -b + 2cs, -a + bs - cs^2)$. Hence $F$ and $\tau(F)$ are equivalent, but not necessarily properly equivalent. If $F$ is reduced, then $\tau(F)$ is also reduced. The cycle of $F$ is the sequence $((\tau\rho^i)(G))$ for $i \in \mathbb{Z}$, where $G = (k, l, m)$ is a reduced form with $k > 0$ which is equivalent to $F$. Similarly, the proper cycle of $F$ is the sequence $(\rho^i(G))$ for $i \in \mathbb{Z}$, where $G$ is a reduced form which is properly equivalent to $F$. The cycle and the proper cycle of $F$ are invariants of the equivalence class of $F$. We represent the cycle (or proper cycle) of $F$ by its period

$$F_0 \sim F_1 \sim \cdots \sim F_{l-1}$$

of length $l$. We explain how to compute the cycle of $F$: Let $F_0 = F = (a_0, b_0, c_0)$ and let

(1.3) $$s_i = s(F_i) = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor.$$ 

Then

$$F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1})$$

(1.4) $$= (|c_i|, -b_i + 2s_i|c_i|, -a_i - b_i s_i - c_i s_i^2)$$

for $0 \leq i \leq l - 2$. Hence $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ is the cycle of $F$ of length $l$. The proper cycle of $F$ is given by the following lemma.
1.1. Lemma. [1, p. 106] Let $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ be the cycle of $F$ of length $l$.

(1) If $l$ is odd, then the proper cycle of $F$ is the cycle

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length $2l$. In this case the equivalence class of $F$ is equal to the proper equivalence class of $F$.

(2) If $l$ is even, then the proper cycle of $F$ is the cycle

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length $l$. In this case the equivalence class of $F$ is the disjoint union of the proper equivalence class of $F$ and the proper equivalence class of $\tau(F)$.

The right neighbor of $F = (a, b, c)$, denoted by $R(F)$, is the form $(A, B, C)$ determined by the three conditions:

i. $A = c$,
ii. $b + B \equiv 0 \pmod{2A}$ and $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$,
iii. $B^2 - 4AC = \Delta$.

It is clear from definition that

$$R(F) = (A, B, C) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (a, b, c)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & -\delta \end{pmatrix} (a, b, c),$$

where

$$\delta = \frac{b + B}{2a}.$$ 

Therefore, $F$ is properly equivalent to its right neighbor $R(F)$ (see [2, p.129]).

2. Base Points of Indefinite Quadratic Forms.

We assume that $F = (a, b, c)$ is integral and indefinite throughout this paper. The base point of $F$ is

$$z = z(F) = \frac{-b + \sqrt{\Delta}}{2a},$$

which is one of the zeros of $F(x, 1) = ax^2 + bx + 1$. The negative of $F$ is $-F = (-a, -b, -c)$, and its base point is

$$\bar{z} = \bar{z}(-F) = \frac{b + \sqrt{\Delta}}{-2a} = \frac{-b - \sqrt{\Delta}}{2a}.$$

We recall that an element $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ is called hyperbolic if $|r + u| > 2$, parabolic if $|r + u| = 2$, and elliptic if $|r + u| < 2$.

A complex number $z$ is a fixed point of $g$ if

$$gz = z \iff \frac{rz + s}{tz + u} = z \iff tz^2 + (u - r)z - s = 0.$$
Note that \( g^{-1} = \begin{pmatrix} u & -s \\ -t & r \end{pmatrix} \) and hence \((g^{-1})^t = \begin{pmatrix} u & -t \\ -s & r \end{pmatrix} \in \text{GL}(2, \mathbb{Z})\). Similarly, a complex number \( z \) is a fixed point of \((g^{-1})^t\) if
\[
(g^{-1})^t z = z \iff \frac{uz - t}{-sz + r} = z \iff sz^2 + (u - r)z - t = 0.
\]
So, \( g \) fixes
\[
z = \frac{r - u \pm \sqrt{(u - r)^2 + 4ts}}{2t} = \frac{r - u \pm \sqrt{(u + r)^2 + 4s}}{2s},
\]
and \((g^{-1})^t\) fixes
\[
z = \frac{r - u \pm \sqrt{(u - r)^2 + 4ts}}{2s} = \frac{r - u \pm \sqrt{(u + r)^2 + 4s}}{2s}.
\]
Since \( \text{GL}(2, \mathbb{Z}) \) is discrete, the stabilizer
\[\{ g \in \text{GL}(2, \mathbb{Z}) : gz = z \}\]
of any complex number \( z \) in \( \text{GL}(2, \mathbb{Z}) \) is a cyclic subgroup of \( \text{GL}(2, \mathbb{Z}) \). Hence we can call fixed points hyperbolic, parabolic or elliptic according to whether the matrices fixing them are hyperbolic, parabolic or elliptic, respectively.

