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ON THE GENERALIZED BESSEL HEAT EQUATION RELATED TO THE GENERALIZED BESSEL DIAMOND OPERATOR

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Abstract

In this article, we study the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2 \otimes_B^{m,k} u(x,t)$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}_n^+$. Here the operator $\otimes_B^{m,k}$ is called the Generalized Bessel Diamond Operator, iterated k times, and is defined by

$$\otimes_B^{m,k} = \Big[\big(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\big)^m - \big(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\big)^m \Big]^k,$$

where k and m are positive integers, $p+q=n$, $B_{x_i} = \dfrac{\partial^2}{\partial x_i^2} + \dfrac{2v_i}{x_i}\dfrac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \dots, n$, n being the dimension of the space $\mathbb{R}_n^+$, $u(x,t)$ is an unknown function of the form $(x,t) = (x_1, \dots, x_n, t) \in \mathbb{R}_n^+ \times (0,\infty)$, $f(x)$ is a given generalized function and c a constant. We obtain the solution of this equation, which is related to the spectrum and the kernel, the so called generalized Bessel diamond heat kernel. Moreover, the generalized Bessel diamond heat kernel is shown to have interesting properties and to be related to the kernel of an extension of the heat equation.

Keywords: Heat kernel, Dirac-delta distribution, Bessel Diamond Operator, Spectrum.

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1. Introduction

The casual fundamental solution $h(x,t)$ is the particular solution of

$$\frac{\partial E}{\partial t} - a\Delta E = \delta(x)\delta(t),$$

which vanishes identically for $t < 0$. Thus $h(x,t)$ satisfies

$$\frac{\partial h}{\partial t} - a\Delta h = \delta(x)\delta(t),\ h \equiv 0 \text{ for } t < 0.$$

The causal fundamental solution $h(x,t)$ has a direct physical interpretation; it is the temperature distribution in a medium, which is at zero temperature up to the time $t = 0$, when a concentrated source is introduced at $x = 0$, this source instantaneously releasing a unit of heat. Although h is defined for all t and x, its calculation presents a problem only for $t > 0$ ($h = 0$ for $t < 0$). This immediately suggests a slightly different point of view; for $t > 0$ no sources are present, so that h satisfies the homogeneous equation and must reduce, at $t = 0+$, to a certain initial temperature. This initial temperature is the one to which the medium has been raised just after the introduction of the instantaneous concentrated source of unit strength. We now show that this initial temperature is $\delta(x)$.

It is known that the one-dimensional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is the temperature of some object and D is a constant called the "thermal diffusivity" of the material that makes up the object (we could equally well have modeled the diffusion of chemical by letting $u(x,t)$ represent the concentration of some chemical and D the constant "diffusivity" of the chemical species inside the material that makes up the object). The diffusion equation describes such a physical situation as heat conduction in a one-dimensional solid body, spread of a die in a stationary fluid, population dispersion, and other similar processes. In [2], Chou and Jiang described the diffusion onto a small surface patch on a spherical molecule with an attractive potential all around it. A similar model has been presented by Zhou, who takes into account the attractive interaction and the influence from the heterogeneous surface reactivity only in a thin spherical shell around the target molecule [19]. In this way, the interaction required to hold the reactants together long enough for them to find the reactive site can be estimated. Both of these models indicate that the short range Van der Waals' force could provide sufficient interaction to overcome the orientational constraint of the target molecule. For a recent discussion of these and some other models for heterogeneous surface reactivity see also Chou and Zhou [4]. We refer the reader to the papers [1, 3, 20, 21] for these subjects.

It is known that for the heat equation

$$\frac{\partial u(x,t)}{\partial t} = c^2\Delta u(x,t), \tag{1.1}$$

with the initial condition $u(x,0) = f(x)$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

denotes the Laplacian operator and $u(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$. We can obtain the solution as

$$u(x,t) = \frac{1}{(4c^2\pi t)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} f(x-y)e^{-\frac{|x|^2}{4c^2 t}}\, dy,$$

or in the classical convolution form

$$u(x,t) = E(x,t) * f(x), \tag{1.2}$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4c^2 t}} \tag{1.3}$$

and the symbol $*$ denotes the classical convolution.

