On partially $\tau$-quasinormal subgroups of finite groups

Changwen Li*, Xuemei Zhang† and Xiaolan Yi‡

Abstract
Let $H$ be a subgroup of a group $G$. We say that: (1) $H$ is $\tau$-quasinormal in $G$ if $H$ permutes with every Sylow subgroup $Q$ of $G$ such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$; (2) $H$ is partially $\tau$-quasinormal in $G$ if $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of $H$ which are $\tau$-quasinormal in $G$. In this paper, we find a condition under which every chief factor of $G$ below a normal subgroup $E$ of $G$ is cyclic by using the partial $\tau$-quasinormality of some subgroups.


Keywords: $S$-quasinormal, partially $\tau$-quasinormal, cyclic.

Received 05 : 06 : 2013 : Accepted 03 : 10 : 2013 Doi : 10.15672/HJMS.2104437528

1. Introduction
All groups considered in the paper are finite. The notations and terminology in this paper are standard, as in [4] and [6]. $G$ always denotes a finite group, $\pi(G)$ denotes the set of all prime dividing $|G|$ and $F^\infty(G)$ is the generalized Fitting subgroup of $G$, i.e., the product of all normal quasinilpotent subgroups of $G$.

Normal subgroup plays an important role in the study of the structure of groups. Many authors are interested to extend the concept of normal subgroup. For example, a subgroup $H$ of $G$ is said to be $S$-quasinormal [7] in $G$ if $H$ permutes with every Sylow subgroup of $G$. As a generalization of $S$-quasinormality, a subgroup $H$ of $G$ is said to be $\tau$-quasinormal [11] in $G$ if $H$ permutes with every Sylow subgroup $Q$ of $G$ such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$. On the other hand, Wang [17] extended normality as

*School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China
Email: lcw2000@126.com

†Department of Basic Sciences, Yancheng Institute of Technology, Yancheng, 224051, China

‡School of Science, Zhejiang Sci-Tech University, Hangzhou, 310018, China
The project is supported by the Natural Science Foundation of China (No:11401264) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.
follows: a subgroup $H$ of $G$ is said to be $c$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $HK = G$ and $H \cap K \leq H_G$, where $H_G$ is the maximal normal subgroup of $G$ contained in $H$. In the literature, many people have studied the influence of the $\tau$-quasinormality and $c$-normality on the structure of finite groups and obtained many interesting results (see [2, 5, 8, 11, 12, 17, 19]). As a development, we now introduce a new concept:

1.1. Definition. A subgroup $H$ of a group $G$ is said to be partially $\tau$-quasinormal in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is $\delta$-quasinormal in $G$ and $H \cap T \leq H_G$, where $H_G$ is the subgroup generated by all those subgroups of $H$ which are $\tau$-quasinormal in $G$.

Clearly, partially $\tau$-quasinormal subgroup covers both the concepts of $\tau$-quasinormal subgroup and $c$-normal subgroup. However, the following examples show that the converse is not true.

1.2. Example. Let $G = S_4$ be the symmetric group of degree 4.

1) Let $H$ be a Sylow 3-subgroup of $G$ and $N$ the normal abelian 2-subgroup of $G$ of order 4. Then $HN = A_4 \trianglelefteq G$ and $H \cap N = 1$. Hence $H$ is a partially $\tau$-quasinormal subgroup of $G$. But, obviously, $H$ is not $c$-normal in $G$.

2) Let $H = \langle (14) \rangle$. Obviously, $H A_4 = G$ and $H \cap A_4 = 1$. Hence $H$ is partially $\tau$-quasinormal in $G$. But, obviously, $H$ is not $\tau$-quasinormal in $G$.

A normal subgroup $E$ of a group $G$ is said to be hypercyclically embedded in $G$ if every chief factor of $G$ below $E$ is cyclic. The product of all normal hypercyclically embedded subgroups of $G$ is denoted by $Z_\psi(G)$. In [15] and [16], Skiba gave some characterizations of normal hypercyclically embedded subgroups related to $S$-quasinormal subgroups. The main purpose of this paper is to give a new characterization by using partially $\tau$-quasinormal property of maximal subgroups of some Sylow subgroups. We obtain the following result.

