RESULTS ON THE COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF DISTRIBUTIONS

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Abstract

Let \( Li(x) \) denote the dilogarithm integral. The goal of this paper is to evaluate several commutative neutrix convolution products involving the dilogarithm integral and its associated functions \( Li_+(x) \) and \( Li_-(x) \).

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1. Introduction

The dilogarithm integral \( Li(x) \) is defined by

\[
Li(x) = - \int_0^x \frac{\ln |1-t|}{t} dt
\]

(see [1]), and the associated functions \( Li_+(x) \) and \( Li_-(x) \) are defined by

\[
Li_+(x) = H(x) Li(x), \quad Li_-(x) = H(-x) Li(x) = Li(x) - Li_+(x),
\]

where \( H(x) \) denotes Heaviside’s function.

Next, the distribution \( \ln |1-x|^{-1} \) is defined by

\[
\ln |1-x|^{-1} = - Li'(x),
\]

and its associated distributions \( \ln |1-x|^{-1}_+ \) and \( \ln |1-x|^{-1}_- \) are defined by

\[
\ln |1-x|^{-1}_+ = H(x) \ln |1-x|^{-1} = - Li'_+(x), \quad \ln |1-x|^{-1}_- = H(-x) \ln |1-x|^{-1} = - Li'_-(x).
\]

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We define the function $I_r(x)$ by

$$I_r(x) = \int_0^x u^r \ln |1 - u| \, du$$

for $r = 0, 1, 2, \ldots$, and it was shown in [5] that

$$I_r(x) = \frac{1}{r + 1} (x^{r+1} - 1) \ln |1 - x| - \frac{1}{r + 1} \sum_{i=0}^{r} \frac{x^{i+1}}{i+1}.$$ 

The classical definition of the convolution product of two functions $f$ and $g$ is as follows:

**1.1. Definition.** Let $f$ and $g$ be functions. Then the convolution $f \ast g$ is defined by

$$(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) \, dt$$

for all points $x$ for which the integral exist.

It follows easily from the definition that if $f \ast g$ exists then $g \ast f$ exists and $f \ast g = g \ast f$.

Further, if $f \ast g$ exists then

$$f' \ast g \ast f = f \ast g' \ast f \ast g.$$

Definition 1.1 can be extended to define the convolution $f \ast g$ of two distributions $f$ and $g$ in $D'$, the space of infinitely differentiable functions with compact support, see Gel’fand and Shilov [6].

**1.2. Definition.** Let $f$ and $g$ be distributions in $D'$. Then the convolution $f \ast g$ is defined by the equation

$$\langle (f \ast g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary $\varphi$ in $D$, provided $f$ and $g$ satisfy either of the conditions

(a) Either $f$ or $g$ has bounded support,

(b) The supports of $f$ and $g$ are bounded on the same side.

Note that if $f$ and $g$ are locally summable functions satisfying either of the above conditions, and the classical convolution $f \ast g$ exists, then it is in agreement with Definition 1.1.

The following convolutions were proved in [5] for $r = 0, 1, 2, \ldots$.

$$\text{Li}_+^r(x) \ast x_+^r = \frac{1}{r + 1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} I_{r-i}(x) x_+^i + \frac{1}{r + 1} x^{r+1}_+ \text{Li}_+^r(x)$$

$$\ln |1 - x|_+^{-1} \ast x_+^r = \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i} I_{r-i-1}(x) x_+^i - x^r \text{Li}_+^r(x).$$

The convolution product of distributions may be defined without any restriction on the supports. One of the best known is the definition given by V.S. Vladimirov, although several other definitions of the convolution product are equivalent to that of Vladimirov. However, the convolution product in the sense of any of these definitions does not exist for many pairs of distributions.

In [3] the commutative neutrix convolution product is defined, and this exists for a considerably larger class of pairs of distributions. In that definition, unit-sequences of function in $D$ are used which allows one to approximate a given distribution by a sequence of distributions of bounded support.

To recall the definition of the commutative neutrix convolution product we first let $\tau$ be a function in $D$, see [7], satisfying the conditions:
The function \( \tau_n \) is then defined by
\[
\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^n - n^{n+1}), & x > n, \\
\tau(n^n x + n^{n+1}), & x < -n,
\end{cases}
\]
for \( n = 1, 2, \ldots \).