On the other hand, in [11], we have studied the solutions of the Bessel Diamond Heat Equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \diamond_B^k u(x,t) \tag{1.4}$$

under the initial condition $u(x,0) = f(x)$, where the k-times iterated *Bessel diamond operator* $\diamond_B^k$ is defined by

$$\diamond_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^2 - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^2\right]^k,$$

$p+q=n$ is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_i > 0,\ i = 1,2,\ldots,n\}$, k is a positive integer, $B_{x_i} = \dfrac{\partial^2}{\partial x_i^2} + \dfrac{2v_i}{x_i}\dfrac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1,2,\ldots,n$, $u(x,t)$ is an unknown function, $f(x)$ is the given generalized function and c is a constant. Moreover, such a Bessel diamond heat kernel has interesting properties and is also related to the kernel of an extension of the heat equation.

We obtain $u(x,t) = E(x,t) *_B f(x)$, the symbol $*_B$ being the B-convolution in (2.1), as a solution of (1.6), which satisfies (1.7), where

$$E(x,t) = C_\upsilon \int_{\Omega^+} e^{c^2 t\left[\left(y_1^2+\ldots+y_p^2\right)^2 - \left(y_{p+1}^2+\ldots+y_{p+q}^2\right)^2\right]^k} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i}\, dy, \tag{1.5}$$

$\Omega^+ \subset \mathbb{R}_n^+$ is the spectrum of $E(x,t)$ for any fixed $t > 0$, and $J_{\upsilon_i - \frac{1}{2}}(x_i y_i)$ is the normalized Bessel function [11].

The purpose of this work is to study the solutions of following equation:

$$\frac{\partial}{\partial t} u(x,t) = c^2 \otimes_B^{m,k} u(x,t) \tag{1.6}$$

under the initial condition

$$u(x,0) = f(x), \text{ for } x \in \mathbb{R}_n^+, \tag{1.7}$$

where the k-times iterated *generalized Bessel diamond operator* $\otimes_B^{m,k}$ is defined by

$$\otimes_B^{m,k} = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^m - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^m\right]^k, \tag{1.8}$$

$p+q=n$ is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_i > 0,\ i = 1,2,\ldots,n\}$, k and m are positive integers, $B_{x_i} = \dfrac{\partial^2}{\partial x_i^2} + \dfrac{2v_i}{x_i}\dfrac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1,2,\ldots,n$, $u(x,t)$ is an unknown function, $f(x)$ is the given generalized function and c is a constant. To this end, we figure out some interesting properties of the generalized Bessel heat kernel which is closely related to the kernel of an extension of the heat equation.

We obtain $u(x,t) = E(x,t) *_B f(x)$, the symbol $*_B$ being the B-convolution in (2.1), as a solution of (1.6), which satisfies (1.7), where

$$E(x,t) = C_\upsilon \int\limits_{\Omega^+} \mathrm{e}^{(-1)^{mk} c^2 t\left[\left(y_1^2+\cdots+y_p^2\right)^m - \left(y_{p+1}^2+\cdots+y_{p+q}^2\right)^m\right]^k} \prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i}\, dy, \tag{1.9}$$

$\Omega^+ \subset \mathbb{R}_n^+$ is the spectrum of $E(x,t)$ for any fixed $t > 0$, and $J_{\upsilon_i - \frac{1}{2}}(x_i y_i)$ is the normalized Bessel function.