Main Theorem. Let $E$ be a normal subgroup of $G$. Suppose that there exists a normal subgroup $X$ of $G$ such that $F^*(E) \leq X \leq E$ and $X$ satisfies the following properties: for every non-cyclic Sylow $p$-subgroup $P$ of $X$, every maximal subgroup of $P$ not having a supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then $E$ is hypercyclically embedded in $G$.

The following theorems are the main stages in the proof of Main Theorem.

1.3. Theorem. Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. If every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is partially $\tau$-quasinormal in $G$, then $G$ is soluble.

1.4. Theorem. Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Then $G$ is $p$-nilpotent if and only if every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is partially $\tau$-quasinormal in $G$.

1.5. Theorem. Let $E$ be a normal subgroup in $G$ and let $P$ be a Sylow $p$-subgroup of $E$, where $p$ is a prime divisor of $|E|$ with $(|E|, p - 1) = 1$. Suppose that every maximal subgroup of $P$ not having a $p$-supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then each chief factor of $G$ between $E$ and $O_p(E)$ is cyclic.

1.6. Theorem. Let $E$ be a normal subgroup of a group $G$. Suppose that for each $p \in \pi(E)$, every maximal subgroup of non-cyclic Sylow $p$-subgroup $P$ of $E$ not having a $p$-supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then every chief factor of $G$ below $E$ is cyclic.
2. Preliminaries

2.1. Lemma ([3] and [7]). Suppose that $H$ is a subgroup of $G$ and $H$ is $S$-quasinormal in $G$. Then

1. If $H \leq K \leq G$, then $H$ is $S$-quasinormal in $K$.
2. If $N$ is a normal subgroup of $G$, then $HN$ is $S$-quasinormal in $G$ and $HN/N$ is $S$-quasinormal in $G/N$.
3. If $K \leq G$, then $H \cap K$ is $S$-quasinormal in $K$.
4. $H$ is subnormal in $G$.
5. If $K \leq G$ and $K$ is $S$-quasinormal in $G$, then $H \cap K$ is $S$-quasinormal in $G$.

2.2. Lemma ([11, Lemmas 2.2 and 2.3]). Let $G$ be a group and $H \leq K \leq G$.

1. If $H$ is $\tau$-quasinormal in $G$, then $H$ is $\tau$-quasinormal in $K$.
2. Suppose that $H$ is normal in $G$ and $\pi(K/H) = \pi(K)$. If $K$ is $\tau$-quasinormal in $G$, then $K/H$ is $\tau$-quasinormal in $G/H$.
3. Suppose that $H$ is normal in $G$. Then $EH/H$ is $\tau$-quasinormal in $G/H$ for every $\tau$-quasinormal subgroup $E$ in $G$ satisfying $(|H|, |E|) = 1$.
4. If $H$ is $\tau$-quasinormal in $G$ and $H \leq O_p(G)$ for some prime $p$, then $H$ is $S$-quasinormal in $G$.
5. $H_G \leq H_K$.
6. Suppose that $K$ is a $p$-group and $H$ is normal in $G$. Then $K \trianglerighteq H$.
7. Suppose that $H$ is normal in $G$. Then $E \cap H/H \leq (EH/H)_{\pi(G/H)}$ for every $\tau$-subgroup $E$ of $G$ satisfying $(|H|, |E|) = 1$.

2.3. Lemma. Let $G$ be a group and $H \leq G$. Then

1. If $H$ is partially $\tau$-quasinormal in $G$ and $H \leq K \leq G$, then $H$ is partially $\tau$-quasinormal in $K$.
2. Suppose that $N \trianglelefteq G$ and $N \leq H$. If $H$ is a $p$-group and $H$ is partially $\tau$-quasinormal in $G$, then $H/N$ is partially $\tau$-quasinormal in $G/N$.
3. Suppose that $H$ is a $p$-subgroup of $G$ and $N$ is a normal $p'$-subgroup of $G$. If $H$ is partially $\tau$-quasinormal in $G$, then $HN/N$ is partially $\tau$-quasinormal in $G/N$.
4. If $H$ is partially $\tau$-quasinormal in $G$ and $H \leq K \leq G$, then there exists $T \leq G$ such that $HT$ is $S$-quasinormal in $G$, $H \cap T \leq H_{rG}$ and $HT \leq K$.