2.1. Theorem. Given any hyperbolic fixed \( z \) of \( g \) in \( \text{GL}(2, \mathbb{Z}) \), there exists an indefinite quadratic form \( F \) whose base point is \( z \).

Proof. Let \( g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \). Then \((g^{-1})^t = \begin{pmatrix} u & -t \\ -s & r \end{pmatrix}\). Further, \( g \) and \((g^{-1})^t\) generate the stabilizer of hyperbolic fixed point \( z \) in \( \text{GL}(2, \mathbb{Z}) \). Then by (2.3), \( gz = z \) gives rise to the equation
\[
(tz^2 + (u - r)z - s = 0,
\]
and by (2.4), \((g^{-1})^t z = z \) gives rise the equation
\[
sz^2 + (u - r)z - t = 0.
\]
If we choose
\[
g_z = \begin{cases} g & \text{if } z = \frac{r - u \pm \sqrt{(u - r)^2 + 4ts}}{2t} \\ (g^{-1})^t & \text{if } z = \frac{r - u \pm \sqrt{(u + r)^2 + 4s}}{2s} \end{cases}
\]
then
\[
F_z = \begin{cases} (t, u - r, -s) & \text{if } g_z = g \\ (s, u - r, -t) & \text{if } g_z = (g^{-1})^t \end{cases}
\]
is an indefinite quadratic form of discriminant
\[
\Delta = (u - r)^2 + 4ts = (u + r)^2 \pm 4
\]
whose base point is \( z \). \(\square\)

2.2. Theorem. Let \( z_1 \) and \( z_2 \) be two hyperbolic numbers and let \( F_{z_1} = (a_1, b_1, c_1) \) and \( F_{z_2} = (a_2, b_2, c_2) \) be two indefinite quadratic forms which correspond to \( z_1 \) and \( z_2 \), respectively. Let \( g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \). Then \( F_{z_1} \) and \( F_{z_2} \) are equivalent if and only if \((g^{-1})^t z_1 = z_2\).
Proof. Suppose that $F_1$ and $F_2$ are (properly) equivalent. Then by (1.2), we have
\[ gF_1 = (a_1 r^2 + b_1 rs + c_1 s^2, 2a_1 rt + b_1 ru + b_1 ts + 2c_1 su, a_1 t^2 + b_1 tu + c_1 u^2) \]
\[ = F_2 = (a_2, b_2, c_2) \]
for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ with $\det g = 1$. So by (2.1), the base point of $F_2$ is
\[ z_2 = -(2a_1 rt + b_1 ru + b_1 ts + 2c_1 su) + \sqrt{\Delta} \]
\[ = \frac{-(2a_1 rt + b_1 ru + b_1 ts + 2c_1 su) + \sqrt{\Delta}}{2(a_1 r^2 + b_1 rs + c_1 s^2)} \].

Note that the base point of $F_1$ is $z_1 = \frac{-b_1 + \sqrt{\Delta}}{2a_1}$ and $(g^{-1})^t = \begin{pmatrix} u & -t \\ -s & r \end{pmatrix}$. So
\[ (g^{-1})^t z_1 = \frac{u \left( \frac{-b_1 + \sqrt{\Delta}}{2a_1} \right) - t}{-s \left( \frac{-b_1 + \sqrt{\Delta}}{2a_1} \right) + r} \]
\[ = \frac{-b_1 u - 2a_1 t + u \sqrt{\Delta}}{b_1 s + 2a_1 r - s \sqrt{\Delta}} \]
\[ = \frac{(-b_1 u - 2a_1 t + u \sqrt{\Delta})(b_1 s + 2a_1 r + s \sqrt{\Delta})}{(b_1 s + 2a_1 r - s \sqrt{\Delta})(b_1 s + 2a_1 r + s \sqrt{\Delta})} \]
\[ = \frac{\left[ (-b_1 u - 2a_1 t)(b_1 s + 2a_1 r) + us \Delta \right]}{(b_1 s + 2a_1 r)^2 - s^2 \Delta} \].