2. Preliminaries

The generalized shift operator T_x^y has the following form [7, 16, 13]:

$$T_x^y \varphi(x) = C_v^* \int\limits_0^\pi \cdots \int\limits_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n}\right) \times \left(\prod_{i=1}^n \sin^{2v_i - 1}\theta_i\right) d\theta_1 \cdots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+$,

$$C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i + 1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$$

and

$$\begin{aligned} \frac{d^2\varphi}{dx_i^2} + \frac{2v_i}{x_i}\frac{d\varphi}{dx_i} &= \frac{d^2\varphi}{dy_i^2} + \frac{2v_i}{y_i}\frac{d\varphi}{dy_i} \\ \varphi(x_i, 0) &= f(x_i) \\ \varphi_{y_i}(x_i, 0) &= 0, \end{aligned}$$

where $x_i, y_i \in \mathbb{R}^+, i = 1, \ldots, n$. We remark that this shift operator is closely connected with the Bessel differential operator and is called the *generalized shift operator* [7].

The convolution operator determined by the T_x^y is as follows

$$(f *_B \varphi)(x) = \int\limits_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.1}$$

The convolution in (2.1) is known as the B-*convolution*. We note the following properties of the B-convolution and of the generalized shift operator:

(a) $T_x^y \cdot 1 = 1$,
(b) $T_x^0 \cdot f(x) = f(x)$,
(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for $x \in \mathbb{R}_n^+$ and

$$\int\limits_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^n y_i^{2v_i}\right) dx < \infty,$$

then

$$\int\limits_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int\limits_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

(e) $(f *_B g)(x) = (g *_B f)(x)$

The Fourier-Bessel transformation and its inverse transformation are defined as follows [13]–[18]:

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) \, y_i^{2v_i} \right) dy, \tag{2.2}$$

$$\left(F_B^{-1} f\right)(x) = (F_B f)(-x), \; C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma \left(v_i + \frac{1}{2} \right) \right)^{-1}, \tag{2.3}$$

where $J_{v_i - \frac{1}{2}} (x_i y_i)$ is the normalized Bessel function, which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation are known (see [6, 5, 16]).

$$F_B \delta(x) = 1$$

$$F_B(f *_B g)(x) = F_B f(x) \cdot F_B g(x). \tag{2.4}$$

2.1. Definition. The *spectrum of the kernel* $E(x,t)$ defined in (1.9), is the bounded support of the Fourier Bessel transform $F_B E(y,t)$ for any fixed $t > 0$.

2.2. Definition. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$. Then

$$\Gamma_+ = \left\{ x \in \mathbb{R}_n^+ : x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 > 0 \right\}$$

denotes the interior of the forward cone, and $\overline{\Gamma}_+$ its closure.

Let Ω^+ be the spectrum of $E(x,t)$ defined by (1.9) and $\Omega^+ \subset \overline{\Gamma}_+$. Let $F_B E(y,t)$ be the Fourier Bessel transform of $E(x,t)$, which is defined by

$$F_B E(y,t) = \begin{cases} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(y_1^2 + \cdots + y_p^2 \right)^m - \left(y_{p+1}^2 + \cdots + y_{p+q}^2 \right)^m \right]^k} & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \notin \Omega^+. \end{cases} \tag{2.5}$$

2.3. Lemma. [Fourier Bessel transform of the operator $\otimes_B^{m,k}$]

$$F_B \otimes_B^{m,k} u(x) = (-1)^{mk} V^k(x) F_B u(x),$$

where $V^k(x) = \left(\left(\sum\limits_{i=1}^{p} x_i^2 \right)^m - \left(\sum\limits_{j=p+1}^{p+q} x_j^2 \right)^m \right)^k.$

Proof. We use mathematical induction. For $k=1$, we have

$$\begin{aligned}
&F_B\left(\otimes_B^{m,1}u\right)(x)\\
&\quad= C_v\int\limits_{\mathbb{R}_n^+}\left(\otimes_B^{m,1}u(y)\right)\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&\quad= C_v\int\limits_{\mathbb{R}_n^+}\left[\left(B_{y_1}+B_{y_2}+\ldots+B_{y_p}\right)^m-\left(B_{y_{p+1}}+\cdots+B_{y_{p+q}}\right)^m\right]u(y)\\
&\qquad\qquad\times\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&\quad= C_v\int\limits_{\mathbb{R}_n^+}\left(B_{y_1}+B_{y_2}+\ldots+B_{y_p}\right)^m u(y)\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&\qquad- C_v\int\limits_{\mathbb{R}_n^+}\left(B_{y_{p+1}}+\cdots+B_{y_{p+q}}\right)^m u(y)\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&\quad= I_1-I_2.
\end{aligned}$$