Proof. (1) Let $N$ be a normal subgroup of $G$ such that $HN$ is $S$-quasinormal in $G$ and $H \cap N \leq H_{rG}$. Then $K \cap N \leq K$, $K(N \cap K) = HN \cap K$ is $S$-quasinormal in $K$ by Lemma 2.1(3) and $K \cap (K \cap N) = H \cap N \leq H_{rG} \leq H_{rK}$ by Lemma 2.2(5). Hence $H$ is partially $\tau$-quasinormal in $K$.

(2) Suppose that $H$ is partially $\tau$-quasinormal in $G$. Then there exists $K \leq G$ such that $HK$ is $S$-quasinormal in $G$ and $H \cap K \leq H_{rG}$. This implies that $KH/N \leq K\cap N \leq H_{rG}$.

(3) Suppose that $H$ is partially $\tau$-quasinormal in $G$. Then there exists $K \leq G$ such that $HK$ is $S$-quasinormal in $G$ and $H \cap K \leq H_{rG}$. Clearly, $KN/N \leq G$ and $(HL/N)(HN/N) = HKN/N$ is $S$-quasinormal in $G/N$ by Lemma 2.1(2). On the other hand, since $(|HN : H|, |HN : N|) = 1$, $HN/N \cap KN/N = (HN \cap K)N/N = (H \cap K)(N \cap K)/N \leq H_{rG}/N$. In view of Lemma 2.2(7), we have $H_{rG}/N \leq (HN/N)_{\pi(G/N)}$. Hence $HN/N$ is partially $\tau$-quasinormal in $G/N$.

(4) Suppose that $H$ is partially $\tau$-quasinormal in $G$. Then there exists $N \trianglelefteq G$ such that $HN$ is $S$-quasinormal in $G$ and $H \cap N \leq H_{rG}$. Let $T = N \cap K$. Then $T \leq G$.

$HT = H(N \cap K) = HN \cap K$ is $S$-quasinormal in $G$ by Lemma 2.1(5), $HT \leq K$ and $H \cap T = H \cap N \cap K \leq H \cap N \leq H_{rG}$. \qed
2.4. Lemma. Let $G$ be a group and $p$ a prime dividing $|G|$ with $(|G|, p - 1) = 1$.

1. If $N$ is normal in $G$ of order $p$, then $N$ lies in $Z(G)$.
2. If $G$ has cyclic Sylow $p$-subgroups, then $G$ is $p$-nilpotent.
3. If $M \leq G$ and $|G : M| = p$, then $M \leq G$.
4. If $G$ is $p$-supersoluble, then $G$ is $p$-nilpotent.

Proof. (1), (2) and (3) can be found in [18, Theorem 2.8]. Now we only prove (4). Let $A/B$ be an arbitrary chief factor of $G$. If $G$ is $p$-supersolvable, then $A/B$ is either a cyclic group with order $p$ or a $p'$-group. If $|A/B| = p$, then $|\text{Aut}(A/B)| = p - 1$. Since $G/C_G(A/B)$ is isomorphic to a subgroup of $\text{Aut}(A/B)$, the order of $G/C_G(A/B)$ must divide $(|G|, p - 1) = 1$, which shows that $G = C_G(A/B)$. Therefore, we have $G$ is $p$-nilpotent. □

2.5. Lemma ([10, Lemma 2.12]). Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. If every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$, then $G$ is $p$-nilpotent.

2.6. Lemma ([13, Theorem A]). If $P$ is an $S$-quasinormal $p$-subgroup of a group $G$ for some prime $p$, then $N_G(P) \geq O^p(G)$.

2.7. Lemma ([6, VI, 4.10]). Assume that $A$ and $B$ are two subgroups of a group $G$ and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either $A$ or $B$ is contained in a nontrivial normal subgroup of $G$.

