ON WEAKLY COMMUTING MAPS AND COMMON FIXED POINT RESULTS FOR FOUR MAPS IN G-METRIC SPACES

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Abstract
In this paper, we introduce the concept of weakly commuting maps in G-metric spaces and prove a common fixed point theorem for four self maps in the setting of generalized metric spaces. We also present an example to support our result.

Keywords: Common fixed Point, Weakly Commuting Maps, Generalized Metric Spaces.

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1. Introduction
The notion of G-metric space was introduced by Z. Mustafa and B. Sims [10] as a generalization of the notion of metric spaces. Mustafa et al. studied many fixed point results in G-metric spaces (see [8, 9, 10, 11, 12]). The study of common fixed point theorems in generalized metric spaces was initiated by Abbas and Rhoades [2], while, Saddati et al. [13] studied some fixed points in generalized partially ordered G-metric spaces. Shatanawi [15] obtained fixed points of Φ-maps in G-metric spaces. Also, Shatanawi [16] obtained a coupled coincidence fixed point theorem in the setting of a generalized metric spaces for two mapping F and g under certain conditions with an assumption of G-continuity of one of the mapping involved therein, see also [3, 17, 1, 4, 18, 5], while Chugh et al. [6] obtained some fixed point results for maps satisfying property p in a G-metric space. In the present paper, we introduce the concept of weakly commuting maps in G-metric spaces and prove a common fixed point theorem for four self maps in the setting of generalized metric spaces.

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2. Preliminaries.

The following definition was introduced by Mustafa and Sims [10].

2.1. Definition. [10] Let \( X \) be a nonempty set and \( G : X \times X \times X \to \mathbb{R}^{+} \) a function satisfying the following properties:

\[
\begin{align*}
(G_1) & \quad G(x, y, z) = 0 \text{ if } x = y = z, \\
(G_2) & \quad 0 < G(x, x, y), \text{ for all } x, y \in X \text{ with } x \neq y, \\
(G_3) & \quad G(x, x, y) \leq G(x, y, z) \text{ for all } x, y, z \in X \text{ with } z \neq y, \\
(G_4) & \quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots, \text{ symmetry in all three variables,} \\
(G_5) & \quad G(x, y, z) \leq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X.
\end{align*}
\]

Then the function \( G \) is called a generalized metric, or, more specifically, a \( G \)-metric on \( X \), and the pair \((X, G)\) is called a \( G \)-metric space.

2.2. Definition. [10] Let \((X, G)\) be a \( G \)-metric space, and let \( \{x_n\} \) be a sequence of points of \( X \). A point \( x \in X \) is said to be the limit of the sequence \( \{x_n\} \), if

\[
\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0,
\]

and we say that the sequence \( \{x_n\} \) is \( G \)-convergent to \( x \) or \( \{x_n\} \) \( G \)-converges to \( x \).

Thus, \( x_n \to x \) in a \( G \)-metric space \((X, G)\) if for any \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( G(x, x_n, x_m) < \varepsilon \) for all \( m, n \geq k \).

2.3. Proposition. [10] Let \((X, G)\) be a \( G \)-metric space. Then the following are equivalent:

\[
\begin{align*}
(1) & \quad \{x_n\} \text{ is } G \text{-convergent to } x. \\
(2) & \quad G(x_n, x_n, x) \to 0 \text{ as } n \to +\infty. \\
(3) & \quad G(x_n, x) \to 0 \text{ as } n \to +\infty. \\
(4) & \quad G(x_n, x_n, x) \to 0 \text{ as } n, m \to +\infty. \quad \square
\end{align*}
\]

2.4. Definition. [10] Let \((X, G)\) be a \( G \)-metric space. A sequence \( \{x_n\} \) is called \( G \)-Cauchy if for every \( \varepsilon > 0 \), there is \( k \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \), for all \( n, m, l \geq k \); that is \( G(x_n, x_m, x_l) \to 0 \) as \( n, m, l \to +\infty \).

