ON THE GENERALIZED BESSEL HEAT EQUATION RELATED TO THE GENERALIZED BESSEL DIAMOND OPERATOR

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Abstract

In this article, we study the equation
\[ \frac{\partial}{\partial t} u(x, t) = c^2 \otimes_{B}^{m,k} u(x, t) \]
with the initial condition \( u(x, 0) = f(x) \) for \( x \in \mathbb{R}^+_n \). Here the operator \( \otimes_{B}^{m,k} \) is called the Generalized Bessel Diamond Operator, iterated \( k \) times, and is defined by
\[ \otimes_{B}^{m,k} = \left[ \left( B_{x_1} + B_{x_2} + \cdots + B_{x_p} \right)^m - \left( B_{x_{p+1}} + \cdots + B_{x_{p+q}} \right)^m \right]^k, \]
where \( k \) and \( m \) are positive integers, \( p + q = n \), \( B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\nu_i}{x_i} \frac{\partial}{\partial x_i} \), \( 2\nu_i = 2\alpha_i + 1 \), \( \alpha_i > -\frac{1}{2} \), \( x_i > 0 \), \( i = 1, 2, \ldots, n \), \( n \) being the dimension of the space \( \mathbb{R}^+_n \), \( u(x, t) \) is an unknown function of the form \( (x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^+_n \times (0, \infty) \), \( f(x) \) is a given generalized function and \( c \) a constant. We obtain the solution of this equation, which is related to the spectrum and the kernel, the so-called Generalized Bessel Diamond heat kernel. Moreover, the Generalized Bessel Diamond heat kernel is shown to have interesting properties and to be related to the kernel of an extension of the heat equation.

Keywords: Heat kernel, Dirac-delta distribution, Bessel Diamond Operator, Spectrum.

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1. Introduction

The casual fundamental solution \( h(x, t) \) is the particular solution of
\[
\frac{\partial E}{\partial t} - a \Delta E = \delta(x) \delta(t),
\]
which vanishes identically for \( t < 0 \). Thus \( h(x, t) \) satisfies
\[
\frac{\partial h}{\partial t} - a \Delta h = \delta(x) \delta(t), \; h \equiv 0 \text{ for } t < 0.
\]
The causal fundamental solution \( h(x, t) \) has a direct physical interpretation; it is the
temperature distribution in a medium, which is at zero temperature up to the time \( t = 0 \),
when a concentrated source is introduced at \( x = 0 \), this source instantaneously releasing
a unit of heat. Although \( h \) is defined for all \( t \) and \( x \), its calculation presents a problem
only for \( t > 0 \) (\( h = 0 \) for \( t < 0 \)). This immediately suggests a slightly different point of
view; for \( t > 0 \) no sources are present, so that \( h \) satisfies the homogeneous equation and
must reduce, at \( t = 0^+ \), to a certain initial temperature. This initial temperature is the
one to which the medium has been raised just after the introduction of the instantaneous
concentrated source of unit strength. We now show that this initial temperature is
\( \delta(x) \).

It is known that the one-dimensional diffusion equation
\[
\frac{\partial u(x, t)}{\partial t} = D \frac{\partial^2 u(x, t)}{\partial x^2},
\]
where \( u(x, t) \) is the temperature of some object and \( D \) is a constant called the “ther-
mal diffusivity” of the material that makes up the object (we could equally well have
modeled the diffusion of chemical by letting \( u(x, t) \) represent the concentration of some
chemical and \( D \) the constant “diffusivity” of the chemical species inside the material
that makes up the object). The diffusion equation describes such a physical situation as
heat conduction in a one-dimensional solid body, spread of a die in a stationary fluid,
population dispersion, and other similar processes. In \([2]\), Chou and Jiang described the
diffusion onto a small surface patch on a spherical molecule with an attractive potential
all around it. A similar model has been presented by Zhou, who takes into account the
attractive interaction and the influence from the heterogeneous surface reactivity only
in a thin spherical shell around the target molecule \([19]\). In this way, the interaction
required to hold the reactants together long enough for them to find the reactive site can
be estimated. Both of these models indicate that the short range Van der Waals’ force
could provide sufficient interaction to overcome the orientational constraint of the target
molecule. For a recent discussion of these and some other models for heterogeneous sur-
face reactivity see also Chou and Zhou \([4]\). We refer the reader to the papers \([1, 3, 20, 21]\)
for these subjects.