After some calculations, we see that
\[ (-b_1 u - 2a_1 t)(b_1 s + 2a_1 r) + us \Delta \]
\[ = -b_1^2 us - 2a_1 b_1 ur - 2a_1 b_1 st - 4a_1^2 rt + us(b_1^2 - 4a_1 c_1) \]
\[ = -b_1^2 us - 2a_1 b_1 ur - 2a_1 b_1 st - 4a_1^2 rt + b_1^2 us - 4a_1 c_1 us \]
\[ = -2a_1 b_1 ur - 2a_1 b_1 st - 4a_1^2 rt + 4a_1 c_1 us \]
\[ = 2a_1 \left[ -(2a_1 rt + b_1 ru + b_1 ts + 2c_1 su) \right] \],
\[ s(-b_1 u - 2a_1 t) + u(b_1 s + 2a_1 r) = -b_1 us - 2a_1 st + b_1 us + 2a_1 ru \]
\[ = 2a_1 ru - 2a_1 st \]
\[ = 2a_1 (ru - st) \]
\[ = 2a_1 \]
and
\[ (b_1 s + 2a_1 r)^2 - s^2 \Delta = b_1^2 s^2 + 4a_1 b_1 rs + 4a_1^2 r^2 - s^2(b_1^2 - 4a_1 c_1) \]
\[ = b_1^2 s^2 + 4a_1 b_1 rs + 4a_1^2 r^2 - b_1^2 s^2 + 4a_1 c_1 s^2 \]
\[ = 4a_1 b_1 rs + 4a_1^2 r^2 + 4a_1 c_1 s^2 \]
\[ = 2a_1 \left[ 2(a_1 r^2 + b_1 rs + c_1 s^2) \right] \].
So, (2.5) becomes
\[
(g^{-1})^t z_1 = \frac{u \left( \frac{-b_1 t + \sqrt{\Delta}}{2a_1} \right) - t}{-s \left( \frac{-b_1 t + \sqrt{\Delta}}{2a_1} \right) + r}
= \frac{-b_1 u - 2a_1 t + u \sqrt{\Delta}}{b_1 s + 2a_1 r - s \sqrt{\Delta}}
= \frac{(-b_1 u - 2a_1 t + u \sqrt{\Delta})(b_1 s + 2a_1 r + s \sqrt{\Delta})}{(b_1 s + 2a_1 r - s \sqrt{\Delta})(b_1 s + 2a_1 r + s \sqrt{\Delta})}
= \frac{[(-b_1 u - 2a_1 t)(b_1 s + 2a_1 r) + us \Delta] + [s(-b_1 u - 2a_1 t) + u(b_1 s + 2a_1 r)] \sqrt{\Delta}}{(b_1 s + 2a_1 r)^2 - s^2 \Delta}
= \frac{2a_1 [-2a_1 rt + b_1 ru + b_1 ts + 2c_1 su] + 2a_1 \sqrt{\Delta}}{2a_1 [2(a_1 r^2 + b_1 rs + c_1 s^2)]}
= \frac{-(2a_1 rt + b_1 ru + b_1 ts + 2c_1 su) + \sqrt{\Delta}}{2(a_1 r^2 + b_1 rs + c_1 s^2)} = z_2.
\]

Conversely, let \((g^{-1})^t z_1 = z_2\). Let \(h\) be one of the two generators of the stabilizer of \(z_1\) in \(GL(2, \mathbb{Z})\). Then \(k = (g^{-1})^t h g\) is one of the two generators of the stabilizer of \(z_2\) in \(GL(2, \mathbb{Z})\). Further \(h\) and \(k\) have the same trace (since the trace of a matrix is preserved by conjugation). Therefore, by Theorem 2.1, \(h z_1 = z_1\) gives rise to \(F_{z_1}\) and \(k z_2 = z_2\) gives rise to \(F_{z_2}\). So \(F_{z_1}\) and \(F_{z_2}\) have the same discriminant \(\Delta\). We see as above that \(gF_{z_1}\) has the hyperbolic number
\[
\frac{-(2a_1 rt + b_1 ru + b_1 ts + 2c_1 su) + \sqrt{\Delta}}{2(a_1 r^2 + b_1 rs + c_1 s^2)} = (g^{-1})^t z_1 = z_2,
\]
and \(gF_{z_1}\) has the hyperbolic number
\[
\frac{-(2a_1 rt + b_1 ru + b_1 ts + 2c_1 su) - \sqrt{\Delta}}{2(a_1 r^2 + b_1 rs + c_1 s^2)} = (g^{-1})^t z_1 = z_2
\]
by (2.2). Consequently, \(gF_{z_1}(x, 1)\) and \(F_{z_2}(x, 1)\) have the same discriminant \(\Delta\) and the same zeros, so they are equal. Hence \(F_{z_1}\) and \(F_{z_2}\) are (properly) equivalent.