Here,

$$\begin{aligned}
I_1 &= C_v\int\limits_{\mathbb{R}_n^+}\left(B_{y_1}+B_{y_2}+\cdots+B_{y_p}\right)^m u(y)\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&= C_v\int\limits_{\mathbb{R}_n^+}\left(B_{y_1}+B_{y_2}+\cdots+B_{y_p}\right)\left(B_{y_1}+B_{y_2}+\cdots+B_{y_p}\right)^{m-1}u(y)\\
&\qquad\qquad\times\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&= C_v\int\limits_{\mathbb{R}_n^+}\left(B_{y_1}+B_{y_2}+\cdots+B_{y_p}\right)g(y)\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy,\\
&= -\left(x_1^2+\ldots+x_p^2\right)F_B(g)(x).
\end{aligned}$$

Note that above we have used the following equality [7],

$$\int\limits_0^\infty u(y)B_{y_i}J_{v_i-\frac{1}{2}}\left(x_i,y_i\right)y_i^{2v_i}\,dy_i=-x_i^2\int\limits_0^\infty u(y)J_{v_i-\frac{1}{2}}\left(x_i,y_i\right)y_i^{2v_i}\,dy_i.$$

Applying the same arguments successively for a total of $(m-1)$ times, we have following equality

$$\begin{aligned}
I_1 &= C_v\int\limits_{\mathbb{R}_n^+}\left(B_{y_1}+B_{y_2}+\cdots+B_{y_p}\right)^m u(y)\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}\left(x_iy_i\right)y_i^{2v_i}\right)dy\\
&= (-1)^m\left(x_1^2+\cdots+x_p^2\right)^m F_B(u)(x),
\end{aligned}$$

where

$$g(y)=\left(B_{y_1}+B_{y_2}+\cdots+B_{y_p}\right)^{m-1}u(y).$$

Similarly,

$$I_2 = C_v \int\limits_{\mathbb{R}_n^+} \left(B_{y_{p+1}} + \cdots + B_{y_{p+q}}\right)^m u(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy$$
$$= (-1)^m \left(x_{p+1}^2 + \ldots + x_{p+q}^2\right)^m F_B\left(u\right)(x).$$

Hence,

$$F_B\left(\otimes_B^{m,1} u\right)(x) = (-1)^m \left[\left(x_1^2 + \cdots + x_p^2\right)^m - \left(x_{p+1}^2 + \cdots + x_{p+q}^2\right)^m\right]$$
$$\times \int\limits_{\mathbb{R}_n^+} u(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy$$
$$= (-1)^m V(x) F_B u(x),$$

where $V(x) = \left(x_1^2 + \cdots + x_p^2\right)^m - \left(x_{p+1}^2 + \cdots + x_{p+q}^2\right)^m$. Then, applying the inverse Fourier transform we finally obtain

$$\otimes_B^{m,1} u(x) = (-1)^m F_B^{-1} V(x) F_B u(x).$$

Now assume the statement is true for $(k-1)$, i.e,

$$\otimes_B^{m,k-1} u(x) = (-1)^{m(k-1)} F_B^{-1} V^{k-1}(x) F_B u(x).$$

Then, we must prove that it is also true for $k \in \mathbb{N}$. So, we have

$$\otimes_B^{m,k} u(x) = \otimes_B^{m,1} u(x) \left(\otimes_B^{m,k-1} u(x)\right)$$
$$= (-1)^m F_B^{-1} V(x) F_B (-1)^{m(k-1)} F_B^{-1} V^{k-1}(x) F_B u(x)$$
$$= (-1)^{mk} F_B^{-1} V^k(x) F_B u(x).$$