It is known that for the heat equation
\[
(1.1) \quad \frac{\partial u(x, t)}{\partial t} = c^2 \Delta u(x, t),
\]
with the initial condition \( u(x, 0) = f(x) \), where
\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}
\]
denotes the Laplacian operator and \( u(x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty) \). We can
obtain the solution as
\[
u(x, t) = \frac{1}{(4c^2 \pi t)^{n/2}} \int_{\mathbb{R}^n} f(x - y)e^{-\frac{|x-y|^2}{4c^2t}} \, dy,
\]
or in the classical convolution form
\[ u(x, t) = E(x, t) * f(x), \]
where
\[ E(x, t) = \frac{1}{(4\pi^2t)^{\frac{d}{2}}} e^{-\frac{x^2}{4c^2t}} \]
and the symbol \(*\) denotes the classical convolution.

On the other hand, in [11], we have studied the solutions of the Bessel Diamond Heat Equation
\[ \frac{\partial}{\partial t} u(x, t) = c^2 \odot_B^k u(x, t) \]
under the initial condition \( u(x, 0) = f(x) \), where the \( k \)-times iterated Bessel diamond operator \( \odot_B^k \) is defined by
\[ \odot_B^k = \left[ \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_p} \right)^2 - \left( \frac{\partial}{\partial x_{p+1}} + \cdots + \frac{\partial}{\partial x_{p+q}} \right)^2 \right]^k, \]
p + q = n is the dimension of \( \mathbb{R}_n^+ = \{ x : x = (x_1, x_2, \ldots, x_n), x_i > 0, i = 1, 2, \ldots, n \} \), \( k \)
is a positive integer, \( B_{x_i} = \frac{\partial^2}{\partial x_i^2} + 2v_i \frac{\partial}{\partial x_i} \), \( v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0, i = 1, 2, \ldots, n \), \( u(x, t) \) is an unknown function, \( f(x) \) is the given generalized function and \( c \) is a constant. Moreover, such a Bessel diamond heat kernel has interesting properties and is also related to the kernel of an extension of the heat equation.

We obtain \( u(x, t) = E(x, t) *_B f(x) \), the symbol \(*_B \) being the \( B \)-convolution in (2.1), as a solution of (1.6), which satisfies (1.7), where
\[ E(x, t) = C_0 \int_{\Omega^+} e^{-\frac{1}{4c^2}(y_1^2 + \cdots + y_n^2) - (y_{p+1}^2 + \cdots + y_{p+q}^2)} \prod_{i=1}^n J_{\nu_i - \frac{1}{2}}(x_i y_i) y_i^{2\nu_i} \, dy, \]
\( \Omega^+ \subset \mathbb{R}_n^+ \) is the spectrum of \( E(x, t) \) for any fixed \( t > 0 \), and \( J_{\nu_i - \frac{1}{2}}(x_i y_i) \) is the normalized Bessel function [11].

The purpose of this work is to study the solutions of following equation:
\[ \frac{\partial}{\partial t} u(x, t) = c^2 \odot_B^{m,k} u(x, t) \]
under the initial condition
\[ u(x, 0) = f(x), \text{ for } x \in \mathbb{R}_n^+, \]
where the \( k \)-times iterated generalized Bessel diamond operator \( \odot_B^{m,k} \) is defined by
\[ \odot_B^{m,k} = \left[ \left( B_{x_1} + B_{x_2} + \cdots + B_{x_p} \right)^m - \left( B_{x_{p+1}} + \cdots + B_{x_{p+q}} \right)^m \right]^k, \]
p + q = n is the dimension of \( \mathbb{R}_n^+ = \{ x : x = (x_1, x_2, \ldots, x_n), x_i > 0, i = 1, 2, \ldots, n \} \), \( k \) and \( m \) are positive integers, \( B_{x_i} = \frac{\partial^2}{\partial x_i^2} + 2v_i \frac{\partial}{\partial x_i} \), \( v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0, i = 1, 2, \ldots, n \), \( u(x, t) \) is an unknown function, \( f(x) \) is the given generalized function and \( c \) is a constant. To this end, we figure out some interesting properties of the generalized Bessel heat kernel which is closely related to the kernel of an extension of the heat equation.
We obtain $u(x, t) = E(x, t) *_{B} f(x)$, the symbol $*_{B}$ being the $B$-convolution in (2.1), as a solution of (1.6), which satisfies (1.7), where

