ALGEBRAIC MODELS OF
SMOOTH MANIFOLDS AND
NON-ZERO HARMONICITY

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Abstract
In this note we give an obstruction in terms of $\text{Im} H^k(X, \mathbb{Z})$ and the Euler characteristic $\chi(X)$, to the harmonicity of products of harmonic forms representing cohomology classes on $X_C$, where $X$ is a real algebraic variety.

Keywords: Real algebraic varieties, Algebraic homology, Harmonic forms.


1. Introduction
In this work, by a real algebraic variety we mean a complex algebraic variety $X$ with an anti-holomorphic involution $\tau : X \to X$ such that $X^* = \{x \in X \mid \tau(x) = x\}$ is the set of real points of $X$. We will denote $X^*$ by $X(\mathbb{R})$ and the set of complex points by $X(\mathbb{C})$.

All real algebraic varieties under consideration in this note are nonsingular. It is well known that real projective varieties are affine ([1, Proposition 2.4.1] or [2, Theorem 3.4.4]). Moreover, compact affine real algebraic varieties are projective [1, Corollary 2.5.14] and, therefore, we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$, a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r]$, $i = 1, \ldots, s$, such that each $g_i$ vanishes nowhere on $X$ and $F = (f_1/g_1, \ldots, f_s/g_s)$. We say $X$ and $Y$ are isomorphic if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = \text{id}_Y$ and $G \circ F = \text{id}_X$. Isomorphic algebraic varieties will be regarded as being the same.

An algebraic homology group $H^k_{\text{alg}}(X, R)$ ($R = \mathbb{Z}$ or $\mathbb{Z}_2$) is defined as the subgroup of $H_k(X, R)$ generated by the classes represented by real algebraic cycles. For a compact nonsingular real algebraic variety $X$ of dimension $n$, let $H^*_{\text{alg}}(X, R)$ be the Poincaré dual of the group $H^{n-k}_{\text{alg}}(X, R)$.

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