This completes the proof. □

2.4. Lemma. *For $t, \upsilon > 0$ and $x, y \in \mathbb{R}^+$, we have*

$$\int\limits_0^\infty \mathrm{e}^{-c^2 x^2 t} x^{2\upsilon} dx = \frac{\Gamma(\upsilon)}{2c^{2\upsilon+1} t^{\upsilon + \frac{1}{2}}} \tag{2.6}$$

and

$$\int\limits_0^\infty \mathrm{e}^{-c^2 x^2 t} J_{\upsilon - \frac{1}{2}}(xy) x^{2\upsilon}\, dx = \frac{\Gamma(\upsilon + \frac{1}{2})}{2(c^2 t)^{\upsilon + \frac{1}{2}}} \mathrm{e}^{-\frac{y^2}{4c^2 t}}, \tag{2.7}$$

where c is a constant. □

3. Main results

In this section, we will state our main results and give their proofs.

3.1. Lemma. *Let the operator L be defined by*

$$L = \frac{\partial}{\partial t} - c^2 \otimes_B^{m,k}, \tag{3.1}$$

where the k-times iterated generalized Bessel diamond operator $\otimes_B^{m,k}$ is given by

$$\otimes_B^{m,k} u(x) = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^m - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^m\right]^k,$$
$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\upsilon_i}{x_i} \frac{\partial}{\partial x_i},$$

$p+q=n$ is the dimension of $\mathbb{R}_n^+$, k and m are positive integers, $(x_1,\dots,x_n)\in\mathbb{R}_n^+$, and c is a constant. Then,

$$E(x,t)=C_v\int\limits_{\Omega^+}\mathrm{e}^{(-1)^{mk}c^2t\left[\left(y_1^2+\cdots+y_p^2\right)^m-\left(y_{p+1}^2+\cdots+y_{p+q}^2\right)^m\right]^k}\prod_{i=1}^{n}J_{\upsilon_i-\frac{1}{2}}(x_iy_i)y_i^{2\upsilon_i}\,dy \tag{3.2}$$

is the elementary solution of (3.1) in the spectrum $\Omega^+\subset\mathbb{R}_n^+$ for $t>0$.

Proof. Let $E(x,t)$ be the kernel of the elementary solution of L and δ the Dirac-delta distribution. Thus, we have

$$\frac{\partial}{\partial t}E(x,t)-c^2\otimes_B^{m,k}E(x,t)=\delta(x)\delta(t).$$

Applying the Fourier Bessel transform, which is defined by (2.2), to both sides of the above equation, and using $F_B\delta(x)=1$ in Lemma 2.3, we obtain

$$\frac{\partial}{\partial t}F_BE(x,t)-(-1)^{mk}c^2\Big[\left(x_1^2+\cdots+x_p^2\right)^m-\left(x_{p+1}^2+\cdots+x_{p+q}^2\right)^m\Big]^kF_BE(x,t)=\delta(t).$$

Thus, we get

$$F_BE(x,t)=H(t)\mathrm{e}^{(-1)^{mk}c^2t\left[\left(x_1^2+\cdots+x_p^2\right)^m-\left(x_{p+1}^2+\cdots+x_{p+q}^2\right)^m\right]^k},$$

where H is the Heaviside function, which satisfies $H(t)=1$ for $t\geq 0$. Therefore,

$$F_BE(x,t)=\mathrm{e}^{(-1)^{mk}c^2t\left[\left(x_1^2+\cdots+x_p^2\right)^m-\left(x_{p+1}^2+\cdots+x_{p+q}^2\right)^m\right]^k},$$