\[\square\]

\[2.3.\text{Remark.}\] Note that in the proof of Theorem 2.2 we assume that \(F_{z_1}\) and \(F_{z_2}\) are properly equivalent. If we assume that \(F_{z_1}\) and \(F_{z_2}\) are improperly equivalent, then the proof is same.

\[2.4.\text{Example.}\] The forms \(F_{z_1} = (1, 7, -6)\) and \(F_{z_2} = (4, 3, -4)\) with discriminant \(\Delta = 73\) are properly equivalent under \(g = \begin{pmatrix} 7 & 9 \\ 10 & 13 \end{pmatrix} \in GL(2, \mathbb{Z})\), that is, \(gF_{z_1} = F_{z_2}\). The base point of \(F_{z_1}\) is \(z_1 = \frac{-7 + \sqrt{73}}{2}\) and the base point of \(F_{z_2}\) is \(z_2 = \frac{-3 + \sqrt{73}}{8}\). Note that \((g^{-1})^t = \begin{pmatrix} 13 & -10 \\ -9 & 7 \end{pmatrix}\), and hence
\[(g^{-1})^t z_1 = \frac{13 \left( \frac{-7 + \sqrt{73}}{2} \right) - 10}{-9 \left( \frac{-7 + \sqrt{73}}{2} \right) + 7} = \frac{-111 + 13\sqrt{73}}{77 - 9\sqrt{73}} = \frac{(-111 + 13\sqrt{73})(77 + 9\sqrt{73})}{(77 - 9\sqrt{73})(77 + 9\sqrt{73})} = \frac{-6 + 2\sqrt{73}}{16} = \frac{-3 + \sqrt{73}}{8} = z_2.

Now we define the following transformations for an indefinite quadratic form \(F = (a, b, c)\).

\[
\begin{align*}
\tau(F) &= (-a, b, -c) \\
\psi(F) &= (-a, -b, -c) = -F \\
\xi(F) &= (c, b, a) \\
\chi(F) &= (-c, b, -a).
\end{align*}
\]

First, we consider the connection between them.

2.5. **Theorem.** Let \(F = (a, b, c)\) be an indefinite quadratic form. Then

\[
\chi\tau = \xi, \quad \xi\tau = \chi, \quad \chi\xi = \tau \quad \text{and} \quad \chi\xi\tau\psi = \psi.
\]

**Proof.** Let \(F = (a, b, c)\) be an indefinite quadratic form. Then

\[
\tau(F) = (-a, b, -c) \iff \chi(\tau(F)) = (c, b, a) = \xi(F).
\]

Similarly, it can be shown that

\[
\begin{align*}
\tau(F) &= (-a, b, -c) \iff \xi(\tau(F)) = (-c, b, -a) = \chi(F). \\
\xi(F) &= (c, b, a) \iff \chi(\xi(F)) = (-a, b, -c) = \tau(F). \\
\chi(F) &= (-c, b, -a) \iff \xi(\chi(F)) = (-a, b, -c) = \tau(F). \\
\psi(F) &= (-a, -b, -c) \iff \tau(\psi(F)) = (a, -b, c) \iff \xi(\tau(\psi(F))) = (c, -b, a) \iff \chi(\xi(\tau(\psi(F)))) = (-a, -b, -c) = \psi(F).
\end{align*}
\]

This completes the proof. \(\square\)

2.6. **Remark.** Note that \(\tau^{-1} = \tau, \psi^{-1} = \psi, \xi^{-1} = \xi\) and \(\chi^{-1} = \chi\). So all the results obtained in Theorem 2.5 can be easily verified.

Now we can give the following results concerning the base points of indefinite quadratic forms in the cycle and proper cycle of \(F\).

2.7. **Theorem.** In the cycle \(F_0 \sim F_1 \sim \cdots \sim F_{l-1}\) of \(F\),

\[
z(F_i) = z(\chi(F_{i-1} - i))
\]

for \(0 \leq i \leq l - 1\).
Proof. Let \( F = F_0 = (a_0, b_0, c_0) \). Then by (1.3) and (1.4), we get

\[
F_0 = (a_0, b_0, c_0) \\
F_1 = (a_1, b_1, c_1) \\
F_2 = (a_2, b_2, c_2) \\
\vdots \\
F_{i-3} = (a_{i-3}, b_{i-3}, c_{i-3}) \\
F_{i-1} = (a_{i-1}, b_{i-1}, c_{i-1}) \\
F_{i-2} = (a_{i-2}, b_{i-2}, c_{i-2}) \\
F_{i-1} = (a_{i-2}, b_{i-2}, c_{i-2}) \\
F_{i-2} = (a_{i-2}, b_{i-2}, c_{i-2}) \\
F_{i-1} = (a_{i-2}, b_{i-2}, c_{i-2}) \\
F_{i-2} = (a_{i-2}, b_{i-2}, c_{i-2}) \\
\]

