Hacettepe Journal of Mathematics and Statistics
Volume 40 (2) (2011), 341 – 348

UPPER AND LOWER NA-CONTINUOUS MULTIFUNCTIONS

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Received 21 : 06 : 2010 : Accepted 04 : 04 : 2011

Abstract

The aim of this paper is to introduce a new class of continuous multifunctions, namely upper and lower na-continuous multifunctions, and to obtain some characterizations concerning upper and lower na-continuous multifunctions. The authors investigate the graph of upper and lower na-continuous multifunctions, and the preservation of properties under upper na-continuous multifunctions. Also, the relationship between upper and lower na-continuous multifunctions and some known types of continuous multifunctions are discussed.

Keywords: α-open sets, δ-open sets, multifunctions, α-compact spaces, δ-connected spaces.

2010 AMS Classification: 54 C 60, 54 C 10.

Communicated by Lawrence M. Brown

1. Introduction

One of the important and basic topics in the theory of classical point set topology and in several branches of mathematics, which has been investigated by many authors, is continuity of functions. This concept has been extended to the setting of multifunctions. A multifunction, or multivalued mapping, has many applications in mathematical programming, probability, statistics, fixed point theorems and even in economics. There are several weak and strong variants of continuity of multifunctions in the literature, for instance continuity [11], strong continuity [2] and super continuity [1].

In 1986 Chae and Noiri [5] introduced the concept of na-continuous functions, and some of its properties were given by the authors. A function $f : (X, \tau) \to (Y, \vartheta)$ is said to be *na-continuous* if for each point $x \in X$ and each α-open set V in Y containing $f(x)$, there exists a δ-open set U in X containing x such that $f(U) \subseteq V$. The purpose of this paper is to extend this concept to multifunctions, and to discuss the results obtained.

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These indicate that the property of na-continuity for multifunctions is stronger than both continuity and super continuity for multifunctions, but weaker than strong continuity for multifunctions.

2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure (resp. the interior) of A is denoted by A^- (resp. A°). A subset A is said to be *regular open* [15] (resp. *regular closed*) if $A = A^{-\circ}$ (resp. $A = A^{\circ -}$). The *δ-interior* [16] of a subset A of X is the union of all regular open sets of X contained in A, and is denoted by A°_δ.

A subset A is called *δ-open* [16] if $A = A^\circ_\delta$, i.e, a set is δ-open, if it is the union of regular open sets. The complement of a δ-open set is called *δ-closed*. Alternatively, a subset A is δ-closed [16] if $A = A^-_\delta$, where $A^-_\delta = \{x \in X : A \cap U^{-\circ} \neq \emptyset,\ U \in \tau \text{ and } x \in U\}$.

A subset A is said to be *α-open* [9] if $A \subseteq A^{\circ-\circ}$. The complement of an α-open set is called *α-closed*. Following the notion introduced by Nijastad [9] we denote the family of all α-open sets in (X,τ) by τ^α. Nijastad proved that τ^α is a topology on X.

The family of all regular open (δ-open, α-open) sets of X is denoted by $RO(X)$ ($\delta O(X)$, $\alpha O(X)$), respectively. The family of all regular open (δ-open, α-open) sets of X containing a point $x \in X$ is denoted by $RO(X,x)$ ($\delta O(X,x)$, $\alpha O(X,x)$), respectively. The intersection of all α-closed (resp. δ-closed) sets of X containing A is called the *α-closure* (*δ-closure*) of A, and is denoted by A^-_α (resp. A^-_δ).

For a space (X,τ), the collection of all δ-open sets of (X,τ) forms a topology for X which is usually called the *semiregularization* of τ, and denoted by τ_s. In general $\tau_s \subseteq \tau$, and if $\tau_s = \tau$ then (X,τ) is called a *semiregular space*.

