

## Base and subbase in intuitionistic $I$ -fuzzy topological spaces

Chengyu Liang\* and Conghua Yan †

### Abstract

In this paper, the concepts of the base and subbase in intuitionistic  $I$ -fuzzy topological spaces are introduced, and use them to discuss fuzzy continuous mapping and fuzzy open mapping. We also study the base and subbase in the product of intuitionistic  $I$ -fuzzy topological spaces, and  $T_2$  separation in product intuitionistic  $I$ -fuzzy topological spaces. Finally, the relation between the generated product intuitionistic  $I$ -fuzzy topological spaces and the product generated intuitionistic  $I$ -fuzzy topological spaces are studied.

**Keywords:** Intuitionistic  $I$ -fuzzy topological space; Base; Subbase;  $T_2$  separation; Generated Intuitionistic  $I$ -fuzzy topological spaces.

*2000 AMS Classification:* 54A40

### 1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was first introduced by Atanassov [1]. From then on, this theory has been studied and applied in a variety areas ([4, 14, 18], etc). Among of them, the research of the theory of intuitionistic fuzzy topology is similar to the the theory of fuzzy topology. In fact, Çoker [4] introduced the concept of intuitionistic fuzzy topological spaces, this concept is originated from the fuzzy topology in the sense of Chang [3](in this paper we call it intuitionistic  $I$ -topological spaces). Based on Çoker's work [4], many topological properties of intuitionistic  $I$ -topological spaces has been discussed ([5, 10, 11, 12, 13]). On the other hand, Šostak [17] proposed a new notion of fuzzy topological spaces, and this new fuzzy topological structure has been accepted widely. Influenced by Šostak's work [17], Çoker [7] gave the notion of intuitionistic fuzzy topological spaces in the sense of Šostak. By the standardized terminology introduced in [16], we will call it intuitionistic  $I$ -fuzzy

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\*Department of Mathematics, School of Science, Beijing Institute of Technology, Beijing 100081, PR China, Email: liangchengyu87@163.com

†Corresponding Author, Institute of Math., School oh Math. Sciences, Nanjing Normal University, Nanjing, Jiangsu 210023, PR China,

topological spaces in this paper. In [15], the authors studied the compactness in intuitionistic  $I$ -fuzzy topological spaces.

Recently, Yan and Wang [19] generalized Fang and Yue's work ([8, 21]) from  $I$ -fuzzy topological spaces to intuitionistic  $I$ -fuzzy topological spaces. In [19], they introduced the concept of intuitionistic  $I$ -fuzzy quasi-coincident neighborhood systems of intuitionistic fuzzy points, and construct the notion of generated intuitionistic  $I$ -fuzzy topology by using fuzzifying topologies. As an important result, Yan and Wang proved that the category of intuitionistic  $I$ -fuzzy topological spaces is isomorphic to the category of intuitionistic  $I$ -fuzzy quasi-coincident neighborhood spaces in [19].

It is well known that base and subbase are very important notions in classical topology. They also discussed in  $I$ -fuzzy topological spaces by Fang and Yue [9]. As a subsequent work of Yan and Wang [19], the main purpose of this paper is to introduce the concepts of the base and subbase in intuitionistic  $I$ -fuzzy topological spaces, and use them to discuss fuzzy continuous mapping and fuzzy open mapping. Then we also study the base and subbase in the product of intuitionistic  $I$ -fuzzy topological spaces, and  $T_2$  separation in product intuitionistic  $I$ -fuzzy topological spaces. Finally, we obtain that the generated product intuitionistic  $I$ -fuzzy topological spaces is equal to the product generated intuitionistic  $I$ -fuzzy topological spaces.

Throughout this paper, let  $I = [0, 1]$ ,  $X$  a nonempty set, the family of all fuzzy sets and intuitionistic fuzzy sets on  $X$  be denoted by  $I^X$  and  $\zeta^X$ , respectively. The notation  $\text{pt}(I^X)$  denotes the set of all fuzzy points on  $X$ . For all  $\lambda \in I$ ,  $\underline{\lambda}$  denotes the fuzzy set on  $X$  which takes the constant value  $\lambda$ . For all  $A \in \zeta^X$ , let  $A = \langle \mu_A, \gamma_A \rangle$ . (For the relating to knowledge of intuitionistic fuzzy sets and intuitionistic  $I$ -fuzzy topological spaces, we may refer to [1] and [19].)

## 2. Some preliminaries

**2.1. Definition.** ([20]) A fuzzifying topology on a set  $X$  is a function  $\tau : 2^X \rightarrow I$ , such that

- (1)  $\tau(\emptyset) = \tau(X) = 1$ ;
- (2)  $\forall A, B \subseteq X, \tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ ;
- (3)  $\forall A_t \subseteq X, t \in T, \tau(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \tau(A_t)$ .

The pair  $(X, \tau)$  is called a fuzzifying topological space.

**2.2. Definition.** ([1, 2]) Let  $a, b$  be two real numbers in  $[0, 1]$  satisfying the inequality  $a + b \leq 1$ . Then the pair  $\langle a, b \rangle$  is called an intuitionistic fuzzy pair.

Let  $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle$  be two intuitionistic fuzzy pairs, then we define

- (1)  $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$  if and only if  $a_1 \leq a_2$  and  $b_1 \geq b_2$ ;
- (2)  $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$  if and only if  $a_1 = a_2$  and  $b_1 = b_2$ ;
- (3) if  $\langle a_j, b_j \rangle_{j \in J}$  is a family of intuitionistic fuzzy pairs, then  $\bigvee_{j \in J} \langle a_j, b_j \rangle = \langle \bigvee_{j \in J} a_j, \bigwedge_{j \in J} b_j \rangle$ , and  $\bigwedge_{j \in J} \langle a_j, b_j \rangle = \langle \bigwedge_{j \in J} a_j, \bigvee_{j \in J} b_j \rangle$ ;
- (4) the complement of an intuitionistic fuzzy pair  $\langle a, b \rangle$  is the intuitionistic fuzzy pair defined by  $\overline{\langle a, b \rangle} = \langle b, a \rangle$ ;

In the following, for convenience, we will use the symbols  $1^\sim$  and  $0^\sim$  denote the intuitionistic fuzzy pairs  $\langle 1, 0 \rangle$  and  $\langle 0, 1 \rangle$ . The family of all intuitionistic fuzzy pairs is denoted by  $\mathcal{A}$ . It is easy to find that the set of all intuitionistic fuzzy pairs with above order forms a complete lattice, and  $1^\sim, 0^\sim$  are its top element and bottom element, respectively.

**2.3. Definition.** ([4]) Let  $X, Y$  be two nonempty sets and  $f : X \rightarrow Y$  a function, if  $B = \{\langle y, \mu_B(y), \gamma_B(y) \rangle : y \in Y\} \in \zeta^Y$ , then the preimage of  $B$  under  $f$ , denoted by  $f^{\leftarrow}(B)$ , is the intuitionistic fuzzy set defined by

$$f^{\leftarrow}(B) = \{\langle x, f^{\leftarrow}(\mu_B)(x), f^{\leftarrow}(\gamma_B)(x) \rangle : x \in X\}.$$

Here  $f^{\leftarrow}(\mu_B)(x) = \mu_B(f(x))$ ,  $f^{\leftarrow}(\gamma_B)(x) = \gamma_B(f(x))$ . (This notation is from [16]).