We now have the following definition of the commutative neutrix convolution product.

**1.3. Definition.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \), and let \( f_n = f \tau_n \) and \( g_n = g \tau_n \) for \( n = 1, 2, \ldots \). Then the commutative neutrix convolution product \( f \ast g \) is defined as the neutrix limit of the sequence \( \{f_n \ast g_n\} \), provided that the limit \( h \) exists in the sense that
\[
\lim_{n \to \infty} \langle f_n \ast g_n, \phi \rangle = \langle h, \phi \rangle
\]
for all \( \phi \) in \( \mathcal{D} \), where \( N \) is the neutrix (see van der Corput [2]), having domain \( N' = \{1, 2, \ldots, n, \ldots\} \) and range \( N'' \) the real numbers, with negligible functions finite linear sums of the functions
\[
n^\lambda \ln^{r-1} n, \ln^r n, \quad (\lambda \neq 0, \ r = 1, 2, \ldots),
\]
and all functions which converge to zero in the usual sense as \( n \) tends to infinity.

Note that in this definition the convolution product \( f_n \ast g_n \) is in the sense of Definition 1.1, the distributions \( f_n \) and \( g_n \) having bounded support since the support of \( \tau_n \) is contained in the interval \([-n - n^{-1}, n + n^{-1}]\). This neutrix convolution product is also commutative.

It is obvious that any results proved with the original definition hold with the new definition. The following theorem (proved in [3]) therefore holds, showing that the commutative neutrix convolution product is a generalization of the convolution product. So the idea of a neutrix lies in neglecting certain numerical sequences diverging to \( \pm \infty \), which makes for a wider class of pairs of distributions \( f \) and \( g \) for which the product exists. It should be noted that, in general, the definition of a commutative neutrix convolution product depends on the choice of the sequence \( \tau_n \) as well as the set of negligible sequences.

**1.4. Theorem.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \), satisfying either condition (a) or condition (b) of Gel’fand and Shilov’s definition. Then the commutative neutrix convolution product \( f \ast g \) exists and
\[
f \ast g = f \ast g.
\]

Note, however, that \((f \ast g)'\) is not necessarily equal to \( f' \ast g \), although we do have the following theorem proved in [4].

**1.5. Theorem.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and suppose that the commutative neutrix convolution product \( f \ast g \) exists. If \( \lim_{\nu \to \infty} \langle (f \tau_\nu'), g_n, \phi \rangle \) exists and equals \( \langle h, \phi \rangle \) for all \( \phi \) in \( \mathcal{D} \), then \( f' \ast g \) exists and
\[
(f \ast g)' = f' \ast g + h.
\]
2. Main result

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions \(n^s \text{Li}(n^r)\), for \(s = 0, 1, 2, \ldots\) and \(r = 1, 2, \ldots\). Before proving some further results we need the following lemma proved in [5]:

2.1. Lemma.

\[
\lim_{n \to \infty} I_n(x) = -\frac{1}{(r + 1)^2}
\]

2.2. Theorem. The commutative neutrix convolution product \(\text{Li}_+(x) \square x^r\) exists and

\[
\text{Li}_+(x) \square x^r = \frac{1}{r + 1} \sum_{i=0}^{r} \binom{r + 1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2} x^i
\]

for \(r = 0, 1, 2, \ldots\).

Proof. We put \([\text{Li}_+(x)]_n = \text{Li}_+(x) \tau_n(x)\) and \([x^r]_n = x^r \tau_n(x)\) for \(n = 1, 2, \ldots\). Since these functions have compact support, the classical convolution product \([\text{Li}_+(x)]_n * [x^r]_n\) exists by Definition 1.1, and we have:

\[
[\text{Li}_+(x)]_n * [x^r]_n = \int_{-\infty}^{\infty} \text{Li}_+(t)(x-t)^r \tau_n(x-t) \tau_n(t) \, dt
\]

\[
= \int_0^n \text{Li}(t)(x-t)^r \tau_n(x-t) \, dt
\]

\[
+ \int_n^{n+n-n} \text{Li}(t)(x-t)^r \tau_n(x-t) \tau_n(t) \, dt
\]

(6) 

Thus using Lemma 2.1 we have,

\[
\lim_{n \to \infty} I_1 = -\frac{1}{(r + 1)^2}
\]