Hence it is easily seen that

\[
F_0 = \chi(F_{i-1}), \quad F_1 = \chi(F_{i-2}), \quad F_2 = \chi(F_{i-3}), \ldots, F_{i-3} = \chi(F_{i-1}), \\
F_{i-1} = \chi(F_{i-2}), \quad F_{i-2} = \chi(F_{i-3}), \ldots, F_{i-3} = \chi(F_{i-1}), \\
F_{i-2} = \chi(F_{i-1}) \quad \text{and} \quad F_{i-1} = \chi(F_0).
\]

Therefore \( F_i = \chi(F_{i-1}) \) for \( 0 \leq i \leq l - 1 \). Consequently \( z(F_i) = z(\chi(F_{i-1})) \) for \( 0 \leq i \leq l - 1 \). \( \square \)

The proper cycle of \( F \) can be derived by using its consecutive right neighbors. Let \( R^i(F_0) \) be the \( i \)-th right neighbor of \( F = F_0 \). We proved in [3] that if \( l \) is odd, then the proper cycle of \( F \) is

\[
F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)
\]

of length \( 2l \), and if \( l \) is even, then the proper cycle of \( F \) is

\[
F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)
\]

of length \( l \). From now on, we assume that the length of the proper cycle of \( F \) is odd throughout this paper.

2.8. Theorem. In the proper cycle of \( F \),

\[
z(R^i(F_0)) = z\left(\tau(R^{i+1}(F_0))\right) \quad \text{for} \quad 0 \leq i \leq l - 1, \\
z(R^i(F_0)) = z(\tau(F_0)) \\
z\left(R^i(F_0)\right) = z\left(\tau(R^{i-1}(F_0))\right) \quad \text{for} \quad l + 1 \leq i \leq 2l - 1.
\]
Proof. Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6), we get
\[ F_0 = (a_0, b_0, c_0) \]
\[ R^1(F_0) = (a_1, b_1, c_1) \]
\[ R^2(F_0) = (a_2, b_2, c_2) \]
\[ \cdots \cdots \cdots \]
\[ R^{2l-1}(F_0) = (a_{2l-1}, b_{2l-1}, c_{2l-1}) \]
\[ R^l(F_0) = (a_l, b_l, c_l) \]
\[ R^{l+1}(F_0) = (a_{l+1}, b_{l+1}, c_{l+1}) \]
\[ \cdots \cdots \cdots \]
\[ R^{2l-3}(F_0) = (a_{2l-3}, b_{2l-3}, c_{2l-3}) \]
\[ R^{2l-2}(F_0) = (a_{2l-2}, b_{2l-2}, c_{2l-2}) \]
\[ R^{2l-1}(F_0) = (a_{2l-1}, b_{2l-1}, c_{2l-1}) \]

Hence, it is easily seen that $R^i(F_0) = \tau(R^{i+1}(F_0))$ for $0 \leq i \leq l - 1$, $R^i(F_0) = \tau(F_0)$ and $R^i(F_0) = \tau(R^{i-1}(F_0))$ for $l + 1 \leq i \leq 2l - 1$.

Therefore $z(R^i(F_0)) = z(\tau(R^{i+1}(F_0)))$ for $0 \leq i \leq l - 1$, $z(R^i(F_0)) = z(\tau(F_0))$ and $z(R^i(F_0)) = z(\tau(R^{i-1}(F_0)))$ for $l + 1 \leq i \leq 2l - 1$, as we claimed. \hfill \Box

2.9. Theorem. In the proper cycle of $F$,
\[ z(R^i(F_0)) = z(\xi(R^{2l-1-i}(F_0))) \]
for $0 \leq i \leq 2l - 1$.

Proof. Let $F = F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6), we get
\[ F_0 = (a_0, b_0, c_0) \]
\[ R^1(F_0) = (a_1, b_1, c_1) \]
\[ R^2(F_0) = (a_2, b_2, c_2) \]
\[ \cdots \cdots \cdots \]
\[ R^{2l-1}(F_0) = (a_{2l-1}, b_{2l-1}, c_{2l-1}) \]
\[ R^l(F_0) = (c_l, b_l, a_l) \]
\[ R^{l+1}(F_0) = (c_{l+1}, b_{l+1}, a_{l+1}) \]
\[ \cdots \cdots \cdots \]
\[ R^{2l-3}(F_0) = (c_{2l-3}, b_{2l-3}, a_{2l-3}) \]
\[ R^{2l-2}(F_0) = (c_{2l-2}, b_{2l-2}, a_{2l-2}) \]
\[ R^{2l-1}(F_0) = (c_{2l-1}, b_{2l-1}, a_{2l-1}) \]