If  $A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X\} \in \zeta^X$ , then the image  $A$  under  $f$ , denoted by  $f^{\rightarrow}(A)$  is the intuitionistic fuzzy set defined by

$$f^{\rightarrow}(A) = \{\langle y, f^{\rightarrow}(\mu_A)(y), (\underline{1} - f^{\rightarrow}(\underline{1} - \gamma_A))(y) \rangle : y \in Y\}.$$

Where

$$f^{\rightarrow}(\mu_A)(y) = \begin{cases} \sup_{x \in f^{\leftarrow}(y)} \mu_A(x), & \text{if } f^{\leftarrow}(y) \neq \emptyset, \\ 0, & \text{if } f^{\leftarrow}(y) = \emptyset. \end{cases}$$

$$\underline{1} - f^{\rightarrow}(\underline{1} - \gamma_A)(y) = \begin{cases} \inf_{x \in f^{\leftarrow}(y)} \gamma_A(x), & \text{if } f^{\leftarrow}(y) \neq \emptyset, \\ 1, & \text{if } f^{\leftarrow}(y) = \emptyset. \end{cases}$$

**2.4. Definition.** ([7]) Let  $X$  be a nonempty set,  $\delta : \zeta^X \rightarrow \mathcal{A}$  satisfy the following:

- (1)  $\delta(\langle \underline{0}, \underline{1} \rangle) = \delta(\langle \underline{1}, \underline{0} \rangle) = 1^\sim$ ;
- (2)  $\forall A, B \in \zeta^X, \delta(A \wedge B) \geq \delta(A) \wedge \delta(B)$ ;
- (3)  $\forall A_t \in \zeta^X, t \in T, \delta(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \delta(A_t)$ .

Then  $\delta$  is called an intuitionistic  $I$ -fuzzy topology on  $X$ , and the pair  $(X, \delta)$  is called an intuitionistic  $I$ -fuzzy topological space. For any  $A \in \zeta^X$ , we always suppose that  $\delta(A) = \langle \mu_\delta(A), \gamma_\delta(A) \rangle$  later, the number  $\mu_\delta(A)$  is called the openness degree of  $A$ , while  $\gamma_\delta(A)$  is called the nonopenness degree of  $A$ . A fuzzy continuous mapping between two intuitionistic  $I$ -fuzzy topological spaces  $(\zeta^X, \delta_1)$  and  $(\zeta^Y, \delta_2)$  is a mapping  $f : X \rightarrow Y$  such that  $\delta_1(f^{\leftarrow}(A)) \geq \delta_2(A)$ . The category of intuitionistic  $I$ -fuzzy topological spaces and fuzzy continuous mappings is denoted by  $II\text{-FTOP}$ .

**2.5. Definition.** ([6, 11, 12]) Let  $X$  be a nonempty set. An intuitionistic fuzzy point, denoted by  $x_{(\alpha, \beta)}$ , is an intuitionistic fuzzy set  $A = \{\langle y, \mu_A(y), \gamma_A(y) \rangle : y \in X\}$ , such that

$$\mu_A(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

and

$$\gamma_A(y) = \begin{cases} \beta, & \text{if } y = x, \\ 1, & \text{if } y \neq x. \end{cases}$$

Where  $x \in X$  is a fixed point, the constants  $\alpha \in I_0$ ,  $\beta \in I_1$  and  $\alpha + \beta \leq 1$ . The set of all intuitionistic fuzzy points  $x_{(\alpha, \beta)}$  is denoted by  $\text{pt}(\zeta^X)$ .

**2.6. Definition.** ([12]) Let  $x_{(\alpha, \beta)} \in \text{pt}(\zeta^X)$  and  $A, B \in \zeta^X$ . We say  $x_{(\alpha, \beta)}$  quasi-coincides with  $A$ , or  $x_{(\alpha, \beta)}$  is quasi-coincident with  $A$ , denoted  $x_{(\alpha, \beta)} \hat{q}A$ , if  $\mu_A(x) + \alpha > 1$  and  $\gamma_A(x) + \beta < 1$ . Say  $A$  quasi-coincides with  $B$  at  $x$ , or say  $A$  is quasi-coincident with  $B$  at  $x$ ,  $A \hat{q}B$  at  $x$ , in short, if  $\mu_A(x) + \mu_B(x) > 1$  and  $\gamma_A(x) + \gamma_B(x) < 1$ . Say  $A$  quasi-coincides with  $B$ , or  $A$  is quasi-coincident with  $B$ , if  $A$  is quasi-coincident with  $B$  at some point  $x \in X$ .

Relation “does not quasi-coincides with” or “is not quasi-coincident with ” is denoted by  $\neg \hat{q}$ .

It is easily to know for  $\forall x_{(\alpha, \beta)} \in \text{pt}(\zeta^X)$ ,  $x_{(\alpha, \beta)} \hat{q} < \underline{1}, \underline{0} >$  and  $x_{(\alpha, \beta)} \neg \hat{q} < \underline{0}, \underline{1} >$ .

**2.7. Definition.** ([19]) Let  $(X, \delta)$  be an intuitionistic  $I$ -fuzzy topological space. For all  $x_{(\alpha, \beta)} \in \text{pt}(\zeta^X)$ ,  $U \in \zeta^X$ , the mapping  $Q_{x_{(\alpha, \beta)}}^\delta : \zeta^X \rightarrow \mathcal{A}$  is defined as follows

$$Q_{x_{(\alpha, \beta)}}^\delta(U) = \begin{cases} \bigvee_{x_{(\alpha, \beta)} \hat{q} V \leq U} \delta(V), & x_{(\alpha, \beta)} \hat{q} U; \\ 0^\sim, & x_{(\alpha, \beta)} \neg \hat{q} U. \end{cases}$$

The set of  $Q^\delta = \{Q_{x_{(\alpha, \beta)}}^\delta : x_{(\alpha, \beta)} \in \text{pt}(\zeta^X)\}$  is called intuitionistic  $I$ -fuzzy quasi-coincident neighborhood system of  $\delta$  on  $X$ .

**2.8. Theorem.** ([19]) Let  $(X, \delta)$  be an intuitionistic  $I$ -fuzzy topological space,  $Q^\delta = \{Q_{x_{(\alpha, \beta)}}^\delta : x_{(\alpha, \beta)} \in \text{pt}(\zeta^X)\}$  of maps  $Q_{x_{(\alpha, \beta)}}^\delta : \zeta^X \rightarrow \mathcal{A}$  defined in Definition 2.7 satisfies:  $\forall U, V \in \zeta^X$ ,

- (1)  $Q_{x_{(\alpha, \beta)}}^\delta(\langle \underline{1}, \underline{0} \rangle) = 1^\sim$ ,  $Q_{x_{(\alpha, \beta)}}^\delta(\langle \underline{0}, \underline{1} \rangle) = 0^\sim$ ;
- (2)  $Q_{x_{(\alpha, \beta)}}^\delta(U) > 0^\sim \Rightarrow x_{(\alpha, \beta)} \hat{q} U$ ;
- (3)  $Q_{x_{(\alpha, \beta)}}^\delta(U \wedge V) = Q_{x_{(\alpha, \beta)}}^\delta(U) \wedge Q_{x_{(\alpha, \beta)}}^\delta(V)$ ;
- (4)  $Q_{x_{(\alpha, \beta)}}^\delta(U) = \bigvee_{x_{(\alpha, \beta)} \hat{q} V \leq U} \bigwedge_{y_{(\lambda, \rho)} \hat{q} V} Q_{y_{(\lambda, \rho)}}^\delta(V)$ ;
- (5)  $\delta(U) = \bigwedge_{x_{(\alpha, \beta)} \hat{q} U} Q_{x_{(\alpha, \beta)}}^\delta(U)$ .

**2.9. Lemma.** ([21]) Suppose that  $(X, \tau)$  is a fuzzifying topological space, for each  $A \in I^X$ , let  $\omega(\tau)(A) = \bigwedge_{r \in I} \tau(\sigma_r(A))$ , where  $\sigma_r(A) = \{x : A(x) > r\}$ . Then  $\omega(\tau)$  is an  $I$ -fuzzy topology on  $X$ , and  $\omega(\tau)$  is called induced  $I$ -fuzzy topology determined by fuzzifying topology  $\tau$ .

**2.10. Definition.** ([19]) Let  $(X, \tau)$  be a fuzzifying topological space,  $\omega(\tau)$  is an induced  $I$ -fuzzy topology determined by fuzzifying topology  $\tau$ . For each  $A \in \zeta^X$ , let  $\text{I}\omega(\tau)(A) = \langle \mu^\tau(A), \gamma^\tau(A) \rangle$ , where  $\mu^\tau(A) = \omega(\tau)(\mu_A) \wedge \omega(\tau)(\underline{1} - \gamma_A)$ ,  $\gamma^\tau(A) = 1 - \mu^\tau(A)$ . We say that  $(\zeta^X, \text{I}\omega(\tau))$  is a generated intuitionistic  $I$ -fuzzy topological space by fuzzifying topological space  $(X, \tau)$ .