If \(0 \leq x \leq n\) we see that

\[
I_1 = \int_0^n \text{Li}(t)(x-t)^r \tau_n(x-t) \, dt
\]

\[
= -\int_0^n (x-t)^r \int_0^t \ln|1-u| \, du \, dt
\]

\[
= -\int_0^n \ln\frac{1-u}{u} \int_u^n (x-t)^r \, dt \, du
\]

\[
= \frac{1}{r+1} \sum_{i=0}^{r+1} (-1)^{r-i} x^i \binom{r+1}{i} \int_0^n u^{i-1} \ln|1-u| \{u^{r-i+1} - n^{r-i+1}\} \, du
\]

\[
= \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} x^i I_{r-i}(n) + \frac{1}{r+1} x^{r+1} \text{Li}(n) +
\]

\[
+ \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} x^i \text{Li}(n)n^{r-i+1}.
\]

Thus using Lemma 2.1 we have,

\[
\lim_{n \to \infty} I_1 = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2} x^i.
\]
Next, if \(-n \leq x \leq 0\) we have

\[
I_1 = \int_{0}^{x} \operatorname{Li}(t)(x-t)^r \tau_n(x-t) \, dt
\]

(8)

\[
= \int_{0}^{x} \operatorname{Li}(t)(x-t)^r \, dt + \int_{x}^{x+n} \operatorname{Li}(t)(x-t)^r \tau_n(x-t) \, dt,
\]

where

\[
\int_{0}^{x} \operatorname{Li}(t)(x-t)^r \, dt = - \int_{0}^{x} (x-t)^r \int_{0}^{t} u^{-1} \ln |1-u| \, du \, dt
\]

\[
= - \int_{0}^{x} t^{-1} \ln |1-u| \int_{u}^{x} (x-t)^r \, dt \, du
\]

(9)

\[
= \frac{1}{r+1} \int_{0}^{x} u^{-1} (1-u) \left[ (-n)^{r+1} - (x-u)^{r+1} \right] du
\]

\[
= \frac{1}{r+1} \left( (-n)^{r+1} - x^{r+1} \right) \operatorname{Li}(x+n)
\]

\[
+ \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} x^{r-i} I_{r-i}(x+n).
\]

Further, it is easily seen that, for each fixed \(x\) and \(K = \sup \{ |\operatorname{Li}_+(x)| \} \) we have

\[
\left| \int_{x}^{x+n} \operatorname{Li}(t)(x-t)^r \tau_n(x-t) \, dt \right| \leq K \int_{x}^{x+n} |x-t|^r \, dt
\]

and so

(10) \quad \lim_{n \to \infty} \int_{x}^{x+n} \operatorname{Li}(t)(x-t)^r \tau(x-t) \, dt = 0.

Using the fact that

\[
N - \lim_{n \to \infty} I_r(x+n) = N - \lim_{n \to \infty} I_r(n) = -\frac{1}{(r+1)^2},
\]

and noting that the function \(\operatorname{Li}(x+n)\) is negligible we have from (8), (9) and (10) that in this case also equation (7) holds.

Further, it is easily seen that for each fixed \(x\)

(11) \quad \lim_{n \to \infty} I_2 = \lim_{n \to \infty} \int_{x}^{x+n} \operatorname{Li}(t)(x-t)^r \tau_n(x-t) \, dt = 0

and now (5) follows immediately from (6), (7) and (11), so proving the theorem. \(\square\)

### 2.3. Corollary

The commutative neutrix convolution product \(\operatorname{Li}_+(x) \square x^r \) exists and

\[
\operatorname{Li}_+(x) \square x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^i \left[ \frac{1}{(r-i+1)x^i - I_{r-i}(x)x^i} \right]
\]

(12)

\[
+ \left( -1 \right)^{r+1} \frac{r+1}{r+1} \operatorname{Li}_+(x) \square x^r
\]

for \(r = 0, 1, 2, \ldots\)

Proof. Since the commutative neutrix convolution product is distributive with respect to addition, we have

\[
\operatorname{Li}_+(x) \square x^r = \operatorname{Li}_+(x) \square x^r + (-1)^r \operatorname{Li}_+(x) \square x^r,
\]

and equation (12) follows from equation (5) and (3). \(\square\)
2.4. Theorem. The commutative neutrix convolution product \( \ln |1 - x| x^{-1} \star x^r \) exists and

\[
\ln |1 - x| x^{-1} \star x^r = \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^{r-i}}{(r-i)^{2}} x^i
\]

for \( r = 0, 1, 2, \ldots \)