\[(1.9) \quad E(x, t) = C_{v} \int_{\Omega^{+}} e^{(-1)^{m_{k}\pi^{2}t}}[(y_{1}^{2} + \cdots + y_{p}^{2})^{m} - (y_{p+1}^{2} + \cdots + y_{p+q}^{2})^{m}] \prod_{i=1}^{n} J_{v_{i} - \frac{1}{2}}(x_{i}y_{i}) y_{i}^{2v_{i}} dy.
\]

$\Omega^{+} \subset \mathbb{R}_{+}^{n}$ is the spectrum of $E(x, t)$ for any fixed $t > 0$, and $J_{v_{i} - \frac{1}{2}}(x_{i}y_{i})$ is the normalized Bessel function.

2. Preliminaries

The generalized shift operator $T_{y}^{x}$ has the following form [7, 16, 13]:

\[T_{y}^{x} \varphi(x) = C_{v}^{*} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \varphi\left(\sqrt{x_{1}^{2} + y_{1}^{2} - 2x_{1}y_{1} \cos \theta_{1}}, \ldots, \sqrt{x_{n}^{2} + y_{n}^{2} - 2x_{n}y_{n} \cos \theta_{n}}\right)\]
\[\times \left(\prod_{i=1}^{n} \sin^{2v_{i} - 1} \theta_{i}\right) d\theta_{1} \cdots d\theta_{n},\]

where $x, y \in \mathbb{R}_{+}^{n}$,

\[C_{v}^{*} = \prod_{i=1}^{n} \frac{\Gamma(v_{i} + 1)}{\Gamma(\frac{1}{2}) \Gamma(v_{i})}\]

and

\[\frac{d^{2} \varphi}{dx_{i}^{2}} + \frac{2v_{i}}{x_{i}} \frac{d \varphi}{dx_{i}} = \frac{d^{2} \varphi}{dy_{i}^{2}} + \frac{2v_{i}}{y_{i}} \frac{d \varphi}{dy_{i}},\]
\[\varphi(x_{i}, 0) = f(x_{i}),\]
\[\varphi_{y_{i}}(x_{i}, 0) = 0,\]

where $x_{i}, y_{i} \in \mathbb{R}^{+}, i = 1, \ldots, n$. We remark that this shift operator is closely connected with the Bessel differential operator and is called the generalized shift operator [7].

The convolution operator determined by the $T_{y}^{x}$ is as follows

\[(2.1) \quad (f *_{B} \varphi)(x) = \int_{\mathbb{R}_{+}^{n}} f(y) T_{y}^{x} \varphi(x) \left(\prod_{i=1}^{n} y_{i}^{2v_{i}}\right) dy.
\]

The convolution in (2.1) is known as the $B$-convolution. We note the following properties of the $B$-convolution and of the generalized shift operator:

(a) $T_{y}^{x} \cdot 1 = 1$,
(b) $T_{y}^{x} : f(x) = f(x)$,
(c) If $f(x), g(x) \in C(\mathbb{R}_{+}^{n}), g(x)$ is a bounded function for $x \in \mathbb{R}_{+}^{n}$ and

\[\int_{\mathbb{R}_{+}^{n}} |f(x)| \left(\prod_{i=1}^{n} y_{i}^{2v_{i}}\right) dx < \infty,
\]

then

\[\int_{\mathbb{R}_{+}^{n}} T_{y}^{x} f(x) g(y) \left(\prod_{i=1}^{n} y_{i}^{2v_{i}}\right) dy = \int_{\mathbb{R}_{+}^{n}} f(y) T_{y}^{x} g(x) \left(\prod_{i=1}^{n} y_{i}^{2v_{i}}\right) dy.
\]
(d) From (c), we have the following equality for \( g(x) = 1 \),

\[
\int_{\mathbb{R}^n_+} T^y_x f(x) \left( \prod_{i=1}^n y_i^{2v_i} \right) \, dy = \int_{\mathbb{R}^n_+} f(y) \left( \prod_{i=1}^n y_i^{2v_i} \right) \, dy.
\]