By a *multifunction* $F : X \to Y$ we mean a point-to-set correspondence from X into Y, and assume that $F(x) \neq \emptyset$ for all $x \in X$. Following [3,4], for a multifunction $F : X \to Y$ we denote the *upper and lower inverses* of a set B of Y by $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, respectively. For each $A \subset X$, $F(A) = \bigcup_{x\in A} F(x)$. Also, F is said to be a *surjection* if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists a $x \in X$ such that $y \in F(x)$. Moreover $F : (X,\tau) \to (Y,\vartheta)$ is called *upper semi continuous* [13] (renamed *upper continuous* [12]) (resp. *lower semi continuous* [13] (renamed *lower continuous* [12])) if $F^+(V)$ (resp. $F^-(V)$) is open in X for each open set V of Y.

3. Characterizations and basic properties

3.1. Definition. A multifunction $F : (X,\tau) \to (Y,\vartheta)$ is said to be:

a) *Upper na-continuous* (briefly u.na.c.) *at a point* $x \in X$ if for each α-open set V of Y such that $x \in F^+(V)$, there exists $U \in \delta O(X,x)$ such that $U \subseteq F^+(V)$,
b) *Lower na-continuous* (briefly l.na.c.) *at a point* $x \in X$ if for each α-open set V of Y such that $x \in F^-(V)$, there exists $U \in \delta O(X,x)$ such that $U \subseteq F^-(V)$,
c) u.na.c (l.na.c.) if F has this property at each point of X.
d) *Na-continuous* if it is both u.na.c and l.na.c.

3.2. Theorem. *The following conditions are equivalent for a multifunction* $F : (X,\tau) \to (Y,\vartheta)$*:*

(1) F *is u.na.c.*
(2) *For each* $x \in X$ *and each α-open set* V *of* Y *such that* $x \in F^+(V)$*, there exists* $U \in RO(X,x)$ *such that* $U \subseteq F^+(V)$*,*
(3) $F^+(V) \in \delta O(X)$ *for any α-open set* V *in* Y*,*
(4) $F^-(F) \in \delta C(X)$ *for any α-closed set* F *in* Y*,*

(5) $F(A_\delta^-) \subset [F(A)]_\alpha^-$ *for any subset* A *of* X,
(6) $[F^-(B)]_\delta^- \subset F^-(B_\alpha^-)$ *for any subset* B *of* Y.

Proof. (1) $\Longrightarrow$ (2) Since δ-open sets are a union of regular open sets, the proof is obvious.

(2) $\Longrightarrow$ (3) Let V be an α-open set in Y and $x \in F^+(V)$. Then there exists $U_x \in RO(X,x)$ such that $U_x \subset F^+(V)$. Hence $F^+(V) = \bigcup_{x\in F^+(V)} U_x$, and $F^+(V)$ is δ-open.

(3) $\Longrightarrow$ (4) Obvious.

(4) $\Longrightarrow$ (5) For any subset A of X, $[F(A)]_\alpha^-$ is an α-closed set in Y. It can be seen that

$$A \subset F^-(F(A)) \subset F^-([F(A)]_\alpha^-) \text{ and } A_\delta^- \subset F^-([F(A)]_\alpha^-)$$

Hence we obtain $F(A_\delta^-) \subset [F(A)]_\alpha^-$.

(5) $\Longrightarrow$ (6) Let B be a subset of Y. By (5) we have

$$F([F^-(B)]_\delta^-) \subset [F(F^-(B))]_\alpha^- \subset B_\alpha^-.$$

Hence, $[F^-(B)]_\delta^- \subset F^-(B_\alpha^-)$

(6) $\Longrightarrow$ (1) Let $x \in X$ and let V be any α-open set of Y such that $x \in F^+(V)$. Then $X - V$ is an α-closed set of Y. By (6) we have $[F^-(X-V)]_\delta^- \subset F^-((X-V)_\alpha^-) = F^-(X-V)$. This shows that $F^-(X-V) = X - F^+(V)$ is δ-closed and $F^+(V)$ is δ-open in X. □