which coincides with (2.5). Thus from (2.3), we have

$$E(x,t)=C_v\int\limits_{\mathbb{R}_n^+}\mathrm{e}^{(-1)^{mk}c^2t\left[\left(x_1^2+\ldots+x_p^2\right)^m-\left(x_{p+1}^2+\cdots+x_{p+q}^2\right)^m\right]^k}\prod_{i=1}^{n}J_{\upsilon_i-\frac{1}{2}}(x_iy_i)y_i^{2\upsilon_i}\,dy,$$

where Ω^+ is the spectrum of $E(x,t)$. Thus for $t>0$, we have

$$E(x,t)=C_v\int\limits_{\Omega^+}\mathrm{e}^{(-1)^{mk}c^2t\left[\left(x_1^2+\cdots+x_p^2\right)^m-\left(x_{p+1}^2+\cdots+x_{p+q}^2\right)^m\right]^k}\prod_{i=1}^{n}J_{\upsilon_i-\frac{1}{2}}(x_iy_i)y_i^{2\upsilon_i}\,dy.$$

□

3.2. Theorem. *Let us consider the equation*

$$\frac{\partial}{\partial t}u(x,t)-c^2\otimes_B^{m,k}u(x,t)=0 \tag{3.3}$$

under the initial condition

$$u(x,0)=f(x), \tag{3.4}$$

where the k-times iterated generalized Bessel diamond operator $\otimes_B^{m,k}$ is defined by

$$\otimes_B^{m,k}=\left[\left(B_{x_1}+B_{x_2}+\cdots+B_{x_p}\right)^m-\left(B_{x_{p+1}}+\cdots+B_{x_{p+q}}\right)^m\right]^k,$$
$$B_{x_i}=\frac{\partial^2}{\partial x_i^2}+\frac{2\upsilon_i}{x_i}\frac{\partial}{\partial x_i},$$

$p+q=n$ is the dimension of $\mathbb{R}_n^+$, k and m are positive integers, $u(x,t)$ is an unknown function for $(x,t)=(x_1,\dots,x_n,t)\in\mathbb{R}_n^+\times(0,\infty)$, $f(x)$ is the given generalized function, and c is a constant. Then

$$u(x,t)=E(x,t)*_Bf(x) \tag{3.5}$$

is a solution of (3.3) which satisfies (3.4), where $E(x,t)$ is given by (3.2).

Proof. Taking the Fourier Bessel transform, defined by (2.2), of both sides of (3.3) for $x \in \mathbb{R}_n^+$, and using Lemma 2.3, we obtain

$$\frac{\partial}{\partial t} F_B u(x,t) = (-1)^{mk} c^2 \left[\left(x_1^2 + \cdots + x_p^2\right)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k F_B u(x,t). \tag{3.6}$$

Thus, if we consider the initial condition (2.7) then we have the following equality for (3.6)

$$u(x,t) = f(x) *_B F_B^{-1} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(x_1^2 + \cdots + x_p^2\right)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2 \right]^k}. \tag{3.7}$$

Here, if we use (2.2) and (2.3), then we have

$$\begin{aligned} u(x,t) &= f(x) *_B F_B^{-1} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(x_1^2 + \cdots + x_p^2\right)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k} \\ &= \int_{\mathbb{R}_n^+} F_B^{-1} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(x_1^2 + \cdots + x_p^2\right)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2\upsilon_i} \right) dy \\ &= \int_{\mathbb{R}_n^+} \left[C_\upsilon \int_{\mathbb{R}_n^+} \mathrm{e}^{(-1)^{mk} c^2 t V^k(z)} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(y_i z_i) z_i^{2\upsilon_i} \, dz \right] T_x^y f(x) \left(\prod_{i=1}^n y_i^{2\upsilon_i} \right) dy, \end{aligned} \tag{3.8}$$

where $V(z) = \left(z_1^2 + z_2^2 + \cdots + z_p^2\right)^m - \left(z_{p+1}^2 + z_{p+2}^2 + \cdots + z_{p+q}^2\right)^m$. Set

$$E(x,t) = C_\upsilon \int_{\mathbb{R}_n^+} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(y_1^2 + \cdots + y_n^2\right)^m - (y_{p+1}^2 + \cdots + y_{p+q}^2)^m \right]^k} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \, dy. \tag{3.9}$$