(e) \((f \ast_B g)(x) = (g \ast_B f)(x)\)

The Fourier-Bessel transformation and its inverse transformation are defined as follows [13]-[18]:

\[
(F_B f)(x) = C_v \int_{\mathbb{R}^n_+} f(y) \left( \prod_{i=1}^n J_{v_i} \left( \frac{x_i y_i}{2} \right) \right) \, dy,
\]

\[
(F_B^{-1} f)(x) = (F_B f)(-x), \quad C_v = \left( \prod_{i=1}^n 2^{v_i} \frac{\Gamma \left( v_i + \frac{1}{2} \right)}{\Gamma \left( v_i \right)} \right)^{-1},
\]

where \( J_{v_i} \left( \frac{x_i y_i}{2} \right) \) is the normalized Bessel function, which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation are known (see [6, 5, 16]).

\[
F_B \delta(x) = 1
\]

\[
F_B(f \ast_B g)(x) = F_B f(x) \cdot F_B g(x).
\]

2.1. Definition. The spectrum of the kernel \( E(x, t) \) defined in (1.9), is the bounded support of the Fourier Bessel transform \( F_B E(y, t) \) for any fixed \( t > 0 \).

2.2. Definition. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \). Then

\[
\Gamma_+ = \{ x \in \mathbb{R}^n_+ : x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 > 0 \}
\]

denotes the interior of the forward cone, and \( \overline{\Gamma}_+ \) its closure.

Let \( \Omega^+ \) be the spectrum of \( E(x, t) \) defined by (1.9) and \( \Omega^+ \subset \overline{\Gamma}_+ \). Let \( F_B E(y, t) \) be the Fourier Bessel transform of \( E(x, t) \), which is defined by

\[
F_B E(y, t) = \begin{cases} e^{(-1)^m \sum_{i=1}^n \frac{x_i^2}{2} - \left( \sum_{i=p+1}^{p+q} x_j^2 \right)^m} & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \notin \Omega^+. \end{cases}
\]

2.3. Lemma. [Fourier Bessel transform of the operator \( \otimes_B^{m,k} \)]

\[
F_B \otimes_B^{m,k} u(x) = (-1)^m V^k(x) F_B u(x),
\]

where \( V^k(x) = \left( \sum_{i=1}^p x_i^2 \right)^m - \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^m \).
Proof. We use mathematical induction. For \( k = 1 \), we have
\[
F_{B} \left( \otimes_B^{m,1} u \right) (x)
\]
\[
= C_v \int_{\mathbb{R}^+} \left( \otimes_B^{m,1} u(y) \right) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
= C_v \int_{\mathbb{R}^+} \left[ \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right)^m \left( B_{y_{p+1}} + \cdots + B_{y_{p+q}} \right)^m \right] u(y)
\]
\[
\times \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
= C_v \int_{\mathbb{R}^+} \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right)^m u(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
- C_v \int_{\mathbb{R}^+} \left( B_{y_{p+1}} + \cdots + B_{y_{p+q}} \right)^m u(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
= I_1 - I_2.
\]
Here,
\[
I_1 = C_v \int_{\mathbb{R}^+} \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right)^m u(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
= C_v \int_{\mathbb{R}^+} \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right) \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right)^{m-1} u(y)
\]
\[
\times \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
= C_v \int_{\mathbb{R}^+} \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right) g(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy,
\]
Note that above we have used the following equality [7],
\[
\int_{0}^{\infty} u(y) B_{y_i} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} dy_i = -x_i^2 \int_{0}^{\infty} u(y) J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} dy_i.
\]
Applying the same arguments successively for a total of \((m - 1)\) times, we have following equality
\[
I_1 = C_v \int_{\mathbb{R}^+} \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right)^m u(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right)^{2v_i} \right) dy
\]
\[
= (-1)^m \left( x_1^2 + \cdots + x_p^2 \right)^m F_{B} \left( u \right) (x),
\]
where
\[
g(y) = \left( B_{y_1} + B_{y_2} + \cdots + B_{y_p} \right)^{m-1} u(y).
\]
Similarly,
\[ I_2 = C_v \int_{\mathbb{R}_+^n} (B_{y_{p+1}} + \cdots + B_{y_{p+q}})^m u(y) \left( \prod_{i=1}^n J_{\nu_i - \frac{1}{2}} (x_i y_i) \right) dy \]
\[ = (-1)^m (x_{p+1}^2 + \cdots + x_{p+q}^2)^m F_B(u)(x). \]