**2.11. Lemma.** ([19]) Let  $(X, \tau)$  be a fuzzifying topological space, then

- (1)  $\forall A \subseteq X$ ,  $\mu^\tau(\langle 1_A, 1_{A^c} \rangle) = \tau(A)$ .
- (2)  $\forall A = \langle \underline{\alpha}, \underline{\beta} \rangle \in \zeta^X$ ,  $\text{I}\omega(\tau)(A) = 1^\sim$ .

**2.12. Lemma.** ([19]) *Suppose that  $(\zeta^X, \delta)$  is an intuitionistic  $I$ -fuzzy topological space, for each  $A \subseteq X$ , let  $[\delta](A) = \mu_\delta(\langle 1_A, 1_{A^c} \rangle)$ . Then  $[\delta]$  is a fuzzifying topology on  $X$ .*

**2.13. Lemma.** ([19]) *Let  $(X, \tau)$  be a fuzzifying topological space and  $(X, I\omega(\tau))$  a generated intuitionistic  $I$ -fuzzy topological space. Then  $[I\omega(\tau)] = \tau$ .*

### 3. Base and subbase in Intuitionistic $I$ -fuzzy topological spaces

**3.1. Definition.** Let  $(X, \tau)$  be an intuitionistic  $I$ -fuzzy topological space and  $\mathcal{B} : \zeta^X \rightarrow \mathcal{A}$ .  $\mathcal{B}$  is called a base of  $\tau$  if  $\mathcal{B}$  satisfies the following condition

$$\tau(U) = \bigvee_{\substack{\bigvee_{\lambda \in K} B_\lambda = U \\ \lambda \in K}} \bigwedge_{\lambda \in K} \mathcal{B}(B_\lambda), \forall U \in \zeta^X.$$

**3.2. Definition.** Let  $(X, \tau)$  be an intuitionistic  $I$ -fuzzy topological space and  $\varphi : \zeta^X \rightarrow \mathcal{A}$ ,  $\varphi$  is called a subbase of  $\tau$  if  $\varphi^{(\cap)} : \zeta^X \rightarrow \mathcal{A}$  is a base, where  $\varphi^{(\cap)}(A) = \bigvee_{\cap\{B_\lambda : \lambda \in E\} = A} \bigwedge_{\lambda \in E} \varphi(B_\lambda)$ , for all  $A \in \zeta^X$  with  $(\cap)$  standing for “finite intersection”.

**3.3. Theorem.** *Suppose that  $\mathcal{B} : \zeta^X \rightarrow \mathcal{A}$ . Then  $\mathcal{B}$  is a base of some intuitionistic  $I$ -fuzzy topology, if  $\mathcal{B}$  satisfies the following condition*

- (1)  $\mathcal{B}(0_\sim) = \mathcal{B}(1_\sim) = 1_\sim$ ,
- (2)  $\forall U, V \in \zeta^X, \mathcal{B}(U \wedge V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$ .

*Proof.* For  $\forall A \in \zeta^X$ , let  $\tau(A) = \bigvee_{\substack{\bigvee_{\lambda \in K} B_\lambda = A \\ \lambda \in K}} \bigwedge_{\lambda \in K} \mathcal{B}(B_\lambda)$ . To show that  $\mathcal{B}$  is a base

of  $\tau$ , we only need to prove  $\tau$  is an intuitionistic  $I$ -fuzzy topology on  $X$ . For all  $U, V \in \zeta^X$ ,

$$\begin{aligned} \tau(U) \wedge \tau(V) &= \left( \bigvee_{\alpha \in K_1} \bigwedge_{A_\alpha = U} \mathcal{B}(A_\alpha) \right) \wedge \left( \bigvee_{\beta \in K_2} \bigwedge_{B_\beta = V} \mathcal{B}(B_\beta) \right) \\ &= \bigvee_{\alpha \in K_1} \bigvee_{\beta \in K_2} \left( \left( \bigwedge_{A_\alpha = U} \mathcal{B}(A_\alpha) \right) \wedge \left( \bigwedge_{B_\beta = V} \mathcal{B}(B_\beta) \right) \right) \\ &\leq \bigvee_{\alpha \in K_1, \beta \in K_2} \left( \bigwedge_{(A_\alpha \wedge B_\beta) = U \wedge V} \mathcal{B}(A_\alpha \wedge B_\beta) \right) \\ &\leq \bigvee_{\lambda \in K} \bigwedge_{C_\lambda = U \wedge V} \mathcal{B}(C_\lambda) \\ &= \tau(U \wedge V). \end{aligned}$$

For all  $\{A_\lambda : \lambda \in K\} \subseteq \zeta^X$ , Let  $\mathcal{B}_\lambda = \{\{B_{\delta_\lambda} : \delta_\lambda \in K_\lambda\} : \bigvee_{\delta_\lambda \in K_\lambda} B_{\delta_\lambda} = A_\lambda\}$ , then

$$\tau\left(\bigvee_{\lambda \in K} A_\lambda\right) = \bigvee_{\delta \in K_1} \bigwedge_{\substack{B_\delta = \bigvee_{\lambda \in K} A_\lambda \\ \delta \in K_1}} \mathcal{B}(B_\delta).$$

For all  $f \in \prod_{\lambda \in K} \mathcal{B}_\lambda$ , we have

$$\bigvee_{\lambda \in K} \bigvee_{B_{\delta_\lambda} \in f(\lambda)} B_{\delta_\lambda} = \bigvee_{\lambda \in K} A_\lambda.$$

Therefore,

$$\begin{aligned} \mu_{\tau(\bigvee_{\lambda \in K} A_\lambda)} &= \bigvee_{\delta \in K_1} \bigwedge_{B_\delta = \bigvee_{\lambda \in K} A_\lambda} \bigwedge_{\delta \in K_1} \mu_{\mathcal{B}(B_\delta)} \\ &\geq \bigvee_{f \in \prod_{\lambda \in K} \mathcal{B}_\lambda} \bigwedge_{\lambda \in K} \bigwedge_{B_{\delta_\lambda} \in f(\lambda)} \mu_{\mathcal{B}(B_{\delta_\lambda})} \\ &= \bigwedge_{\lambda \in K} \bigvee_{\{B_{\delta_\lambda} : \delta_\lambda \in K_\lambda\} \in \mathcal{B}_\lambda} \bigwedge_{\delta_\lambda \in K_\lambda} \mu_{\mathcal{B}(B_{\delta_\lambda})} \\ &= \bigwedge_{\lambda \in E} \mu_{\tau(A_\lambda)}. \end{aligned}$$

Similarly, we have

$$\gamma_{\tau(\bigvee_{\lambda \in K} A_\lambda)} \leq \bigvee_{\lambda \in K} \gamma_{\tau(A_\lambda)}.$$

Hence

$$\tau(\bigvee_{\lambda \in K} A_\lambda) \geq \bigwedge_{\lambda \in K} \tau(A_\lambda).$$

This means that  $\tau$  is an intuitionistic  $I$ -fuzzy topology on  $X$  and  $\mathcal{B}$  is a base of  $\tau$ .  $\square$

**3.4. Theorem.** *Let  $(X, \tau), (Y, \delta)$  be two intuitionistic  $I$ -fuzzy topology spaces and  $\delta$  generated by its subbase  $\varphi$ . The mapping  $f : (X, \tau) \rightarrow (Y, \delta)$  satisfies  $\varphi(U) \leq \tau(f^{\leftarrow}(U))$ , for all  $U \in \zeta^Y$ . Then  $f$  is fuzzy continuous, i.e.,  $\delta(U) \leq \tau(f^{\leftarrow}(U)), \forall U \in \zeta^Y$ .*