2.5. Proposition. [10] Let \((X, G)\) be a \( G \)-metric space. Then the following are equivalent:

\[
\begin{align*}
(1) & \quad \text{The sequence } \{x_n\} \text{ is } G \text{-Cauchy.} \\
(2) & \quad \text{For every } \varepsilon > 0, \text{ there is } k \in \mathbb{N} \text{ such that } G(x_n, x_m, x) < \varepsilon, \text{ for all } n, m, k \geq k \quad \square
\end{align*}
\]

2.6. Definition. [10] Let \((X, G)\) and \((X', G')\) be \( G \)-metric spaces, and let \( f : (X, G) \to (X', G') \) be a function. Then \( f \) is said to be \( G \)-continuous at a point \( a \in X \) if and only if for every \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( x, y \in X \) and \( G(a, x, y) < \delta \) implies \( G'(f(a), f(x), f(y)) < \varepsilon \). A function \( f \) is \( G \)-continuous on \( X \) if and only if it is \( G \)-continuous at all \( a \in X \).

2.7. Proposition. [10] Let \((X, G)\) be a \( G \)-metric space. Then the function \( G(x, y, z) \) is jointly continuous in all three of its variables.

Every \( G \)-metric on \( X \) defines a metric \( d_G \) on \( X \) by

\[
d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X.
\]

For a symmetric \( G \)-metric space

\[
d_G(x, y) = 2G(x, y, y), \text{ for all } x, y \in X.
\]

However, if \( G \) is not symmetric, then the following inequality holds:

\[
\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \text{ for all } x, y \in X.
\]

The following are examples of \( G \)-metric spaces.
2.8. Example. [10] Let \((\mathbb{R}, d)\) be the usual metric space. Define \(G_s\) by

\[ G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z) \]

for all \(x, y, z \in \mathbb{R}\). Then it is clear that \((\mathbb{R}, G_s)\) is a \(G\)-metric space.

2.9. Example. [10] Let \(X = \{a, b\}\). Define \(G\) on \(X \times X \times X\) by

\[ G(a, a, a) = G(b, b, b) = 0, \quad G(a, a, b) = 1, \quad G(a, b, b) = 2 \]

and extend \(G\) to \(X \times X \times X\) by using the symmetry in the variables. Then it is clear that \((X, G)\) is a \(G\)-metric space.

2.10. Definition ([10]). A \(G\)-metric space \((X, G)\) is called \(G\)-complete if every \(G\)-Cauchy sequence in \((X, G)\) is \(G\)-convergent in \((X, G)\).

3. Main Results.

In 1982, Sessa [14] introduced the concept of weakly commuting maps in metric spaces as follows

3.1. Definition. Let \((X, d)\) be a metric space and \(f, g\) be two self mappings of \(X\). Then \(f\) and \(g\) are called weakly commuting if

\[ d(fgx, gfx) \leq d(fx, gx) \]

holds for all \(x \in X\).

Following Sessa [14], the concept of weakly commuting maps in \(G\)-metric space is defined as:

3.2. Definition. Let \((X, G)\) be a \(G\)-metric space and \(f, g\) two self mappings of \(X\). Then the pair \(\{f, g\}\) is called weakly commuting if

\[ G(fgx, gfx, gfx) \leq G(fx, gx, gx) \]

holds for all \(x \in X\).

Now, we study a common fixed point for four maps satisfying a set of conditions in a \(G\)-metric space; in addition we introduce an example of our main result.