Hence,
\[ F_B (\otimes_B^{m,1} u)(x) = (-1)^m \left( (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right) \]
\[ \times \int_{\mathbb{R}_+^n} u(y) \left( \prod_{i=1}^n J_{\nu_i - \frac{1}{2}} (x_i y_i) \right) dy \]
\[ = (-1)^m V(x) F_B u(x), \]
where \( V(x) = (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m. \) Then, applying the inverse Fourier transform we finally obtain
\[ \otimes_B^{m,1} u(x) = (-1)^m F_B^{-1} V(x) F_B u(x). \]

Now assume the statement is true for \((k-1),\) i.e,
\[ \otimes_B^{m,k-1} u(x) = (-1)^m F_B^{-1} V^{k-1}(x) F_B u(x). \]

Then, we must prove that it is also true for \(k \in \mathbb{N}.\) So, we have
\[ \otimes_B^{m,k} u(x) = \otimes_B^{m,1} u(x) \left( \otimes_B^{m,k-1} u(x) \right) \]
\[ = (-1)^m F_B^{-1} V(x) F_B (-1)^m F_B^{-1} V^{k-1}(x) F_B u(x) \]
\[ = (-1)^m F_B^{-1} V^k(x) F_B u(x). \]

This completes the proof. \(\square\)

2.4. Lemma. For \(t, \nu > 0\) and \(x, y \in \mathbb{R}^+,\) we have
\[ \int_0^\infty e^{-c x^2} x^{2\nu} dx = \frac{\Gamma(\nu)}{2 c^{\nu+1} t^{\nu+\frac{1}{2}}} \]
and
\[ \int_0^\infty e^{-c x^2} J_{\nu - \frac{1}{2}} (x y) x^{2\nu} dx = \frac{\Gamma(\nu + \frac{1}{2})}{2 (c y^2)^{\nu+\frac{1}{2}}} e^{-\frac{y^2}{2 c}}, \]
where \(c\) is a constant. \(\square\)

3. Main results

In this section, we will state our main results and give their proofs.

3.1. Lemma. Let the operator \(L\) be defined by
\[ L = \frac{\partial}{\partial t} - c^2 \otimes_B^{m,k}, \]
where the \(k\)-times iterated generalized Bessel diamond operator \(\otimes_B^{m,k}\) is given by
\[ \otimes_B^{m,k} u(x) = \left( (B_{x_1} + B_{x_2} + \cdots + B_{x_p})^m - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})^m \right) \]
\[ B_{x_i} = \frac{\partial^2}{\partial x_i^2} + 2 \nu_i \frac{\partial}{\partial x_i}. \]
$p + q = n$ is the dimension of $\mathbb{R}_n^+$, $k$ and $m$ are positive integers, $(x_1, \ldots, x_n) \in \mathbb{R}_n^+$, and $c$ is a constant. Then,

\begin{equation}
E(x, t) = C_x \int_{\Omega^+} e^{(-1)^m k c^2 t} [(x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m]^{k} \prod_{i=1}^{n} J_{v_i} \cdot \frac{1}{2} (x_i y_i) y_i^{2v_i} \ dy
\end{equation}

is the elementary solution of (3.1) in the spectrum $\Omega^+ \subset \mathbb{R}_n^+$ for $t > 0$.