*Proof.*  $\forall U \in \zeta^Y$ ,

$$\begin{aligned} \delta(U) &= \bigvee_{\lambda \in K} \bigwedge_{A_\lambda = U} \bigwedge_{\lambda \in K} \bigvee_{\{B_\mu : \mu \in K_\lambda\} = A_\lambda} \bigwedge_{\mu \in K_\lambda} \varphi(B_\mu) \\ &\leq \bigvee_{\lambda \in K} \bigwedge_{A_\lambda = U} \bigwedge_{\lambda \in K} \bigvee_{\{B_\mu : \mu \in K_\lambda\} = A_\lambda} \bigwedge_{\mu \in K_\lambda} \tau(f^{\leftarrow}(B_\mu)) \\ &\leq \bigvee_{\lambda \in K} \bigwedge_{A_\lambda = U} \bigwedge_{\lambda \in K} \tau(f^{\leftarrow}(A_\lambda)) \\ &\leq \bigvee_{\lambda \in K} \bigwedge_{A_\lambda = U} \tau(f^{\leftarrow}(\bigvee_{\lambda \in K} A_\lambda)) \\ &= \tau(f^{\leftarrow}(U)). \end{aligned}$$

This completes the proof.  $\square$

**3.5. Theorem.** Suppose that  $(X, \tau)$ ,  $(Y, \delta)$  are two intuitionistic I-fuzzy topology spaces and  $\tau$  is generated by its base  $\mathcal{B}$ . If the mapping  $f : (X, \tau) \rightarrow (Y, \delta)$  satisfies  $\mathcal{B}(U) \leq \delta(f \rightarrow(U))$ , for all  $U \in \zeta^X$ . Then  $f$  is fuzzy open, i.e.,  $\forall W \in \zeta^X, \tau(W) \leq \delta(f \rightarrow(W))$ .

*Proof.*  $\forall W \in \zeta^X$ ,

$$\begin{aligned} \tau(W) &= \bigvee_{\lambda \in K} \bigwedge_{A_\lambda = W} \mathcal{B}(A_\lambda) \\ &\leq \bigvee_{\lambda \in K} \bigwedge_{A_\lambda = W} \delta(f \rightarrow(A_\lambda)) \\ &\leq \bigvee_{\lambda \in K} \delta(f \rightarrow(\bigvee_{\lambda \in K} A_\lambda)) \\ &= \delta(f \rightarrow(W)). \end{aligned}$$

Therefore,  $f$  is open.  $\square$

**3.5. Theorem.** Let  $(X, \tau), (Y, \delta)$  be two intuitionistic I-fuzzy topology spaces and  $f : (X, \tau) \rightarrow (Y, \delta)$  intuitionistic I-fuzzy continuous,  $Z \subseteq X$ . Then  $f|_Z : (Z, \tau|_Z) \rightarrow (Y, \delta)$  is continuous, where  $(f|_Z)(x) = f(x)$ ,  $(\tau|_Z)(A) = \vee\{\tau(U) : U|_Z = A\}$ , for all  $x \in Z, A \in \zeta^Z$ .

*Proof.*  $\forall W \in \zeta^Z, (f|_Z) \leftarrow(W) = f \leftarrow(W)|_Z$ , we have

$$\begin{aligned} (\tau|_Z)((f|_Z) \leftarrow(W)) &= \vee\{\tau(U) : U|_Z = (f|_Z) \leftarrow(W)\} \\ &\geq \tau(f \leftarrow(W)) \\ &\geq \delta(W). \end{aligned}$$

Then  $f|_Z$  is intuitionistic I-fuzzy continuous.  $\square$

**3.6. Theorem.** Let  $(X, \tau)$  be an intuitionistic I-fuzzy topology space and  $\tau$  generated by its base  $\mathcal{B}$ ,  $\mathcal{B}|_Y(U) = \vee\{\mathcal{B}(W) : W|_Y = U\}$ , for  $Y \subseteq X, U \in \zeta^Y$ . Then  $\mathcal{B}|_Y$  is a base of  $\tau|_Y$ .

*Proof.* For  $\forall U \in \zeta^X, (\tau|_Y)(U) = \bigvee_{V|_Y=U} \tau(V) = \bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mathcal{B}(A_\lambda)$ . It

remains to show the following equality

$$\bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mathcal{B}(A_\lambda) = \bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

In one hand, for all  $V \in \zeta^X$  with  $V|_Y = U$ , and  $\bigvee_{\lambda \in K} A_\lambda = V$ , we have  $\bigvee_{\lambda \in K} A_\lambda|_Y = U$ . Put  $B_\lambda = A_\lambda|_Y$ , clearly  $\bigvee_{\lambda \in K} B_\lambda = U$ . Then

$$\bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W) \geq \bigwedge_{\lambda \in K} \mathcal{B}(A_\lambda).$$

Thus,

$$\bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mathcal{B}(A_\lambda) \leq \bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

On the other hand,  $\forall a \in (0, 1], a < \bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mu_{\mathcal{B}(W)}$ , there exists a family of  $\{B_\lambda : \lambda \in K\} \subseteq \zeta^Y$ , such that

$$(1) \bigvee_{\lambda \in K} B_\lambda = U;$$

(2)  $\forall \lambda \in K$ , there exists  $W_\lambda \in \zeta^X$  with  $W_\lambda|_Y = B_\lambda$  such that  $a < \mu_{\mathcal{B}(W_\lambda)}$ .

Let  $V = \bigvee_{\lambda \in E} W_\lambda$ , it is clear  $V|_Y = U$  and  $\bigwedge_{\lambda \in K} \mu_{\mathcal{B}(W_\lambda)} \geq a$ . Then

$$\bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mu_{\mathcal{B}(A_\lambda)} \geq a.$$

By the arbitrariness of  $a$ , we have

$$\bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mu_{\mathcal{B}(A_\lambda)} \geq \bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mu_{\mathcal{B}(W)}.$$

Similarly, we may obtain that

$$\bigwedge_{V|_Y=U} \bigwedge_{\lambda \in K} \bigvee_{A_\lambda=V} \gamma_{\mathcal{B}(A_\lambda)} \leq \bigwedge_{\lambda \in K} \bigvee_{B_\lambda=U} \bigwedge_{W|_Y=B_\lambda} \gamma_{\mathcal{B}(W)}.$$

So we have

$$\bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mathcal{B}(A_\lambda) \geq \bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

Therefore,

$$\bigvee_{V|_Y=U} \bigvee_{\lambda \in K} \bigwedge_{A_\lambda=V} \mathcal{B}(A_\lambda) = \bigvee_{\lambda \in K} \bigwedge_{B_\lambda=U} \bigvee_{W|_Y=B_\lambda} \mathcal{B}(W).$$

This means that  $\mathcal{B}|_Y$  is a base of  $\tau|_Y$ . □

**3.7. Theorem.** Let  $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in J}$  be a family of intuitionistic I-fuzzy topology spaces and  $P_\beta : \prod_{\alpha \in J} X_\alpha \rightarrow X_\beta$  the projection. For all  $W \in \zeta^{\prod_{\alpha \in J} X_\alpha}$ ,  $\varphi(W) = \bigvee_{\alpha \in J} \bigvee_{P_\alpha^+(U)=W} \tau_\alpha(U)$ . Then  $\varphi$  is a subbase of some intuitionistic I-fuzzy topology  $\tau$ , here  $\tau$  is called the product intuitionistic I-fuzzy topologies of  $\{\tau_\alpha : \alpha \in J\}$  and denoted by  $\tau = \prod_{\alpha \in J} \tau_\alpha$ .