3.3. Theorem. *The following conditions are equivalent for a multifunction* $F : (X,\tau) \to (Y,\vartheta)$*:*

(1) F *is l.na.c.*
(2) *For each* $x \in X$ *and for each* α*-open set* V *of* Y *such that* $x \in F^-(V)$*, there exists* $U \in RO(X,x)$ *such that* $U \subseteq F^-(V)$,
(3) $F^-(V) \in \delta O(X)$ *for any* α*-open set* V *in* Y,
(4) $F^+(F) \in \delta C(X)$ *for any* α*-closed set* F *in* Y,
(5) $F(A_\delta^-) \subset [F(A)]_\alpha^-$ *for any subset* A *of* X,
(6) $[F^+(B)]_\delta^- \subset F^+(B_\alpha^-)$ *for any subset* B *of* Y.

Proof. Dual to the proof of Theorem 3.2. □

3.4. Definition. [5] A net $(x_\lambda)_{\lambda\in D}$ in X is said to δ*-converge* (*sf converge*) to a point x in X if the net is eventually in each regular open (α-open) set containing x.

3.5. Theorem. *A multifunction* $F : (X,\tau) \to (Y,\vartheta)$ *is u.na.c. (l.na.c.) if and only if for each* $x \in X$ *and each net* (x_λ) *which* δ*-converges to* x*,* $F(x_\lambda)$ *sf converges to* $F(x)$*.*

Proof. $\Longrightarrow$ Let (x_λ) be a net which δ-converges to x in X, and let V be an α-open set such that $x \in F^+(V)$. Since F is an u.na.c. multifunction, there exists $U \in RO(X,x)$ such that $U \subseteq F^+(V)$. Since (x_λ) δ-converges to x, (x_λ) is eventually in U. Hence $F(x_\lambda)$ is eventually in V.

$\Longleftarrow$ Suppose that F is not an u.na.c. multifunction. Then there exists a point x and an α-open set V with $x \in F^+(V)$ such that $U \not\subset F^+(V)$ for each $U \in \delta O(X,x)$. Let $x_u \in U$ and $x_u \notin F^+(V)$. Then for the δ-neighbourhood net (x_u), (x_u) is δ-convergent to x but $F(x_u)$ is not sf convergent to $F(x)$. This is a contradiction. Thus F is an u.na.c. multifunction.

The proof for l.na.c. is similar. □

3.6. Definition. [2] A multifunction $F : (X,\tau) \to (Y,\vartheta)$ is said to be:

(1) *Upper strongly continuous* if $F^+(V)$ is clopen in X for each subset V of Y,
(2) *Lower strongly continuous* if $F^-(V)$ is clopen in X for each subset V of Y,
(3) *Strongly continuous* if it is both upper strongly continuous and lower strongly continuous

3.7. Definition. [1] A multifunction $F : (X, \tau) \to (Y, \vartheta)$ is said to be:

(1) *Upper super continuous* if $F^+(V)$ is δ-open in X for each open subset V of Y,
(2) *Lower super continuous* if $F^-(V)$is δ-open in X for each open subset V of Y,
(3) *Super continuous* if it is both upper super continuous and lower super continuous.

3.8. Remark. For a multifunction $F : (X, \tau) \to (Y, \vartheta)$ the following implications hold:

$$\begin{array}{ccccc} \text{upper (lower) strongly cont.} & \longrightarrow & \text{u.na.c. (l.na.c)} & \longrightarrow & \text{upper (lower) super cont.} \\ & & & & \downarrow \\ & & & & \text{upper (lower) cont.} \end{array}$$

where none of these implications is reversible as shown by Examples 3.9, 3.10 and 3.11.

3.9. Example. Let $X = \{a, b, c\}$, $Y = \{p, q, r, s\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, and $\vartheta = \{\emptyset, Y, \{p, q, r\}\}$. Define a multifunction $F : (X, \tau) \to (Y, \vartheta)$ as follows: $F(a) = \{p, q\}$, $F(b) = \{q, r, s\}$, $F(c) = \{r\}$. Then F is u.na.c., but not upper strongly continuous.