3.3. Theorem. Let \(X\) be a complete \(G\)-metric space, and let \(A, B, S, T : X \rightarrow X\) be mappings satisfying:

\[ G(Sx, Ty, Ty) \leq pG(Ax, By, By) + qG(Sx, Sx, Ax) + rG(Ty, Ty, Ty) \quad (3.1) \]

and

\[ G(Sx, Sx, Ty) \leq pG(Ax, Ax, By) + qG(Sx, Sx, Ax) + rG(Ty, Ty, By) \quad (3.2) \]

Assume the maps \(A, B, S\) and \(T\) satisfy the following conditions:

1. \(TX \subseteq AX\) and \(SX \subseteq BX\).
2. The mappings \(A\) and \(B\) are sequentially continuous, and
3. The pairs \(\{A, S\}\) and \(\{B, T\}\) are weakly commuting.

If \(p, q, r \geq 0\) with \(p + q + r \in [0, 1)\), then \(A, B, S\) and \(T\) have a unique common fixed point.
Proof. If $X$ is a symmetric $G$-metric space, then by adding the above two inequalities we obtain
\[
G(Sx, Ty, Ty) + G(Sx, Sx, Ty) \leq p[G(Ax, By, By) + G(Ax, Ax, By)] + 2q[G(Sx, Sx, Ax)] + 2r[G(Ty, Ty, By)],
\]
which further implies that
\[
d_C(Sx, Ty) \leq pd_C(Ax, By) + qd_C(Sx, Ax) + rd_C(Ty, By),
\]
for all $x, y \in X$ with $0 \leq p + q + r < 1$ and the fixed point of $A, B, S$ and $T$ follows from the result for metric spaces, see [14].

Now if $X$ is not a symmetric $G$-metric space then by the definition of the metric $(X, d_C)$ and Inequalities (3.1) and (3.2), we obtain
\[
d_C(Sx, Ty) = G(Sx, Ty, Ty) + G(Sx, Sx, Ty) \leq p[G(Ax, By, By) + G(Ax, Ax, By)] + qG(Sx, Sx, Ax) + G(Sx, Sx, Ax) + rG(Ty, Ty, By)]\]
\[
\leq pd_C(Ax, By) + \frac{1}{2}qd_C(Sx, Ax) + \frac{1}{4}rd_C(Ty, By).\]