*Proof.* We need to prove  $\varphi^{(\cap)}$  is a subbase of  $\tau$ .

$$\begin{aligned}\varphi^{(\cap)}(1_{\sim}) &= \bigvee_{\cap\{B_\lambda:\lambda\in E\}=1_{\sim}} \bigwedge_{\lambda\in E} \varphi(B_\lambda) \\ &= \bigvee_{\cap\{B_\lambda:\lambda\in E\}=1_{\sim}} \bigwedge_{\lambda\in E} \bigvee_{\alpha\in J} \bigvee_{P_\alpha^-(U)=B_\lambda} \tau_\alpha(U) \\ &= 1_{\sim}.\end{aligned}$$

Similarly,  $\varphi^{(\cap)}(0_{\sim}) = 1_{\sim}$ . For all  $U, V \in \zeta^{\prod_{\alpha\in J} X_\alpha}$ , we have

$$\begin{aligned}\varphi^{(\cap)}(U) \wedge \varphi^{(\cap)}(V) &= \left( \bigvee_{\cap\{B_\alpha:\alpha\in E_1\}=U} \bigwedge_{\alpha\in E_1} \varphi(B_\alpha) \right) \wedge \left( \bigvee_{\cap\{C_\beta:\beta\in E_2\}=V} \bigwedge_{\beta\in E_2} \varphi(C_\beta) \right) \\ &= \bigvee_{\cap\{B_\alpha:\alpha\in E_1\}=U} \bigvee_{\cap\{C_\beta:\beta\in E_2\}=V} \left( \bigwedge_{\alpha\in E_1} \varphi(B_\alpha) \right) \wedge \left( \bigwedge_{\beta\in E_2} \varphi(C_\beta) \right) \\ &\leq \bigvee_{\cap\{B_\lambda:\lambda\in E\}=U\wedge V} \bigwedge_{\lambda\in E} \varphi(B_\lambda) \\ &= \varphi^{(\cap)}(U \wedge V).\end{aligned}$$

Hence,  $\varphi^{(\cap)}$  is a base of  $\tau$ , i.e.,  $\varphi$  is a subbase of  $\tau$ . And by Theorem 3.3 we have

$$\begin{aligned}\tau(A) &= \bigvee_{\substack{\bigvee_{\lambda\in K} B_\lambda=A \\ \lambda\in K}} \bigwedge_{\lambda\in K} \varphi^{(\cap)}(B_\lambda) \\ &= \bigvee_{\substack{\bigvee_{\lambda\in K} B_\lambda=A \\ \lambda\in K}} \bigwedge_{\lambda\in K} \bigvee_{\cap\{C_\rho:\rho\in E\}=B_\lambda} \bigwedge_{\rho\in E} \varphi(C_\rho) \\ &= \bigvee_{\substack{\bigvee_{\lambda\in K} B_\lambda=A \\ \lambda\in K}} \bigwedge_{\lambda\in K} \bigvee_{\cap\{C_\rho:\rho\in E\}=B_\lambda} \bigwedge_{\rho\in E} \bigvee_{\alpha\in J} \bigvee_{P_\alpha^-(V)=C_\rho} \tau_\alpha(V).\end{aligned}$$

□

By the above discussions, we easily obtain the following corollary.

**3.8. Corollary.** Let  $(\prod_{\alpha\in J} X_\alpha, \prod_{\alpha\in J} \tau_\alpha)$  be the product space of a family of intuitionistic I-fuzzy topology spaces  $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in J}$ . Then  $P_\beta : (\prod_{\alpha\in J} X_\alpha, \prod_{\alpha\in J} \tau_\alpha) \rightarrow (X_\beta, \tau_\beta)$  is continuous, for all  $\beta \in J$ .

*Proof.*  $\forall U \in \zeta^{X_\beta}$ ,

$$\begin{aligned}\tau(P_\beta^-(U)) &= \bigvee_{\substack{\bigvee_{\lambda\in K} B_\lambda=P_\beta^-(U) \\ \lambda\in K}} \bigwedge_{\lambda\in K} \bigvee_{\cap\{C_\rho:\rho\in E\}=B_\lambda} \bigwedge_{\rho\in E} \bigvee_{\alpha\in J} \bigvee_{P_\alpha^-(V)=C_\rho} \tau_\alpha(V) \\ &\geq \tau_\beta(U)\end{aligned}$$

Therefore,  $P_\beta$  is continuous. □

#### 4. Applications in product Intuitionistic I-fuzzy topological space

**4.1. Definition.** Let  $(X, \tau)$  be an intuitionistic  $I$ -fuzzy topology space. The degree to which two distinguished intuitionistic fuzzy points  $x_{(\alpha, \beta)}, y_{(\lambda, \rho)} \in \text{pt}(\zeta^X) (x \neq y)$  are  $T_2$  is defined as follows

$$T_2(x_{(\alpha, \beta)}, y_{(\lambda, \rho)}) = \bigvee_{U \wedge V = 0_{\sim}} (Q_{x_{(\alpha, \beta)}}(U) \wedge Q_{y_{(\lambda, \rho)}}(V)).$$

The degree to which  $(X, \tau)$  is  $T_2$  is defined by

$$T_2(X, \tau) = \bigwedge \{T_2(x_{(\alpha, \beta)}, y_{(\lambda, \rho)}) : x_{(\alpha, \beta)}, y_{(\lambda, \rho)} \in \text{pt}(\zeta^X), x \neq y\}.$$

**4.2. Theorem.** Let  $(X, \text{I}\omega(\tau))$  be a generated intuitionistic  $I$ -fuzzy topological space by fuzzifying topological space  $(X, \tau)$  and  $T_2(X, \text{I}\omega(\tau)) \triangleq \langle \mu_{T_2(X, \text{I}\omega(\tau))}, \gamma_{T_2(X, \text{I}\omega(\tau))} \rangle$ . Then  $\mu_{T_2(X, \text{I}\omega(\tau))} = T_2(X, \tau)$ .

*Proof.* For all  $x, y \in X, x \neq y$ , and each  $a < \bigwedge \{ \bigvee_{U \wedge V = 0_{\sim}} (\mu_{Q_{x_{(\alpha, \beta)}}}(U) \wedge \mu_{Q_{y_{(\lambda, \rho)}}}(V)) : x_{(\alpha, \beta)}, y_{(\lambda, \rho)} \in \text{pt}(\zeta^X), x \neq y \}$ , there exists  $U, V \in \zeta^X$  with  $U \wedge V = 0_{\sim}$  such that  $a < \mu_{Q_{x_{(1,0)}}}(U), a < \mu_{Q_{y_{(1,0)}}}(V)$ . Then there exists  $U_1, V_1 \in \zeta^X$ , such that

$$\begin{aligned} x_{(1,0)} \widehat{q} U_1 &\leq U, \quad a < \omega(\tau)(\mu_{U_1}), \\ y_{(1,0)} \widehat{q} V_1 &\leq V, \quad a < \omega(\tau)(\mu_{V_1}). \end{aligned}$$

Denote  $A = \sigma_0(\mu_{U_1}), B = \sigma_0(\mu_{V_1})$ , it is clear that  $x \in A, y \in B$ . From the fact  $U \wedge V = 0_{\sim}$ , it implies  $\mu_{U_1} \wedge \mu_{V_1} = \underline{0}$ . Then we have  $\sigma_0(\mu_{U_1}) \wedge \sigma_0(\mu_{V_1}) = \emptyset$ , i.e.,  $A \wedge B = \emptyset$ .

$$a < \omega(\tau)(\mu_{U_1}) = \bigwedge_{r \in I} \tau(\sigma_r(\mu_{U_1})) \leq \tau(\sigma_0(\mu_{U_1})) = \tau(A).$$

Thus

$$a < \bigvee_{x \in U \subseteq A} \tau(U) = N_x(A).$$

Similarly, we have  $a < N_y(B)$ . Hence

$$a < \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)).$$

Then

$$a \leq \bigwedge \{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y \}.$$

Therefore,

$$\begin{aligned} &\bigwedge \{ \bigvee_{U \wedge V = 0_{\sim}} (\mu_{Q_{x_{(\alpha, \beta)}}}(U) \wedge \mu_{Q_{y_{(\lambda, \rho)}}}(V)) : x_{(\alpha, \beta)}, y_{(\lambda, \rho)} \in \text{pt}(\zeta^X), x \neq y \} \\ &\leq \bigwedge \{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y \}. \end{aligned}$$