It is easily seen that $R^i(F_0) = \xi(R^{2l-1-i}(F_0))$ for $0 \leq i \leq 2l - 1$. Therefore $z(R^i(F_0)) = z(\xi(R^{2l-1-i}(F_0)))$ for $0 \leq i \leq 2l - 1$. \hfill \Box

2.10. Theorem. In the proper cycle of $F$,
\[ z(R^i(F_0)) = z(\chi(R^{l-1-i}(F_0))) \]
for $0 \leq i \leq l - 1$
\[ z(R^i(F_0)) = z(\chi(R^{2l-1-i}(F_0))) \]
for $l \leq i \leq 2l - 1$. 

Proof. Let \( F = F_0 = (a_0, b_0, c_0) \). Then by (1.5) and (1.6), we get

\[ F_0 = (a_0, b_0, c_0) \]
\[ R^1(F_0) = (a_1, b_1, c_1) \]
\[ R^2(F_0) = (a_2, b_2, c_2) \]

\[ R^{i-3}(F_0) = (a_{i-3}, b_{i-3}, c_{i-3}) \]
\[ R^{i-2}(F_0) = (a_{i-2}, b_{i-2}, c_{i-2}) \]
\[ R^{i-1}(F_0) = (a_{i-1}, b_{i-1}, c_{i-1}) \]
\[ R^0(F_0) = (a_0, b_0, c_0) \]
\[ R^1(F_0) = (a_1, b_1, c_1) \]
\[ R^2(F_0) = (a_2, b_2, c_2) \]
\[ R^{i-3}(F_0) = (a_{i-3}, b_{i-3}, c_{i-3}) \]
\[ R^{i-2}(F_0) = (a_{i-2}, b_{i-2}, c_{i-2}) \]
\[ R^{i-1}(F_0) = (a_{i-1}, b_{i-1}, c_{i-1}) \]
\[ R^0(F_0) = (a_0, b_0, c_0) \]
\[ R^1(F_0) = (a_1, b_1, c_1) \]

Hence, \( R^l(F_0) = \chi(R^{i-1-l}(F_0)) \) for \( 0 \leq i \leq l - 1 \), and \( R^l(F_0) = \chi(R^{i-1-l}(F_0)) \) for \( l \leq i \leq 2l - 1 \). Therefore \( z(R^l(F_0)) = z(\chi(R^{i-1-l}(F_0))) \) for \( 0 \leq i \leq l - 1 \) and \( z(R^l(F_0)) = z(\chi(R^{i-1-l}(F_0))) \) for \( l \leq i \leq 2l - 1 \). \( \square \)

2.11. Theorem. In the proper cycle of \( F \),

\[ z(R^l(F_0)) = z(\chi(R^{i-1-l}(F_0))) \text{ for } 1 \leq i \leq 2l - 2 \]
\[ z(R^{2l-1}(F_0)) = z(\chi(R^{i-1-l}(F_0))) \]
\[ z(R^l(F_0)) = z(\chi(R^{i-1-l}(F_0))) \text{ for } l \leq i \leq l - 1 \]

(1)

\[ z(R^l(F_0)) = z(\chi(R^{i-1-l}(F_0))) \text{ for } l + 1 \leq i \leq 2l - 1. \]