**Proof.** Let $E(x, t)$ be the kernel of the elementary solution of $L$ and $\delta$ the Dirac-delta distribution. Thus, we have

$$\frac{\partial}{\partial t} E(x, t) - c^2 \mathbb{D}_{B}^{m, k} E(x, t) = \delta(x) \delta(t).$$

Applying the Fourier Bessel transform, which is defined by (2.2), to both sides of the above equation, and using $F_B \delta(x) = 1$ in Lemma 2.3, we obtain

$$\frac{\partial}{\partial t} F_B E(x, t) - (-1)^m c^2 [\left( x_1^2 + \cdots + x_p^2 \right)^m - \left( x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^m]^{k} F_B E(x, t) = \delta(t).$$

Thus, we get

$$F_B E(x, t) = H(t) e^{(-1)^m k c^2 t} \left[ \left( x_1^2 + \cdots + x_p^2 \right)^m - \left( x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^m \right]^{k},$$

where $H$ is the Heaviside function, which satisfies $H(t) = 1$ for $t \geq 0$. Therefore,

$$F_B E(x, t) = e^{(-1)^m k c^2 t} \left[ \left( x_1^2 + \cdots + x_p^2 \right)^m - \left( x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^m \right]^{k},$$

which coincides with (2.5). Thus from (2.3), we have

$$E(x, t) = C_x \int_{\mathbb{R}_n^+} e^{(-1)^m k c^2 t} [\left( x_1^2 + \cdots + x_p^2 \right)^m - \left( x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^m]^{k} \prod_{i=1}^{n} J_{v_i} \cdot \frac{1}{2} (x_i y_i) y_i^{2v_i} \ dy,$$

where $\Omega^+$ is the spectrum of $E(x, t)$. Thus for $t > 0$, we have

$$E(x, t) = C_x \int_{\Omega^+} e^{(-1)^m k c^2 t} [\left( x_1^2 + \cdots + x_p^2 \right)^m - \left( x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^m]^{k} \prod_{i=1}^{n} J_{v_i} \cdot \frac{1}{2} (x_i y_i) y_i^{2v_i} \ dy.$$

\[\square\]

**3.2. Theorem.** Let us consider the equation

\begin{equation}
\frac{\partial}{\partial t} u(x, t) - c^2 \mathbb{D}_{B}^{m, k} u(x, t) = 0
\end{equation}

under the initial condition

\begin{equation}
u(x, 0) = f(x),
\end{equation}

where the $k$-times iterated generalized Bessel diamond operator $\mathbb{D}_{B}^{m, k}$ is defined by

$$\mathbb{D}_{B}^{m, k} = \left[ (B_{x_1} + B_{x_2} + \cdots + B_{x_p})^m - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})^m \right]^{k},$$

$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\nu_i}{x_i} \frac{\partial}{\partial x_i},$

$p + q = n$ is the dimension of $\mathbb{R}_n^+$, $k$ and $m$ are positive integers, $u(x, t)$ is an unknown function for $(x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0, \infty)$, $f(x)$ is the given generalized function, and $c$ is a constant. Then

\begin{equation}
u(x, t) = E(x, t) * B f(x)
\end{equation}

is a solution of (3.3) which satisfies (3.4), where $E(x, t)$ is given by (3.2).
Proof. Taking the Fourier Bessel transform, defined by (2.2), of both sides of (3.3) for \( x \in \mathbb{R}^n_+ \), and using Lemma 2.3, we obtain

\[
\frac{\partial}{\partial t} F_B u(x,t) = (-1)^m k^2 \left[ (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k F_B u(x,t). \tag{3.6}
\]

Thus, if we consider the initial condition (2.7) then we have the following equality for (3.6)

\[
u(x,t) = f(x) * B \left[ (-1)^m k^2 \left[ (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k \right]. \tag{3.7}
\]

Here, if we use (2.2) and (2.3), then we have

\[
u(x,t) = f(x) * B \left[ (-1)^m k^2 \left[ (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k \right]
= \int_{\mathbb{R}^n_+} F_B^{-1} e^{(-1)^m k^2 \left[ (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m \right]^k} T_x^y f(x) \left( \prod_{i=1}^n y_i^{2u_i} \right) dy \tag{3.8}
\]

where \( V(z) = (z_1^2 + \cdots + z_p^2)^m - (z_{p+1}^2 + \cdots + z_{p+q}^2)^m \). Set

\[ E(x,t) = C_v \int_{\mathbb{R}^n_+} e^{(-1)^m k^2 \left[ (x_1^2 + y_1^2)^m - (y_{p+1}^2 + \cdots + y_{p+q}^2)^m \right]^k} \prod_{i=1}^n J_{u_i-\frac{1}{2}} (x_i y_i) y_i^{2u_i} dy. \tag{3.9} \]