On the other hand, for all  $x_{(\alpha, \beta)}, y_{(\lambda, \rho)} \in \text{pt}(\zeta^X), x \neq y$ , and  $a < \bigwedge \{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y \}$ , there exists  $A, B \in 2^X, A \wedge B = \emptyset$ , such that  $a < N_x(A), a < N_y(B)$ . Then there exists  $A_1, B_1 \in 2^X$ , such that

$$x \in A_1 \subseteq A, \quad a < \tau(A_1),$$

$$y \in B_1 \subseteq B, a < \tau(B_1).$$

Let  $U = \langle 1_{A_1}, 1_{A_1^c} \rangle, V = \langle 1_{B_1}, 1_{B_1^c} \rangle$ , where  $A_1^c$  is the complement of  $A_1$ , then  $x_{(\alpha,\beta)} \widehat{q} U, y_{(\lambda,\rho)} \widehat{q} V$ . In fact,  $1_{A_1}(x) = 1 > 1 - \alpha, 1_{A_1^c}(x) = 0 < 1 - \beta$ . Thus  $x_{(\alpha,\beta)} \widehat{q} U$ . Similarly, we have  $y_{(\lambda,\rho)} \widehat{q} V$ . By  $A \wedge B = \emptyset$ , we have  $A_1 \wedge B_1 = \emptyset$ . Then for all  $z \in X$ , we obtain

$$\begin{aligned} (1_{A_1} \wedge 1_{B_1})(z) &= 1_{A_1}(z) \wedge 1_{B_1}(z) = 0, \\ (1_{A_1^c} \vee 1_{B_1^c})(z) &= 1_{A_1^c}(z) \vee 1_{B_1^c}(z) = 1. \end{aligned}$$

Hence

$$1_{A_1} \wedge 1_{B_1} = \underline{0}, 1_{A_1^c} \vee 1_{B_1^c} = \underline{1}.$$

Since  $\forall r \in I_1, \sigma_r(1_{A_1}) = A_1$ , we have

$$\omega(\tau)(1_{A_1}) = \bigwedge_{r \in I_1} \tau(\sigma_r(1_{A_1})) = \tau(A_1).$$

By  $\underline{1} - 1_{A_1^c} = 1_{A_1}$ , and  $a < \tau(A_1)$ , we have

$$\begin{aligned} a &< \omega(\tau)(1_{A_1}) \wedge \omega(\tau)(\underline{1} - 1_{A_1^c}) \\ &= \omega(\tau)(\mu_U) \wedge \omega(\tau)(\underline{1} - \gamma_U). \end{aligned}$$

So,

$$a < \bigvee_{x_{(\alpha,\beta)} \widehat{q} W \subseteq U} (\omega(\tau)(\mu_W) \wedge \omega(\tau)(\underline{1} - \gamma_W)) = \mu_{Q_{x_{(\alpha,\beta)}}(U)}.$$

Similarly, we have  $a < \mu_{Q_{y_{(\lambda,\rho)}}(V)}$ . This deduces that

$$a < \bigvee_{U \wedge V = 0_{\sim}} (\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}).$$

Furthermore, we may obtain

$$a \leq \bigwedge \left\{ \bigvee_{U \wedge V = 0_{\sim}} (\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \text{pt}(\zeta^X), x \neq y \right\}.$$

Hence

$$\begin{aligned} &\bigwedge \left\{ \bigvee_{U \wedge V = 0_{\sim}} (\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \text{pt}(\zeta^X), x \neq y \right\} \\ &\geq \bigwedge \left\{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y \right\}. \end{aligned}$$

This means that  $\bigwedge \left\{ \bigvee_{U \wedge V = 0_{\sim}} (\mu_{Q_{x_{(\alpha,\beta)}}(U)} \wedge \mu_{Q_{y_{(\lambda,\rho)}}(V)}) : x_{(\alpha,\beta)}, y_{(\lambda,\rho)} \in \text{pt}(\zeta^X), x \neq y \right\} = \bigwedge \left\{ \bigvee_{A \cap B = \emptyset} (N_x(A) \wedge N_y(B)) : x, y \in X, x \neq y \right\}$ . Therefore we have

$$\mu_{T_2(X, \omega(\tau))} = T_2(X, \tau).$$

□

**4.3. Lemma.** Let  $(\prod_{j \in J} X_j, \prod_{j \in J} \tau_j)$  be the product space of a family of intuitionistic  $I$ -fuzzy topology spaces  $\{(X_j, \tau_j)\}_{j \in J}$ . Then  $\tau_j(A_j) \leq (\prod_{j \in J} \tau_j)(P_j^{\leftarrow}(A_j))$ , for all  $j \in J, A_j \in \zeta^{X_j}$ .

*Proof.* Let  $\prod_{j \in J} \tau_j = \delta$ ,  $x_{(\alpha, \beta)} \widehat{q} f^{\leftarrow}(U) \Leftrightarrow f^{\rightarrow}(x_{(\alpha, \beta)}) \widehat{q} U$ . Then for all  $j \in J$ ,  $A_j \in \zeta^{X_j}$ , we have

$$\begin{aligned}
\delta(P_j^{\leftarrow}(A_j)) &= \bigwedge_{x_{(\alpha, \beta)} \widehat{q} P_j^{\leftarrow}(A_j)} Q_{x_{(\alpha, \beta)}}^{\delta}(P_j^{\leftarrow}(A_j)) \\
&\geq \bigwedge_{x_{(\alpha, \beta)} \widehat{q} P_j^{\leftarrow}(A_j)} Q_{P_j^{\rightarrow}(x_{(\alpha, \beta)})}^{\tau_j}(A_j) \\
&= \bigwedge_{P_j^{\rightarrow}(x_{(\alpha, \beta)}) \widehat{q} A_j} Q_{P_j^{\rightarrow}(x_{(\alpha, \beta)})}^{\tau_j}(A_j) \\
&\geq \bigwedge_{x_{(\alpha, \beta)}^j \widehat{q} A_j} Q_{x_{(\alpha, \beta)}^j}^{\tau_j}(A_j) \\
&= \tau_j(A_j).
\end{aligned}$$

This completes the proof.  $\square$

**4.4. Theorem.** Let  $(\prod_{j \in J} X_j, \prod_{j \in J} \tau_j)$  be the product space of a family of intuitionistic I-fuzzy topology spaces  $\{(X_j, \tau_j)\}_{j \in J}$ . Then  $\bigwedge_{j \in J} T_2(X_j, \tau_j) \leq T_2(\prod_{j \in J} X_j, \prod_{j \in J} \tau_j)$ .

*Proof.* For all  $g_{(\alpha, \beta)}, h_{(\lambda, \rho)} \in \text{pt}(\zeta^{\prod_{j \in J} X_j})$  and  $g \neq h$ . Then there exists  $j_0 \in J$  such that  $g(j_0) \neq h(j_0)$ , where  $g(j_0), h(j_0) \in X_{j_0}$ .