(2)
Proof. (1) Let \( F = F_0 = (a_0, b_0, c_0) \). Then by (1.5) and (1.6),
\[
\begin{align*}
F_0 &= (a_0, b_0, c_0) \\
R^1(F_0) &= (a_1, b_1, c_1) \\
R^2(F_0) &= (a_2, b_2, c_2) \\
R^3(F_0) &= (a_3, b_3, c_3) \\
&\ldots \\
R^{2l-3}(F_0) &= (a_{2l-3}, b_{2l-3}, c_{2l-3}) \\
R^{2l-2}(F_0) &= (a_{2l-2}, b_{2l-2}, c_{2l-2}) \\
R^{2l-1}(F_0) &= (a_{2l-1}, b_{2l-1}, c_{2l-1}) \\
R^{2l}(F_0) &= (a_{2l}, b_{2l}, c_{2l}) \\
R^{2l+1}(F_0) &= (a_{2l+1}, b_{2l+1}, c_{2l+1}) \\
R^{2l+2}(F_0) &= (a_{2l+2}, b_{2l+2}, c_{2l+2}) \\
R^{2l+3}(F_0) &= (a_{2l+3}, b_{2l+3}, c_{2l+3}) \\
&\ldots \\
R^{2l+1}(F_0) &= (a_{2l+1}, b_{2l+1}, c_{2l+1}) \\
R^{2l+2}(F_0) &= (a_{2l+2}, b_{2l+2}, c_{2l+2}) \\
R^{2l+3}(F_0) &= (a_{2l+3}, b_{2l+3}, c_{2l+3}) \\
&\ldots \\
R^{2l+4}(F_0) &= (a_{2l+4}, b_{2l+4}, c_{2l+4}) \\
R^{2l+3}(F_0) &= (a_{2l+3}, b_{2l+3}, c_{2l+3}) \\
R^{2l+2}(F_0) &= (a_{2l+2}, b_{2l+2}, c_{2l+2}) \\
R^{2l+1}(F_0) &= (a_{2l+1}, b_{2l+1}, c_{2l+1}) \\
R^{2l}(F_0) &= (a_{2l}, b_{2l}, c_{2l}). 
\end{align*}
\]
It is clear that \( R^i(F_0) = \chi(\tau(R^{2l+1-i}(F_0))) \) for \( 1 \leq i \leq 2l - 2 \), and \( R^{2l-1}(F_0) = \chi(\tau(F_0)) \). Consequently \( z(R^i(F_0)) = z(\chi(\tau(R^{2l+1-i}(F_0)))) \) for \( 1 \leq i \leq 2l - 2 \) and \( z(R^{2l+1}(F_0)) = z(\chi(\tau(F_0))) \).

(2) This can be proved as for (1). \( \square \)

Now we split the proper cycle
\( F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0) \)
of \( F \) into two equal part as follows:
\[
(2.6) \quad F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)
\]
and
\[
(2.7) \quad R^l(F_0) \sim R^{l+1}(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0),
\]
each of length \( l \). We call (2.6) the first part and (2.7) the second part of the proper cycle of \( F \).

2.12. Theorem. Let \( F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0) \) be the proper cycle of \( F \).

(1) In the first part,
\[
z(\chi(R^i(F_0))) = z(R^{l+1-i}(F_0)) \quad \text{for} \quad 1 \leq i \leq l - 2 \\
z(\chi(R^{l-1}(F_0))) = z(F_0).
\]

(2) In the second part,
\[
z(\chi(R^i(F_0))) = z(R^{2l-1-i}(F_0)) \quad \text{for} \quad l \leq i \leq 2l - 1.
\]
Proof. (1) Let $F_0 = (a_0, b_0, c_0)$. Then by (1.5) and (1.6),

$$
F_0 = (a_0, b_0, c_0)
$$
$$
R^1(F_0) = (a_1, b_1, c_1)
$$
$$
R^2(F_0) = (a_2, b_2, c_2)
$$
$$
R^3(F_0) = (a_3, b_3, c_3)
$$

\[R^{i-3}(F_0) = \left(\frac{a_{i-2}}{2}, \frac{b_{i-2}}{2}, \frac{c_{i-2}}{2}\right)\]
\[R^{i-1}(F_0) = \left(\frac{a_{i-1}}{2}, \frac{b_{i-1}}{2}, -\frac{a_{i-1}}{2}\right)\]
\[R^{i+1}(F_0) = \left(-\frac{c_{i-2}}{2}, \frac{b_{i-2}}{2}, -\frac{a_{i-2}}{2}\right)\]

\[R^{i-4}(F_0) = (-c_3, b_3, -a_3)\]
\[R^{i-3}(F_0) = (-c_2, b_2, -a_2)\]
\[R^{i-2}(F_0) = (-c_1, b_1, -a_1)\]
\[R^{i-1}(F_0) = (-c_0, b_0, -a_0)\].

It is easily seen that $\chi(R^i(F_0)) = R^{i-1-i}(F_0)$ for $1 \leq i \leq l - 2$, and $\chi(R^{l-1}(F_0)) = F_0$.

So $z(\chi(R^i(F_0))) = z(R^{i-1-i}(F_0))$ for $1 \leq i \leq l - 2$ and $z(\chi(R^{l-1}(F_0))) = z(F_0)$.