Since the integral in (3.9) is divergent, we choose \( \Omega^+ \subset \mathbb{R}^n_+ \) to be the spectrum of \( E(x,t) \), and by (3.2) we have

\[ E(x,t) = C_v \int_{\Omega^+} e^{(-1)^m k^2 \left[ (x_1^2 + y_1^2)^m - (y_{p+1}^2 + \cdots + y_{p+q}^2)^m \right]^k} \prod_{i=1}^n J_{u_i-\frac{1}{2}} (x_i y_i) y_i^{2u_i} dy \tag{3.10} \]

Thus (3.8) can be written in the convolution form

\[
u(x,t) = E(x,t) * B f(x).
\]

Moreover, since \( E(x,t) \) exists, we see that

\[
\lim_{t \to 0} E(x,t) = C_v \int_{\Omega^+} \prod_{i=1}^n J_{u_i-\frac{1}{2}} (x_i y_i) y_i^{2u_i} dy \tag{3.11}
= C_v \int_{\mathbb{R}^n_+} \prod_{i=1}^n J_{u_i-\frac{1}{2}} (x_i y_i) y_i^{2u_i} dy
= \delta(x),
\]

for \( x \in \mathbb{R}^n_+ \) (also, see [13]).
Thus, for the solution $u(x, t) = E(x, t) * f(x)$ of (3.3) we have

$$
\begin{align*}
u(x, 0) &= \lim_{t \to 0} u(x, t) \\
&= \lim_{t \to 0} E(x, t) * f(x) \\
&= \delta * f(x) \\
&= f(x),
\end{align*}
$$

which satisfies (3.4). This completes the proof. \(\square\)

### 3.3. Theorem

The kernel $E(x, t)$ defined by (3.10) has the following properties:

i. $E(x, t) \in C^\infty(\mathbb{R}^+_n \times (0, \infty))$, the space of infinitely many times differentiable functions,

ii. $(\partial^m / \partial t^m - c^2 \otimes^{m,k}_B)E(x, t) = 0$ for all $x \in \mathbb{R}^+_n$, $t > 0$,

iii. $\lim_{t \to \infty} E(x, t) = \delta(x)$ for all $x \in \mathbb{R}^+_n$.

**Proof.**

i. From (3.10), and

$$\frac{\partial^m}{\partial t^m} E(x, t) = C_\nu \int_{\Omega^+} \frac{\partial^m}{\partial t^m} (-1)^m c^2 t (y_1^2 + \cdots + y_p^2)^m - (y_{p+1}^2 + \cdots + y_{2p+q}^2)^m \Pi_{i=1}^n J_{\nu_i} - \frac{1}{2} (x_i y_i) y_i^{2\nu_i} \, dy,$$

we have $E(x, t) \in C^\infty$ for $x \in \mathbb{R}^+_n$, $t > 0$.

ii. We have $u(x, t) = E(x, t)$ since $u(x, t) = E(x, t) * f(x)$ holds. Note here that we use the fact $f(x) = \delta(x)$ by the Fourier Bessel transformation. Then, we easily obtain

$$\left(\frac{\partial}{\partial t} - c^2 \otimes^{m,k}_B\right) E(x, t) = 0$$

by direct computation.

iii. This case is obvious by (3.11). \(\square\)

### 3.4. Remark

We consider the operator $\otimes^{m,k}_B$ defined in Lemma 2.3, Theorem 3.2 and Theorem 3.3. Here, as $\nu \to 0$ and $m = 1$, we obtain results in [10].

### 3.5. Remark

We consider the operator $\otimes^{m,k}_B$ defined in Lemma 2.3, Theorem 3.2 and Theorem 3.3. Here, for $m = 2$ we obtain results in [11].

### References