2.8. Lemma ([20, Chap.1, Theorem 7.19]). Let $H$ be a normal subgroup of $G$. Then $H \leq Z_w(G)$ if and only if $H/\Phi(H) \leq Z_w(G/\Phi(H))$.

2.9. Lemma ([14, Lemma 2.11]). Let $N$ be an elementary abelian normal subgroup of a group $G$. Assume that $N$ has a subgroup $D$ such that $1 < |D| < |N|$ and every subgroup $H$ of $N$ satisfying $|H| = |D|$ is $S$-quasinormal in $G$. Then some maximal subgroup of $N$ is normal in $G$.

2.10. Lemma. Let $N$ be a non-identity normal $p$-subgroup of a group $G$. If $N$ is elementary and every maximal subgroup of $N$ is partially $\tau$-quasinormal in $G$, then some maximal subgroup of $N$ is normal in $G$.

Proof. If $|N| = p$, then it is clear. Let $L$ be a non-identity minimal normal $p$-subgroup of $G$ contained in $N$. First we assume that $N \neq L$. By Lemma 2.3(2), the hypothesis still holds on $G/L$. Then by induction some maximal subgroup $M/L$ of $N/L$ is normal in $G/L$. Clearly, $M$ is a maximal subgroup of $N$ and $M$ is normal in $G$. Consequently the lemma follows. Now suppose that $L = N$. Let $M$ be any maximal subgroup of $N$. Then by the hypothesis, there exists $T \leq G$ such that $MT$ is $S$-quasinormal in $G$ and $M \cap T \leq M_{rG}$. Suppose that $M \neq M_{rG}$. Then $MT \neq M$ and $T \neq 1$. If $N \leq MT$, then $N = N \cap MT = M(N \cap T)$. Hence $N \leq T$, which implies that $M = M \cap T = M_{rG}$, a contradiction. If $N \nsubseteq MT$, then $M = M(T \cap N) = MT \cap N$ is $S$-quasinormal in $G$ by Lemma 2.1(5), a contradiction again. Hence $M = M_{rG}$. In view of Lemma 2.2(4), $M$ is $S$-quasinormal in $G$. By Lemma 2.9, some maximal subgroup of $N$ is normal in $G$. Thus the lemma holds. □

2.11. Lemma ([15, Theorem B]). Let $\mathcal{F}$ be any formation and $G$ a group. If $H \triangleleft G$ and $F^*(H) \leq Z_{\mathcal{F}}(G)$, then $H \leq Z_{\mathcal{F}}(G)$. 
3. Proofs of Theorems

Proof of Theorem 1.3. Assume that this theorem is false and let \( G \) be a counterexample with minimal order. We proceed the proof via the following steps.

1. \( O_p(G) = 1 \).
   Assume that \( L = O_p(G) \neq 1 \). Clearly, \( P/L \) is a Sylow \( p \)-subgroup of \( G/L \). Let \( M/L \) be a maximal subgroup of \( P/L \). Then \( M \) is a maximal subgroup of \( P \). If \( M \) has a \( p \)-nilpotent supplement \( D \) in \( G \), then \( M/L \) has a \( p \)-nilpotent supplement \( DL/L \) in \( G/L \). If \( M \) is partially \( \tau \)-quasinormal in \( G \), then \( M/L \) is partially \( \tau \)-quasinormal in \( G/L \) by Lemma 2.3(2). Hence \( G/L \) satisfies the hypothesis of the theorem. The minimal choice of \( G \) implies that \( G/L \) is soluble. Consequently, \( G \) is soluble. This contradiction shows that step (1) holds.

2. \( O_p(G) = 1 \).
   Assume that \( R = O_p(G) \neq 1 \). Then, obviously, \( PR/R \) is a Sylow \( p \)-subgroup of \( G/R \). Suppose that \( M/R \) is a maximal subgroup of \( PR/R \). Then there exists a maximal subgroup \( P_1 \) of \( P \) such that \( M = P_1 R \). If \( P_1 \) has a \( p \)-nilpotent supplement \( D \) in \( G \), then \( M/R \) has a \( p \)-nilpotent supplement \( DR/R \) in \( G/R \). If \( P_1 \) is partially \( \tau \)-quasinormal in \( G \), then \( M/R \) is partially \( \tau \)-quasinormal in \( G/R \) by Lemma 2.3(3). The minimal choice of \( G \) implies that \( G/R \) is soluble. By the well known Feit-Thompson’s theorem, we know that \( R \) is soluble. It follows that \( G \) is soluble, a contradiction.