For all  $U_{j_0}, V_{j_0} \in \zeta^{X_{j_0}}$  with  $U_{j_0} \wedge V_{j_0} = 0_{\sim}^{X_{j_0}}$ , we have

$$P_{j_0}^{\leftarrow}(U_{j_0}) \wedge P_{j_0}^{\leftarrow}(V_{j_0}) = P_{j_0}^{\leftarrow}(U_{j_0} \wedge V_{j_0}) = 0_{\sim}^{\prod_{j \in J} X_j}.$$

Then  $Q_{g(j_0)_{(\alpha, \beta)}}(U_{j_0}) \leq Q_{g_{(\alpha, \beta)}}(P_{j_0}^{\leftarrow}(U_{j_0}))$ . In fact, if  $g(j_0)_{(\alpha, \beta)} \widehat{q} U_{j_0}$ , then  $g_{(\alpha, \beta)} \widehat{q} P_{j_0}^{\leftarrow}(U_{j_0})$ . For all  $V \leq U_{j_0}$ , we have  $P_{j_0}^{\leftarrow}(V) \leq P_{j_0}^{\leftarrow}(U_{j_0})$ . On account of Lemma 4.3, we have

$$\begin{aligned}
\bigvee_{g(j_0)_{(\alpha, \beta)} \widehat{q} V \leq U_{j_0}} \tau_{j_0}(V) &\leq \bigvee_{g_{(\alpha, \beta)} \widehat{q} P_{j_0}^{\leftarrow}(V) \leq P_{j_0}^{\leftarrow}(U_{j_0})} (\prod_{j \in J} \tau_j)(P_{j_0}^{\leftarrow}(V)) \\
&\leq \bigvee_{g_{(\alpha, \beta)} \widehat{q} G \leq P_{j_0}^{\leftarrow}(U_{j_0})} (\prod_{j \in J} \tau_j)(G),
\end{aligned}$$

i.e.,  $Q_{g(j_0)_{(\alpha, \beta)}}(U_{j_0}) \leq Q_{g_{(\alpha, \beta)}}(P_{j_0}^{\leftarrow}(U_{j_0}))$ . Thus,

$$\begin{aligned}
&\bigvee_{U \wedge V = 0_{\sim}^{X_{j_0}}} (Q_{g(j_0)_{(\alpha, \beta)}}(U) \wedge Q_{h(j_0)_{(\lambda, \rho)}}(V)) \\
&\leq \bigvee_{P_{j_0}^{\leftarrow}(U) \wedge P_{j_0}^{\leftarrow}(V) = 0_{\sim}^{\prod_{j \in J} X_j}} (Q_{g_{(\alpha, \beta)}}(P_{j_0}^{\leftarrow}(U)) \wedge Q_{h_{(\lambda, \rho)}}(P_{j_0}^{\leftarrow}(V))) \\
&\leq \bigvee_{G \wedge H = 0_{\sim}^{\prod_{j \in J} X_j}} (Q_{g_{(\alpha, \beta)}}(G) \wedge Q_{h_{(\lambda, \rho)}}(H)).
\end{aligned}$$

So we have

$$T_2(g(j_0)_{(\alpha,\beta)}, h(j_0)_{(\lambda,\rho)}) \leq T_2(g_{(\alpha,\beta)}, h_{(\lambda,\rho)}).$$

Thus

$$T_2(X_{j_0}, \tau_{j_0}) \leq T_2\left(\prod_{j \in J} X_j, \prod_{j \in J} \tau_j\right).$$

Therefore,

$$\bigwedge_{j \in J} T_2(X_j, \tau_j) \leq T_2\left(\prod_{j \in J} X_j, \prod_{j \in J} \tau_j\right).$$

□

**4.5. Lemma.** *Let  $(X, I\omega(\tau))$  be a generated intuitionistic  $I$ -fuzzy topological space by fuzzifying topological space  $(X, \tau)$ . Then*

- (1)  $I\omega(\tau)(A) = 1^\sim$ , for all  $A = \langle \underline{\alpha}, \underline{\beta} \rangle \in \zeta^X$ ;
- (2)  $\forall B \subseteq X, \tau(B) = \mu_{I\omega(\tau)}(\langle 1_B, 1_{B^c} \rangle)$ .

*Proof.* By Lemma 2.11, 2.12 and 2.13, it is easy to prove it. □

**4.6. Lemma.** *Let  $(X, \delta)$  be a stratified intuitionistic  $I$ -fuzzy topological space (i.e., for all  $\langle \alpha, \beta \rangle \in \mathcal{A}, \delta(\langle \underline{\alpha}, \underline{\beta} \rangle) = 1^\sim$ ). Then for all  $A \in \zeta^X$*

$$\bigwedge_{r \in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c} \rangle) \leq \mu_\delta(A).$$

*Proof.* For all  $A \in \zeta^X$ , and for any  $a < \bigwedge_{r \in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c} \rangle), y_{(\alpha,\beta)} \in \text{pt}(\zeta^X)$  with  $y_{(\alpha,\beta)} \widehat{q} A$ , clearly  $\mu_A(y) > 1 - \alpha$ . Then there exists  $\delta > 0$  such that  $\mu_A(y) > 1 - \alpha + \delta$ . Thus  $y \in \sigma_{1-\alpha+\delta}(\mu_A)$ . So we have

$$y_{(\alpha,\beta)} \widehat{q} \langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle.$$

Then

$$\begin{aligned} a &< \mu_\delta(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle) \\ &= \bigwedge_{z_{(\alpha,\beta)} \widehat{q} \langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle} \mu(Q_{z_{(\alpha,\beta)}}(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle)). \end{aligned}$$

Therefore,

$$a < \mu(Q_{y_{(\alpha,\beta)}}(\langle 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle)).$$

Since  $(X, \delta)$  is a stratified intuitionistic  $I$ -fuzzy topological space, we have  $Q_{y_{(\alpha,\beta)}}(\underline{1-\alpha+\delta}, \underline{\alpha-\delta}) = 1^\sim$ . Moreover, it is well known that the following relations hold

$$\underline{1-\alpha+\delta} \wedge 1_{\sigma_{1-\alpha+\delta}(\mu_A)} \leq \mu_A,$$

$$\underline{\alpha-\delta} \vee 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \geq 1 - \mu_A \geq \gamma_A.$$

So we have

$$a < \mu(Q_{y_{(\alpha,\beta)}}(\langle \underline{1-\alpha+\delta} \wedge 1_{\sigma_{1-\alpha+\delta}(\mu_A)}, \underline{\alpha-\delta} \vee 1_{(\sigma_{1-\alpha+\delta}(\mu_A))^c} \rangle)) \leq \mu(Q_{y_{(\alpha,\beta)}}(A)).$$

Then  $a \leq \mu_\delta(A)$ . Therefore,

$$\bigwedge_{r \in I} \mu_\delta(\langle 1_{\sigma_r(\mu_A)}, 1_{(\sigma_r(\mu_A))^c} \rangle) \leq \mu_\delta(A).$$

□

**4.7. Theorem.** Let  $(\prod_{\alpha \in J} X_\alpha, \prod_{\alpha \in J} \tau_\alpha)$  be the product space of a family of fuzzifying topological space  $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in J}$ . Then  $(\prod_{\alpha \in J} \mathbb{I}\omega(\tau_\alpha))(A) = \mathbb{I}\omega(\prod_{\alpha \in J} \tau_\alpha)(A)$ .

*Proof.* Let  $(\prod_{\alpha \in J} \mathbb{I}\omega(\tau_\alpha))(A) = \langle \mu \prod_{\alpha \in J} \mathbb{I}\omega(\tau_\alpha)(A), \gamma \prod_{\alpha \in J} \mathbb{I}\omega(\tau_\alpha)(A) \rangle$ . For all  $a < \mu \prod_{\alpha \in J} \mathbb{I}\omega(\tau_\alpha)(A)$ , there exists  $\{U_j^a\}_{j \in K}$  such that  $\bigvee_{j \in K} U_j^a = A$ , for each  $U_j^a$ , there exists  $\{A_{\lambda,j}^a\}_{\lambda \in E}$  such that  $\bigwedge_{\lambda \in E} A_{\lambda,j}^a = U_j^a$ , where  $E$  is an finite index set. In addition, for every  $\lambda \in E$ , there exists  $\alpha \triangleq \alpha(\lambda) \in J$  and  $W_\alpha \in \zeta^{X_\alpha}$  with  $P_\alpha^{\leftarrow}(W_\alpha) = A_{\lambda,j}^a$  such that  $a < \mu(\mathbb{I}\omega(\tau_\alpha)(W_\alpha))$ . Then we have

$$a < \omega(\tau_\alpha)(\mu_{W_\alpha}),$$

$$a < \omega(\tau_\alpha)(\underline{1} - \gamma_{W_\alpha}).$$

Thus for all  $r \in I$ , we have

$$\begin{aligned} a &< \tau_\alpha(\sigma_r(\mu_{W_\alpha})) \\ &\leq (\prod_{\alpha \in J} \tau_\alpha)(P_\alpha^{\leftarrow}(\sigma_r(\mu_{W_\alpha}))) \\ &= (\prod_{\alpha \in J} \tau_\alpha)(\sigma_r(P_\alpha^{\leftarrow}(\mu_{W_\alpha}))) \\ &= (\prod_{\alpha \in J} \tau_\alpha)(\sigma_r(\mu_{A_{\lambda,j}^a})). \end{aligned}$$