Since the integral in (3.9) is divergent, we choose $\Omega^+ \subset \mathbb{R}_n^+$ to be the spectrum of $E(x,t)$, and by (3.2) we have

$$\begin{aligned} E(x,t) &= C_\upsilon \int_{\mathbb{R}_n^+} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(y_1^2 + \cdots + y_p^2\right)^m - (y_{p+1}^2 + \cdots + y_{p+q}^2)^m \right]^k} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \, dy \\ &= C_\upsilon \int_{\Omega^+} \mathrm{e}^{(-1)^{mk} c^2 t \left[\left(y_1^2 + \cdots + y_p^2\right)^m - (y_{p+1}^2 + \cdots + y_{p+q}^2)^m \right]^k} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \, dy. \end{aligned} \tag{3.10}$$

Thus (3.8) can be written in the convolution form

$$u(x,t) = E(x,t) *_B f(x).$$

Moreover, since $E(x,t)$ exists, we see that

$$\begin{aligned} \lim_{t \to 0} E(x,t) &= C_\upsilon \int_{\Omega^+} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \, dy \\ &= C_\upsilon \int_{\mathbb{R}_n^+} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \, dy \\ &= \delta(x), \end{aligned} \tag{3.11}$$

for $x \in \mathbb{R}_n^+$ (also, see [13]).

Thus, for the solution $u(x,t) = E(x,t) *_B f(x)$ of (3.3) we have

$$\begin{aligned} u(x,0) &= \lim_{t\to 0} u(x,t) \\ &= \lim_{t\to 0} E(x,t) * f(x) \\ &= \delta * f(x) \\ &= f(x), \end{aligned}$$

which satisfies (3.4). This completes the proof. □

3.3. Theorem. *The kernel $E(x,t)$ defined by* (3.10) *has the following properties:*

i. $E(x,t) \in C^\infty(\mathbb{R}_n^+ \times (0,\infty))$, *the space of infinitely many times differentiable functions,*

ii. $(\frac{\partial}{\partial t} - c^2 \otimes_B^{m,k})E(x,t) = 0$ *for all* $x \in \mathbb{R}_n^+$, $t > 0$,

iii. $\lim_{t\to\infty} E(x,t) = \delta(x)$ *for all* $x \in \mathbb{R}_n^+$.

Proof. i. From (3.10), and

$$\begin{aligned} &\frac{\partial^n}{\partial t^n} E(x,t) \\ &\quad = C_\upsilon \int_{\Omega^+} \frac{\partial^n}{\partial t^n} \mathrm{e}^{(-1)^{mk} c^2 t\left[\left(y_1^2+\cdots+y_p^2\right)^m - (y_{p+1}^2+\cdots+y_{p+q}^2)^m\right]^k} \prod_{i=1}^n J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i}\, dy, \end{aligned}$$

we have $E(x,t) \in C^\infty$ for $x \in \mathbb{R}_n^+$, $t > 0$.

ii. We have $u(x,t) = E(x,t)$ since $u(x,t) = E(x,t) *_B f(x)$ holds. Note here that we use the fact $f(x) = \delta(x)$ by the Fourier Bessel transformation. Then, we easily obtain

$$\left(\frac{\partial}{\partial t} - c^2 \otimes_B^{m,k}\right) E(x,t) = 0$$

by direct computation.

iii. This case is obvious by (3.11). □

3.4. Remark. We consider the operator $\otimes_B^{m,k}$ defined in Lemma 2.3, Theorem 3.2 and Theorem 3.3. Here, as $\upsilon \to 0$ and $m = 1$, we obtain results in [10].

3.5. Remark. We consider the operator $\otimes_B^{m,k}$ defined in Lemma 2.3, Theorem 3.2 and Theorem 3.3. Here, for $m = 2$ we obtain results in [11].

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