3. \( P \) is not cyclic.
   If \( P \) is cyclic, then \( G \) is \( p \)-nilpotent by Lemma 2.4, and so \( G \) is soluble, a contradiction.

4. \( N \) is a minimal normal subgroup of \( G \), then \( N \) is not soluble. Moreover, \( G = PN \).
   If \( N \) is \( p \)-soluble, then \( O_p(N) \neq 1 \) or \( O_p(N) \neq 1 \). Since \( O_p(N) \) char \( N \) \( \leq G \), \( O_p(N) \leq O_p(G) \). Analogously \( O_p(N) \leq O_p(G) \). Hence \( O_p(G) \neq 1 \) or \( O_p(G) \neq 1 \), which contradicts step (1) or step (2). Therefore \( N \) is not soluble. Assume that \( PN < G \).

By Lemma 2.3(1), every maximal subgroup of \( P \) not having a \( p \)-nilpotent supplement in \( PN \) is partially \( \tau \)-quasinormal in \( PN \). Thus \( PN \) satisfies the hypothesis. By the minimal choice of \( G \), \( PN \) is soluble and so \( N \) is soluble. This contradiction shows that \( G = PN \).

5. \( G \) has a unique minimal normal subgroup \( N \).
   By step (4), we see that \( G = PN \) for every normal subgroup \( N \) of \( G \). It follows that \( G/N \) is soluble. Since the class of all soluble groups is closed under subdirect product, \( G \) has a unique minimal normal subgroup, say \( N \).

6. The final contradiction.
   If every maximal subgroup of \( P \) has a \( p \)-nilpotent supplement in \( G \), then, in view of Lemma 2.5, \( G \) is \( p \)-nilpotent and so \( G \) is soluble. This contradiction shows that we may choose a maximal subgroup \( P_1 \) of \( P \) such that \( P_1 \) is partially \( \tau \)-quasinormal in \( G \). Then there exists a normal subgroup \( T \) of \( G \) such that \( P_1 T \) is \( S \)-quasinormal in \( G \) and \( P_1 \cap T \leq (P_1)_{rG} \). If \( T = 1 \), then \( P_1 \) is \( S \)-quasinormal in \( G \). In view of Lemma 2.6, \( P_1 \leq \text{PO}^p(G) = G \). By step (5), \( P_1 = 1 \) or \( N \leq P_1 \). Since \( N \) is not soluble by step (4), we have that \( P_1 = 1 \). Consequently, \( P \) is cyclic, which contradicts step (3). Hence \( T \neq 1 \) and \( N \leq T \). It follows that \( P_1 \cap N = (P_1)_{rG} \cap N \).

For any Sylow \( q \)-subgroup \( N_q \) of \( N \) with \( q \neq p \), \( N_q \) is also a Sylow \( q \)-subgroup of \( G \) by step (4). From step (2) it is easy to see that \( (P_1)_{rG} N_q = N_q (P_1)_{rG} \). Then \( (P_1)_{rG} N_q \cap N = N_q ((P_1)_{rG} \cap N) = N_q (P_1 \cap N) \), i.e., \( P_1 \cap N \) is \( \tau \)-quasinormal in \( N \). Since \( N \) is a direct product of some isomorphic non-abelian simple groups, we may assume that \( N \cong N_1 \times \cdots \times N_s \). By Lemma 2.2(1), \( P_1 \cap N \) is \( \tau \)-quasinormal in \( (P_1 \cap N)N_1 \). Thus \( (P_1 \cap N) (N_1)^{n_1} \cap N_1 = (N_1)^{n_1} (P_1 \cap N \cap N_1) = (N_1)^{n_1} (P_1 \cap N_1) \) for any \( n_1 \in N_1 \), where \( N_1 \) is a Sylow \( q \)-subgroup of \( N_1 \) with \( q \neq p \). Since \( (N_1)^{n_1} (P_1 \cap N_1) \neq N_1 \), we have \( N_1 \) is not simple by Lemma 2.7, a contradiction.
Proof of Theorem 1.4. If \( G \) is \( p \)-nilpotent, then \( G \) has a normal Hall \( p' \)-subgroup \( G_p' \). Let \( P_1 \) be any maximal subgroup of \( P \). Then \( |G : P_1 G_p'| = p \). In view of Lemma 2.4(3), \( P_1 G_p' \leq G \). Obviously, \( P_1 \cap G_p' = 1 \). Hence \( P_1 \) is partially \( \tau \)-quasinormal in \( G \).