(2) Similarly it can be shown that

\[R^i(F_0) = (a_i, b_i, c_i)\]
\[R^{i+1}(F_0) = (a_{i+1}, b_{i+1}, c_{i+1})\]
\[R^{i+2}(F_0) = (a_{i+2}, b_{i+2}, c_{i+2})\]

\[R^{2l-3}(F_0) = \left(\frac{a_{2l-3}}{2}, \frac{b_{2l-3}}{2}, \frac{c_{2l-3}}{2}\right)\]
\[R^{2l-1}(F_0) = \left(\frac{a_{2l-1}}{2}, \frac{b_{2l-1}}{2}, -\frac{a_{2l-1}}{2}\right)\]
\[R^{2l+1}(F_0) = \left(-\frac{c_{2l-2}}{2}, \frac{b_{2l-2}}{2}, -\frac{a_{2l-2}}{2}\right)\]

\[R^{2l-3}(F_0) = (-c_{l+2}, b_{l+2}, -a_{l+2})\]
\[R^{2l-2}(F_0) = (-c_{l+1}, b_{l+1}, -a_{l+1})\]
\[R^{2l-1}(F_0) = (-c_l, b_l, -a_l)\].

Hence, $\chi(R^i(F_0)) = R^{2l-1-i}(F_0)$ for $l \leq i \leq 2l - 1$. Consequently $z(\chi(R^i(F_0))) = z(R^{2l-1-i}(F_0))$ for $l \leq i \leq 2l - 1$. □
2.13. Example. The proper cycle of $F = (1, 7, -6)$ is

$$F_0 = (1, 7, -6) \sim R^1(F_0) = (-6, 5, 2) \sim R^2(F_0) = (2, 7, -3) \sim R^3(F_0) = (3, 7, -2) \sim R^4(F_0) = (2, 5, 6) \sim R^5(F_0) = (6, 7, -1) \sim R^6(F_0) = (1, 7, 6) \sim R^{13}(F_0) = (6, 5, -2) \sim R^{14}(F_0) = (-2, 7, 3) \sim R^{15}(F_0) = (3, 5, -4) \sim R^{16}(F_0) = (4, 3, -4) \sim R^{17}(F_0) = (4, 5, -3) \sim R^{18}(F_0) = (3, 7, 2) \sim R^{19}(F_0) = (2, 5, -6) \sim R^{20}(F_0) = (-6, 7, 1)$$

of length 18. In the first part of the cycle,

$$F_0 = (1, 7, -6) \sim R^1(F_0) = (-6, 5, 2) \sim R^2(F_0) = (2, 7, -3) \sim R^3(F_0) = (3, 7, -2) \sim R^4(F_0) = (4, 3, -4) \sim R^5(F_0) = (6, 7, -1)$$

so we have,

$$\chi(R^1(F_0)) = R^1(F_0), \chi(R^2(F_0)) = R^2(F_0), \chi(R^3(F_0)) = R^3(F_0), \chi(R^4(F_0)) = R^4(F_0), \chi(R^5(F_0)) = R^5(F_0)$$

Therefore,

$$z(\chi(R^1(F_0))) = z(R^1(F_0)), \quad z(\chi(R^2(F_0))) = z(R^2(F_0)), \quad z(\chi(R^3(F_0))) = z(R^3(F_0)), \quad z(\chi(R^4(F_0))) = z(R^4(F_0)), \quad z(\chi(R^5(F_0))) = z(R^5(F_0))$$

In the second part of the cycle,

$$R^1(F_0) = (1, 7, 6) \sim R^{13}(F_0) = (6, 5, -2) \sim R^{14}(F_0) = (-2, 7, 3) \sim R^{15}(F_0) = (3, 5, -4) \sim R^{16}(F_0) = (4, 3, -4) \sim R^{17}(F_0) = (4, 5, -3) \sim R^{18}(F_0) = (-6, 7, 1)$$

so we have,

$$\chi(R^{13}(F_0)) = R^{17}(F_0), \chi(R^{14}(F_0)) = R^{16}(F_0), \chi(R^{15}(F_0)) = R^{15}(F_0), \chi(R^{16}(F_0)) = R^{14}(F_0), \chi(R^{17}(F_0)) = R^{13}(F_0)$$

Therefore,

$$z(\chi(R^{13}(F_0))) = z(R^{17}(F_0)), \quad z(\chi(R^{14}(F_0))) = z(R^{16}(F_0)), \quad z(\chi(R^{15}(F_0))) = z(R^{15}(F_0)), \quad z(\chi(R^{16}(F_0))) = z(R^{14}(F_0)), \quad z(\chi(R^{17}(F_0))) = z(R^{13}(F_0))$$
References