Hence

$$\begin{aligned} a &\leq (\prod_{\alpha \in J} \tau_\alpha)(\bigwedge_{\lambda \in E} \sigma_r(\mu_{A_{\lambda,j}^a})) \\ &= (\prod_{\alpha \in J} \tau_\alpha)(\sigma_r(\bigwedge_{\lambda \in E} \mu_{A_{\lambda,j}^a})) \\ &= (\prod_{\alpha \in J} \tau_\alpha)(\sigma_r(\mu_{U_j^a})). \end{aligned}$$

Furthermore

$$\begin{aligned} a &\leq (\prod_{\alpha \in J} \tau_\alpha)(\bigvee_{j \in K} \sigma_r(\mu_{U_j^a})) \\ &= (\prod_{\alpha \in J} \tau_\alpha)(\sigma_r(\bigvee_{j \in K} \mu_{U_j^a})) \\ &= (\prod_{\alpha \in J} \tau_\alpha)(\sigma_r(\mu_A)). \end{aligned}$$

So

$$\begin{aligned} a &\leq \bigwedge_{r \in I} \left( \prod_{\alpha \in J} \tau_\alpha \right) (\sigma_r(\mu_A)) \\ &= \omega \left( \prod_{\alpha \in J} \tau_\alpha \right) (\mu_A). \end{aligned}$$

Similarly, we have

$$a \leq \omega \left( \prod_{\alpha \in J} \tau_\alpha \right) (\underline{1} - \gamma_A).$$

Hence  $a \leq \mu(\text{I}\omega(\prod_{\alpha \in J} \tau_\alpha)(A))$ . By the arbitrariness of  $a$ , we have  $\mu(\prod_{\alpha \in J} \text{I}\omega(\tau_\alpha)(A)) \leq \mu(\text{I}\omega(\prod_{\alpha \in J} \tau_\alpha)(A))$ .

On the other hand, for  $\forall a < \mu(\text{I}\omega(\prod_{\alpha \in J} \tau_\alpha)(A))$ , we have

$$a < \omega \left( \prod_{\alpha \in J} \tau_\alpha \right) (\mu_A) = \bigwedge_{r \in I} \left( \prod_{\alpha \in J} \tau_\alpha \right) (\sigma_r(\mu_A))$$

and

$$a < \omega \left( \prod_{\alpha \in J} \tau_\alpha \right) (\underline{1} - \gamma_A).$$

Then for all  $r \in I$ , we have

$$a < \left( \prod_{\alpha \in J} \tau_\alpha \right) (\sigma_r(\mu_A)).$$

Thus there exists  $\{U_{j,r}^a\}_{j \in K} \subseteq X$  satisfies  $\bigvee_{j \in K} U_{j,r}^a = \sigma_r(\mu_A)$ , and for all  $j \in K$ , there exists  $\{A_{\lambda,j,r}^a\}_{\lambda \in E}$ , where  $E$  is a finite index set, such that  $\bigwedge_{\lambda \in E} A_{\lambda,j,r}^a = U_{j,r}^a$ . For all  $\lambda \in E$ , there exists  $\alpha(\lambda) \in J, W_\alpha \in \zeta^{X_\alpha}$ , such that  $P_\alpha^{\leftarrow}(W_\alpha) = A_{\lambda,j,r}^a$ . By Lemma 4.5 we have

$$\begin{aligned} a < \tau_\alpha(W_\alpha) &= \mu_{\text{I}\omega(\tau_\alpha)}(\langle 1_{W_\alpha}, 1_{W_\alpha^c} \rangle) \\ &\leq \mu \left( \prod_{\alpha \in J} \text{I}\omega(\tau_\alpha) \right) (P_\alpha^{\leftarrow}(\langle 1_{W_\alpha}, 1_{W_\alpha^c} \rangle)) \\ &= \mu \left( \prod_{\alpha \in J} \text{I}\omega(\tau_\alpha) \right) (\langle 1_{P_\alpha^{\leftarrow}(W_\alpha)}, 1_{P_\alpha^{\leftarrow}(W_\alpha^c)} \rangle) \\ &= \mu \left( \prod_{\alpha \in J} \text{I}\omega(\tau_\alpha) \right) (\langle 1_{A_{\lambda,j,r}^a}, 1_{(A_{\lambda,j,r}^a)^c} \rangle) \\ &\leq \mu \left( \prod_{\alpha \in J} \text{I}\omega(\tau_\alpha) \right) (\langle \bigwedge_{\lambda \in E} 1_{A_{\lambda,j,r}^a}, \bigvee_{\lambda \in E} 1_{(A_{\lambda,j,r}^a)^c} \rangle) \\ &= \mu \left( \prod_{\alpha \in J} \text{I}\omega(\tau_\alpha) \right) (\langle 1_{\bigwedge_{\lambda \in E} A_{\lambda,j,r}^a}, 1_{\bigvee_{\lambda \in E} (A_{\lambda,j,r}^a)^c} \rangle) \\ &= \mu \left( \prod_{\alpha \in J} \text{I}\omega(\tau_\alpha) \right) (\langle 1_{U_{j,r}^a}, 1_{(U_{j,r}^a)^c} \rangle). \end{aligned}$$

Then

$$\begin{aligned} a &\leq \mu\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)\left(\left(1_{\bigvee_{j \in K} U_{j,r}^a}, 1_{\left(\bigvee_{j \in K} U_{j,r}^a\right)^c}\right)\right) \\ &= \mu\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)\left(\left(1_{\sigma_r(\mu_A)}, 1_{\left(\sigma_r(\mu_A)\right)^c}\right)\right). \end{aligned}$$

By Lemma 4.6 we have

$$\begin{aligned} a &\leq \bigwedge_{r \in I} \mu\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)\left(\left(1_{\sigma_r(\mu_A)}, 1_{\left(\sigma_r(\mu_A)\right)^c}\right)\right) \\ &\leq \mu\left(\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)(A)\right). \end{aligned}$$

Then

$$\mu\left(\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)(A)\right) \geq \mu\left(\mathbf{I}\omega\left(\prod_{\alpha \in J} \tau_\alpha\right)(A)\right).$$

Hence

$$\mu\left(\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)(A)\right) = \mu\left(\mathbf{I}\omega\left(\prod_{\alpha \in J} \tau_\alpha\right)(A)\right).$$

Then

$$\gamma\left(\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)(A)\right) = \gamma\left(\mathbf{I}\omega\left(\prod_{\alpha \in J} \tau_\alpha\right)(A)\right).$$

Therefore,

$$\left(\prod_{\alpha \in J} \mathbf{I}\omega(\tau_\alpha)\right)(A) = \mathbf{I}\omega\left(\prod_{\alpha \in J} \tau_\alpha\right)(A).$$

□

## 5. Further remarks

As we have shown, the notions of the base and subbase in intuitionistic  $I$ -fuzzy topological spaces are introduced in this paper, and some important applications of them are obtained. Specially, we also use the concept of subbase to study the product of intuitionistic  $I$ -fuzzy topological spaces. In addition, we have proved that the functor  $\mathbf{I}\omega$  preserves the product.

There are two categories in our paper, the one is the category **FYTS** of fuzzifying topological spaces, and the other is the category **IFTS** of intuitionistic  $I$ -fuzzy topological spaces. It is easy to find that  $\mathbf{I}\omega$  is the functor from **FYTS** to **IFTS**. We discussed the property of the functor  $\mathbf{I}\omega$  in Theorem 4.7. A direction worthy of further study is to discuss the properties of the functor  $\mathbf{I}\omega$  in detail. Moreover, we hope to point out that another continuation of this paper is to deal with other topological properties of intuitionistic  $I$ -fuzzy topological spaces.

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