Now we prove the sufficient part. Assume that the assertion is false and let \( G \) be a counterexample with minimal order.

(1) \( G \) is soluble.

It follows directly from Theorem 1.3.

(2) \( G \) has a unique minimal normal subgroup \( N \) such that \( G/N \) is \( p \)-nilpotent. Moreover, \( \Phi(G) = 1 \).

Let \( N \) be a minimal normal subgroup of \( G \). Since \( G \) is solvable by step (1), \( N \) is an elementary abelian subgroup. It is easy to see that \( M \) such that \( G = P_1 G_p' \leq G \). Obviously, \( P_1 \cap G_p' = 1 \). Hence \( P_1 \) is partially \( \tau \)-quasinormal in \( G \).

First, we assume that \( P_1 \) is a maximal subgroup of \( G \). We will show \( P_1 \) is \( p \)-nilpotent in \( G \). Since \( N \) is a minimal normal subgroup of \( G \), we only need to prove \( N \leq P_1 \) when \( P_1 \) is partially \( \tau \)-quasinormal in \( G \). Let \( T \) be a normal subgroup of \( G \) such that \( PT \) is \( S \)-quasinormal in \( G \) and \( P \cap T = (P_1)_{G \cap T} \).

First, we assume that \( T = 1 \), i.e., \( P_1 \) is \( S \)-quasinormal in \( G \). In view of Lemma 2.6, \( (G_1)_{G \cap T} \leq PO(G) = G \). By virtue of Lemma 2.4(2) and step (3), \( P_1 \neq 1 \). Hence \( N \leq P_1 \) by step (2).

Now, assume that \( T \neq 1 \). Then \( N \leq T \). It follows that \( P \cap N = (P_1)_{T \cap G} \cap N \). For any Sylow \( q \)-subgroup \( G_q \) of \( G \) \( (p \neq q) \), \( (P_1)_{T \cap G} G_q = G_q (P_1)_{T \cap G} \) in view of step (4). Then \((P_1)_{T \cap G} \cap N = (P_1)_{T \cap G} G_q \cap N \leq (P_1)_{T \cap G} G_q \). Obviously, \( P_1 \cap N \leq P \). Therefore \( P \cap N \) is normal in \( G \). By the minimality of \( N \), we have \( P_1 \cap N = N \) or \( P_1 \cap N = 1 \). If the latter holds, then the order of \( N \) is \( p \) since \( P_1 \cap N \) is a maximal subgroup of \( N \). Consequently, \( G \) is \( p \)-nilpotent by step (2) and Lemma 2.4(1). This contradiction shows that \( P_1 \cap N = N \) and so \( N \leq P_1 \).

(6) The final contradiction.

Since every maximal subgroup of \( P \) has a \( p \)-nilpotent supplement in \( G \) by step (5), we have \( G \) is \( p \)-nilpotent by Lemma 2.5, a contradiction.

Proof of Theorem 1.5. Assume that this theorem is false and and consider a counterexample \((G, E)\) for which \(|G|/|E|\) is minimal.

(1) \( E \) is \( p \)-nilpotent.