for all $x \in X$. Here, the contractivity factor $p + \frac{1}{2}(q + r)$ may not be less than 1. Therefore the metric gives no information. In this case, for given $x_0 \in X$, choose $x_1 \in X$ such that $Ax_1 = Tx_0$, choose $x_2 \in X$ such that $Sx_1 = Bx_2$, choose $x_3 \in X$ such that $Ax_3 = Tx_2$. Continuing the above process, we can construct a sequence $\{x_n\}$ in $X$ such that $Ax_{2n+1} = Tx_{2n}$, $n \in \mathbb{N} \cup \{0\}$ and $Bx_{2n+2} = Sx_{2n+1}$, $n \in \mathbb{N} \cup \{0\}$. Let
\[
y_{2n} = Ax_{2n+1} = Tx_{2n}, \ n \in \mathbb{N} \cup \{0\}
\]
and
\[
y_{2n+1} = Bx_{2n+2} = Sx_{2n+1}, \ n \in \mathbb{N} \cup \{0\}.
\]
Take $n \in \mathbb{N}$. If $n$ is even, then $n = 2k$ for some $k \in \mathbb{N}$. Then from (3.2), we have
\[
G(y_n, y_{n+1}, y_{n+1}) = G(y_{2k}, y_{2k+1}, y_{2k+1}) = G(Tx_{2k}, Sx_{2k+1}, Sx_{2k+1}) = G(Sx_{2k+1}, Sx_{2k+1}, Tx_{2k}) \leq pG(Ax_{2k+1}, Ax_{2k+1}, Bx_{2k}) + qG(Sx_{2k+1}, Sx_{2k+1}, Ax_{2k+1}) + rG(Tx_{2k}, Tx_{2k}, Bx_{2k}) = pG(y_{2k}, y_{2k+1}, y_{2k+1}) + qG(y_{2k+1}, y_{2k+1}, y_{2k}) + rG(y_{2k}, y_{2k}, y_{2k+1}) = pG(y_n, y_n, y_{n-1}) + qG(y_{n+1}, y_{n+1}, y_{n}) + rG(y_n, y_n, y_{n-1}),
\]
which further implies that
\[
(1 - q)G(y_n, y_{n+1}, y_{n+1}) \leq (p + r)G(y_n, y_n, y_n).
\]
Hence
\[
G(y_n, y_{n+1}, y_{n+1}) \leq \frac{p + r}{1 - q}G(y_{n-1}, y_n, y_n),
\]
or
\[
G(y_n, y_{n+1}, y_{n+1}) \leq \lambda_1 G(y_{n-1}, y_n, y_n), \text{ where } \lambda_1 = \frac{p + r}{1 - q} < 1.
\]
If \( n \) is odd, then \( n = 2k + 1 \) for some \( k \in \mathbb{N} \). Again, from (3.1),
\[
G(y_n, y_{n+1}, y_{n+1}) = G(y_{2k+1}, y_{2k+2}, y_{2k+2}) = G(Sx_{2k+1}, Tx_{2k+2}, Tx_{2k+2})
\leq pG(Ax_{2k+1}, Bx_{2k+2}, Bx_{2k+2})
+ qG(Sx_{2k+1}, Sx_{2k+1}, Ax_{2k+1})
+ rG(Tx_{2k+2}, Tx_{2k+2}, Bx_{2k+2})
= pG(y_{2k+1}, y_{2k+1}, y_{2k+1}) + qG(y_{2k+1}, y_{2k+1}, y_{2k})
+ rG(y_{2k+2}, y_{2k+2}, y_{2k+1})
= pG(y_{n-1}, y_n, y_n) + qG(y_n, y_n, y_{n-1}) + rG(y_{n+1}, y_{n+1}, y_n),
\]
that is
\[
G(y_n, y_{n+1}, y_{n+1}) \leq \frac{p+q}{1-r}G(y_{n-1}, y_n, y_n),
\]
or \( G(y_n, y_{n+1}, y_{n+1}) \leq \lambda_2 G(y_{n-1}, y_n, y_n) \), where \( \lambda_2 = \frac{p+q}{1-r} < 1 \). Choose \( \lambda = \max\{\lambda_1, \lambda_2\} \).
Thus, for each \( n \in \mathbb{N} \), we have
\[
(3.3) \quad G(y_n, y_{n+1}, y_{n+1}) \leq \lambda^n G(y_0, y_1, y_1).
\]
Thus, if \( y_0 = y_1 \), we get \( G(y_n, y_{n+1}, y_{n+1}) = 0 \) for each \( n \in \mathbb{N} \). Hence \( y_n = y_0 \) for each \( n \in \mathbb{N} \). Therefore \( \{y_n\} \) is \( G \)-Cauchy. So we may assume that \( y_0 \neq y_1 \). Let \( n, m \in \mathbb{N} \) with \( m > n \). By axiom \((G_5)\) of the definition of a \( G \)-metric space, we have
\[
G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \ldots + G(y_{m-1}, y_m, y_m).
\]
By Equation (3.3), we get
\[
G(y_n, y_m, y_m) \leq \lambda^n G(y_0, y_1, y_1) + \lambda^{n+1} G(y_0, y_1, y_1) + \ldots + \lambda^{m-1} G(y_0, y_1, y_1)