Proof. Using Theorem 2 we have

\[
\ln |1 - x| x^{-1} \star x^r = r \text{Li}_2(x) x^{r-1} - N \lim_{n \to \infty} [\text{Li}_2(x) \tau_n(x)] \star [x^r]_n
\]

when, on integration by parts we have

\[
[\text{Li}_2(x) \tau_n(x)] \star (x^r)_n = \int_n^{n+h-n} \text{Li}_2(t)(x-t)^r \tau_n(x-t) d\tau_n(t)
\]

\[
= - \text{Li}_2(n)(x-n)^r \tau_n(x-n)
\]

\[
- \int_n^{n+h-n} \ln |1 - t| t^{-1}(x-t)^r \tau_n(x-t) dt
\]

\[
+ r \int_n^{n+h-n} \text{Li}_2(t)(x-t)^{r-1} \tau_n(t) \tau_n(x-t) dt
\]

\[
+ \int_n^{n+h-n} \text{Li}_2(t)(x-t)^{r} \tau_n(t) \tau'_n(x-t) dt.
\]

Now \( \tau_n(x-n) \) is either 0 or 1 for large enough \( n \), and so

\[
N \lim_{n \to \infty} \text{Li}_2(n)(x-n)^r \tau_n(x-n) = 0.
\]

Next we have

\[
\left| \int_n^{n+h-n} \ln |1 - t| t^{-1}(x-t)^r \tau_n(x-t) dt \right| \leq \int_n^{n+h-n} \left| \ln |1 - t| t^{-1}(x-t)^r \right| dt
\]

and it follows that

\[
\lim_{n \to \infty} \int_n^{n+h-n} \ln |1 - t| t^{-1}(x-t)^r \tau_n(x-t) dt = 0.
\]

Similarly,

\[
\lim_{n \to \infty} \int_n^{n+h-n} \text{Li}_2(t)(x-t)^{r-1} \tau_n(t) \tau_n(x-t) dt = 0.
\]

Noting that \( \tau'_n(x-t) = 0 \) for large enough \( n \) and \( x \neq 0 \), it follows that

\[
\lim_{n \to \infty} \int_n^{n+h-n} \text{Li}_2(t)(x-t)^{r} \tau_n(t) \tau'_n(x-t) dt = 0.
\]

If \( x = 0 \), then

\[
\int_n^{n+h-n} \text{Li}_2(t)(x-t)^r \tau_n(t) \tau'_n(-t) dt
\]

\[
= \frac{1}{2} \int_n^{n+h-n} \text{Li}_2(t)(x-t)^r dt \tau^2_n(t) dt
\]

\[
= \frac{1}{2} \text{Li}_2(n)(x-n)^r + \int_n^{n+h-n} \left[ \ln |1 - t| t^{-1}(x-t) - r \text{Li}_2(t) \right](x-t)^{r-1} \tau^2_n(t) dt.
\]
and it follows that
\[
N - \lim_{n \to \infty} \int_{n}^{n+n^{-n}} \log(t)(x-t)^r \tau_n(t)\tau'_n(-t) \, dt = 0.
\]

It now follows from Equations (15) to (20) that
\[
N - \lim_{n \to \infty} [\log(x) \tau_n(x)] \ast [x^r]_n = 0,
\]
and the equation (13) follows directly from (14) and (5), so proving the theorem. \(\square\)

2.5. Corollary. The commutative neutrix convolution product \(\ln|1-x|^{-1} \mathcal{M} x^r\) exists and
\[
\ln|1-x|^{-1} \mathcal{M} x^r = \sum_{i=0}^{r-1} \left( \frac{r}{i} \right) (-1)^i \left( \frac{1}{(r-i)^2} x^i - I_{r-i-1}(x) x^i \right) + (-1)^r x^r \log(x)
\]
for \(r = 0, 1, 2, \ldots\).

Proof. Because we have
\[
\ln|1-x|^{-1} \mathcal{M} x^r = \ln|1-x|^{-1} \ast x^r + (-1)^r \ln|1-x|^{-1} \mathcal{M} x^r,
\]
equation (21) follows from equation (13) and (4). \(\square\)

References