Let \( P_1 \) be a maximal subgroup of \( P \). If \( P \) has a \( p \)-supersolvable supplement \( T \) in \( G \), then \( P_1 \) has a \( p \)-supersolvable supplement \( T \cap E \) in \( E \). Since \(|E|, p - 1 = 1, T \cap E \) is also \( p \)-nilpotent by Lemma 2.4(4). If \( P_1 \) is partially \( \tau \)-quasinormal in \( G \), then \( P_1 \) is also partially \( \tau \)-quasinormal in \( E \) by Lemma 2.3(1). Hence every maximal subgroup of \( P \) not having a \( p \)-nilpotent supplement in \( E \) is partially \( \tau \)-quasinormal in \( E \). In view of Theorem 1.4, \( E \) is \( p \)-nilpotent.

(2) \( P = E \).

By step (1), \( O_p'(E) \) is the normal Hall \( p' \)-subgroup of \( E \). Suppose that \( O_p'(E) \neq 1 \). It is easy to see that the hypothesis of the theorem holds for \((G/O_p'(E), E/O_p'(E))\). By induction, every chief factor of \( G/O_p'(E) \) between \( E/O_p'(E) \) and 1 is cyclic. Consequently, each chief factor of \( G \) between \( E \) and \( O_p'(E) \) is cyclic. This condition shows that \( O_p'(E) = 1 \) and so \( P = E \).
(3) $\Phi(P) = 1$.

Suppose that $\Phi(P) \neq 1$. By Lemma 2.3(2), it is easy to see that the hypothesis of the theorem holds for $(G/\Phi(P), P/\Phi(P))$. By the choice of $(G, E)$, every chief factor of $G/\Phi(P)$ below $P/\Phi(P)$ is cyclic. In view of Lemma 2.8, every chief factor of $G$ below $P$ is cyclic, a contradiction.

(4) Every maximal subgroup of $P$ is partially $\tau$-quasinormal in $G$.

Suppose that there is some maximal subgroup $V$ of $P$ such that $V$ has a $p$-supersolvable supplement $B$ in $G$, then $G = PB$ and $P \cap B \neq 1$. Since $P \cap B \leq B$, we may assume that $B$ has a minimal normal subgroup $N$ contained in $P \cap B$. It is clear that $|N| = p$. Since $P$ is elementary abelian and $G = PB$, we have that $N$ is also normal in $G$. It is easy to see that the hypothesis is still true for $(G/N, P/N)$. Hence every chief factor of $G/N$ below $P/N$ is cyclic by virtue of the choice of $(G, E)$. It follows that every chief factor of $G$ below $P$ is cyclic. This contradiction shows that all maximal subgroups of $P$ are partially $\tau$-quasinormal in $G$.

(5) $P$ is not a minimal normal subgroup of $G$.

Suppose that $P$ is a minimal normal subgroup of $G$, then some maximal subgroup of $P$ is normal in $G$ by Lemma 2.10, which contradicts the minimality of $P$.

(6) If $N$ is a minimal normal subgroup of $G$ contained in $P$, then $P/N \leq Z_w(G/N)$, $N$ is the only minimal normal subgroup of $G$ contained in $P$ and $|N| > p$.

Indeed, by Lemma 2.3(2), the hypothesis holds on $(G/N, P/N)$ for any minimal normal subgroup $N$ of $G$ contained in $P$. Hence every chief factor of $G/N$ below $P/N$ is cyclic by the choice of $(G, E) = (G, P)$. If $|N| = p$, every chief factor of $G$ below $P$ is cyclic, a contradiction. If $G$ has two minimal normal subgroups $R$ and $N$ contained in $P$, then $NR/R \leq P/R$ and from the $G$-isomorphism $NR/R \cong N$ we have $|N| = p$, a contradiction. Hence, (6) holds.