= \lambda^n \sum_{i=0}^{m-1-n} q^i G(y_0, y_1, y_1) \leq \frac{\lambda^n}{1-\lambda} G(y_0, y_1, y_1).
\]
On taking limit \( m, n \to \infty \), we have
\[
\lim_{m, n \to \infty} G(y_n, y_m, y_m) = 0.
\]
So we conclude that \( \{y_n\} \) is a \( G \)-Cauchy sequence in \( X \). Since \( X \) is \( G \)-complete, then it yields that \( \{y_n\} \) and hence any subsequence of \( \{y_n\} \) converges to some \( z \in X \). So that, the subsequences \( \{Ax_{2n+1}\}, \{Bx_{2n+2}\}, \{Sx_{2n+1}\} \) and \( \{Tx_{2n}\} \) converge to \( z \). First suppose that \( A \) is sequentially continuous, so that
\[
\lim_{n \to \infty} A^2 x_{2n+1} = Az \quad \text{and} \quad \lim_{n \to \infty} AS x_{2n+1} = Az.
\]
Since \( \{A, S\} \) is weakly commuting, we have
\[
G(SA x_{2n+1}, AS x_{2n+1}, AS x_{2n+1}) \leq G(Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}).
\]
On taking the limit as \( n \to \infty \), we get that \( G(SA x_{2n+1}, Az, Az) \to 0 \). Thus, we have
\[
\lim_{n \to \infty} SA x_{2n+1} = Az.
\]
Assume \( Az \neq z \), we get
\[
G(SA x_{2n+1}, Tx_{2n}, Tx_{2n}) \leq pG(AAx_{2n+1}, Bx_{2n}, Bx_{2n}) + qG(SAx_{2n+1}, SAx_{2n+1}, AAx_{2n+1})
+ rG(Tx_{2n}, Tx_{2n}, Bx_{2n}).
\]
On letting \( n \to \infty \), we have
\[
G(Az, z, z) \leq pG(Az, z, z) + qG(Az, Az, Az) + rG(z, z, z).
\]
Since \( p < 1 \), we conclude that
\[
G(Az, z, z) < G(Az, z, z),
\]
which is a contradiction. So \( Az = z \). Also,
\[
G(Sz, Sz, Tx_{2n}) \leq pG(Az, Az, Bx_{2n}) + qG(Sz, Sz, Az) + rG(Tx_{2n}, Tx_{2n}, Bx_{2n}).
\]
By taking the limit as \( n \to \infty \), we have
\[
G(Sz, Sz, z) \leq pG(Az, Az, z) + qG(Sz,Sz, Az) + rG(z, z, z) \leq qG(Sz, Sz, z).
\]
Since \( q < 1 \), we get \( G(Sz, Sz, z) = 0 \). So \( Sz = z \). Suppose \( B \) is sequentially continuous,
then
\[
\lim_{n \to \infty} B(Bx_{2n}) = Bz \quad \text{and} \quad \lim_{n \to \infty} B(Tx_{2n}) = Bz.
\]
Assume \( Bz \neq z \). Since
\[
G(Sx_{2n+1}, Tx_{2n}, Tx_{2n}) \leq pG(Ax_{2n+1}, Bx_{2n}, Bx_{2n}) + qG(Sx_{2n+1}, Ax_{2n+1}) + rG(Tx_{2n}, Tx_{2n}, Bx_{2n}),
\]
Again taking the limit as \( n \to \infty \), implies
\[
G(z, Bz, Bz) \leq pG(z, Bz, Bz) + qG(z, z, z) + rG(z, Bz, Bz) < G(z, Bz, Bz),
\]
which is a contradiction. Hence \( Bz = z \). Since
\[
G(Sx_{2n+1}, Tz, Tz) \leq pG(Ax_{2n+1}, Bz, Bz) + qG(Sx_{2n+1}, Ax_{2n+1}) + rG(Tz, Tz, Bz),
\]
on taking the limit as \( n \to \infty \), we get
\[
G(z, Tz, Tz) \leq pG(z, Bz, Bz) + qG(z, z, z) + rG(Tz, Tz, Bz) \leq rG(z, Tz, Tz).
\]
Since \( r < 1 \), we get \( G(z, Tz, Tz) = 0 \). Hence \( Tz = z \). So \( z \) is a common fixed point for \( A, B, S \) and \( T \). To prove that \( z \) is the unique common fixed point let \( w \) be a common fixed point for \( A, B, S \) and \( T \) with \( w \neq z \). Then
\[
G(z, w, w) = G(Sz, Tw, Tw) \leq pG(Az, Bw, Bw) + qG(Sz, Sz, Az) + rG(Tw, Tw, Bw) = pG(z, w, w) + qG(z, z, z) + rG(w, w, w) = pG(z, w, w) < G(z, w, w),
\]
which is a contradiction. So \( z = w \).

\[\square\]

3.4. Corollary. Let \( X \) be a complete \( G \)-metric space, and let \( A, B, S, T : X \to X \) be mappings satisfying:
\[
G(Sx, Ty, Ty) \leq hG(Ax, By, By)
\]
and
\[
G(Sx, Sz, Ty) \leq hG(Ax, Az, By).
\]
Assume the maps $A, B, S$ and $T$ satisfy the following conditions:

1. $TX \subseteq AX$ and $SX \subseteq BX$,
2. The mappings $A$ and $B$ are sequentially continuous, and
3. The pairs $\{A, S\}$ and $\{B, T\}$ are weakly commuting.

If $h \in [0, 1)$, then $A, B, S$ and $T$ have a unique common fixed point. □

3.5. Corollary. Let $X$ be a complete $G$-metric space and let $A : X \to X$ be mappings satisfying:

$$G(Sx, Sy, Sz) \leq kG(Ax, Ay, Az)$$

for all $x, y \in X$. Assume the maps $A$ and $S$ satisfy the following conditions:

1. $SX \subseteq AX$,
2. The map $A$ is sequentially continuous, and
3. The pair $\{A, S\}$ is weakly commuting.

If $k \in [0, 1)$, then $A$ and $S$ have a unique common fixed point.

Proof. Define $B : X \to X$ by $Bx = Ax$ and define $T : X \to X$ by $Tx = Sx$. Then the four maps $A, B, S$ and $T$ satisfy all the hypothesis of Corollary 3.4. So, the result follows from Corollary 3.4. □

3.6. Corollary. Let $X$ be a complete $G$-metric space and let $S : X \to X$ be a mapping satisfying:

$$G(Sx, Sy, Sz) \leq qG(x, y, z)$$

for all $x, y \in X$. If $q \in [0, 1)$, then $S$ has a unique fixed point.

Proof. Follows from Corollary 3.5 by taking $A = B = I$ and $S = T$. □

Now, we introduce an example of Theorem 3.3.

3.7. Example. Let $X = [0, 1]$, Define $A, B, S, T : X \to X$ by $Ax = \frac{1}{2}x$, $Bx = \frac{3}{4}x$, $Sx = \frac{1}{8}x$, and $Tx = \frac{1}{16}x$. Then $TX \subseteq AX$, $SX \subseteq BX$. Note that the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x, y, z) = |x - y| + |x - z| + |y - z|.$$ 

Then $(X, G)$ is a complete $G$-metric. Also

$$G(Sx, Ty, Tz) = 2|Sx - Ty| = \frac{1}{8}|2x - y|,$$
$$G(Ax, By, Bz) = 2|Ax - By| = \frac{1}{2}|2x - y|,$$
$$G(Sx, Sx, Ty) = 2|Sx - Ty| = \frac{1}{8}|2x - y|,$$

and

$$G(Ax, Ax, By) = 2|Ax - By| = \frac{1}{2}|2x - y|.$$ 

So

$$G(Sx, Ty, Tz) \leq \frac{1}{2}G(Ax, By, Bz)$$

and

$$G(Sx, Sx, Ty) \leq \frac{1}{2}G(Ax, Ax, By).$$

Since $AS = SA$ and $BT = TB$, we conclude that the pairs $\{A, S\}$ and $\{B, T\}$ are weakly commuting. Note that $A, B, S$ and $T$ satisfy the hypothesis of Theorem 3.3. Here, $0$ is the unique common fixed point of $A, B, S$ and $T$. 

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