(7) The final contradiction.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$ and $N_1$ any maximal subgroup of $N$. We show that $N_1$ is $S$-quasinormal in $G$. Since $P$ is an elementary abelian $p$-group, we may assume that $D$ is a complement of $N$ in $P$. Let $V = N_1D$. Obviously, $V$ is a maximal subgroup of $P$. By step (4), $V$ is partially $\tau$-quasinormal in $G$. By Lemma 2.3(4), there exist a normal subgroup $T$ of $G$ such that $VT$ is $S$-quasinormal in $G$, $V \cap T \leq V_\tau G$ and $VT \leq P$. In view of Lemma 2.2(4), $V_\tau G$ is an $S$-quasinormal subgroup of $G$. If $T = P$, then $V = V_\tau G$ is $S$-quasinormal in $G$ and hence $V \cap N = N_1D \cap N = N_1(D \cap N) = N_1$ is $S$-quasinormal in $G$ by Lemma 2.1(5). If $T = 1$, then $V = VT$ is $S$-quasinormal in $G$. Consequently, we have also $N_1$ is $S$-quasinormal in $G$. Now we assume that $1 < T < P$. Hence $N \leq T$ by step (6). Then, $N_1 = V \cap N = V_\tau G \cap N$ is $S$-quasinormal in $G$ by virtue of Lemma 2.1(5). Hence some maximal subgroup of $N$ is normal in $G$ by Lemma 2.9. Consequently, $|N| = p$. This contradicts step (6).

Proof of Theorem 1.6. Let $q$ be the smallest prime dividing $|E|$. In view of step (1) of the proof of Theorem 1.5, $E$ is $q$-nilpotent. Let $E_{q'}$ be the normal Hall $q'$-subgroup of $E$. If $E_{q'} \neq 1$, then every chief factor of $G$ below $E$ is cyclic by Theorem 1.5. Hence we may assume that $E_{q'} \neq 1$. Since $E_{q'} \leq G$, we see that $E_{q'} \leq G$. By Lemma 2.3(3), the hypothesis of the theorem holds for $(G/E_{q'}, E/E_{q'})$. By induction, every chief factor of $G/E_{q'}$ below $E/E_{q'}$ is cyclic. On the other hand, $(G, E_{q'})$ also satisfies the hypothesis of the theorem in view of Lemma 2.3(1). By induction again, we have also every chief factor of $G$ below $E_{q'}$ is cyclic. Hence it follows that every chief factor of $G$ below $E$ is cyclic.
Proof of Main Theorem. Applying Theorem 1.6, $X$ is hypercyclically embedded in $G$. Since $F^*(E) \leq X$, we have that $F^*(E)$ is also hypercyclically embedded in $G$. By virtue of Lemma 2.11, $E$ is also hypercyclically embedded in $G$.

4. Some Applications

4.1. Theorem. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathcal{F}$. Suppose that for every non-cyclic Sylow subgroup $P$ of $E$, every maximal subgroup of $P$ not having a supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then $G \in \mathcal{F}$.

Proof. Applying our Main Theorem, every chief factor of $G$ below $E$ is cyclic. Since $\mathcal{F}$ contains $\mathcal{U}$, we know $E$ is contained in the $\mathcal{F}$-hypercentre of $G$. From $G/E \in \mathcal{F}$, it follows that $G \in \mathcal{F}$. □

4.2. Theorem. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathcal{F}$. Suppose that for every non-cyclic Sylow subgroup $P$ of $F^*(E)$, every maximal subgroup of $P$ not having a supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then $G \in \mathcal{F}$.

Proof. The proof is similar to that of Theorem 4.1. □

4.3. Corollary ([9, Theorem 3.4]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of any Sylow subgroup of $F^*(E)$ is $S$-quasinormal in $G$, then $G \in \mathcal{F}$.

4.4. Corollary ([19, Theorem 3.4]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of any Sylow subgroup of $F^*(E)$ is $c$-normal in $G$, then $G \in \mathcal{F}$.

4.5. Corollary ([1, Theorem 1.4]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $E$ a soluble normal subgroup of a group $G$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of any Sylow subgroup of $F(E)$ is $S$-quasinormal in $G$, then $G \in \mathcal{F}$.

4.6. Corollary ([8, Theorem 2]). Let $G$ be a group and $E$ a soluble normal subgroup of $G$ such that $G/E$ is supersolvable. If all maximal subgroups of the Sylow subgroups of $F(E)$ are $c$-normal in $G$, then $G$ is supersolvable.

References