3.10. Example. Let $X = \{a, b, c, d\}$, $Y = \{p, q, r, s\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$, and $\vartheta = \{\emptyset, Y, \{p\}, \{p, q\}\}$. Define a multifunction $F : (X, \tau) \to (Y, \vartheta)$ as follows: $F(a) = F(b) = \{p\}$, $F(c) = \{r\}$, $F(d) = \{q, s\}$. Then F is upper super continuous, but not u.na.c.

3.11. Example. [1] Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{b\}, \{a, b\}\}$, and let $Y = [0, 1]$ with the usual topology. Define a multifunction $F : (X, \tau) \to (Y, \vartheta)$ as follows:

$$F(x) = \begin{cases} [0, \frac{1}{2}) & \text{if } x = a, \\ \{\frac{1}{2}\} & \text{if } x = b, \\ (\frac{1}{2}, 1] & \text{if } x = c. \end{cases}$$

Then F is upper continuous at $x = b$, but not upper super continuous at $x = b$.

3.12. Theorem. *For a multifunction $F : (X, \tau) \to (Y, \vartheta)$ the following are equivalent:*

(1) *$F : (X, \tau) \to (Y, \vartheta)$ is u.na.c (l.na.c.),*
(2) *$F : (X, \tau) \to (Y, \vartheta^\alpha)$ is upper (lower) super continuous,*
(3) *$F : (X, \tau_s) \to (Y, \vartheta^\alpha)$ is upper (lower) continuous.*

Proof. Straightforward. □

3.13. Lemma. [10] *If A is a dense or open subset of (X, τ) and $U \in RO(X)$, then $U \cap A$ is a regular open set in the subspace A.* □

3.14. Theorem. *If $F : (X, \tau) \to (Y, \vartheta)$ is an u.na.c. (l.na.c.) multifunction, and A is an open subset of (X, τ), then the restriction $F_{/A} : (A, \tau_A) \to (Y, \vartheta)$ is an u.na.c. (l.na.c.) multifunction.*

Proof. Let A be an open subset of X, $x \in A$, and let V be an α-open set in Y such that $x \in (F_{/A})^+(V)$. Since F is an u.na.c. multifunction, there exists $U \in RO(X, x)$ such that $U \subset F^+(V)$. By Lemma 3.13, $U \cap A$ is a regular open set in (A, τ_A), and also $U \cap A \subset F^+(V) \cap A = (F_{/A})^+(V)$. This shows that $F_{/A}$ is u.na.c.

The proof for l.na.c. is similar. □

3.15. Definition. [8] A multifunction $F : (X, \tau) \to (Y, \vartheta)$ is said to be,

a) *Upper α-irresolute* if $F : (X, \tau^\alpha) \to (Y, \vartheta^\alpha)$ is upper continuous,
b) *Lower α-irresolute* if $F : (X, \tau^\alpha) \to (Y, \vartheta^\alpha)$ is lower continuous,
c) *α-irresolute* if it is both upper α-irresolute and lower α-irresolute.

3.16. Theorem. *If $F : (X, \tau) \to (Y, \vartheta)$ is an u.na.c. (l.na.c.) multifunction and $G : (Y, \vartheta) \to (Z, \sigma)$ is an upper (lower) α-irresolute multifunction, then GoF is an u.na.c. (l.na.c.) multifunction.*

Proof. The proof of only the first case is given since the proof of the second case is analogous. Let $V \subset Z$ be an α-open set. From the definition of GoF, we have $(GoF)^+(V) = F^+(G^+(V))$. Since G is upper α-irresolute, $G^+(V)$ is α-open. Since F is u.na.c., $F^+(G^+(V))$ is δ-open. Consequently, GoF is u.na.c. □

3.17. Corollary. *If $F : (X, \tau) \to (Y, \vartheta)$ is an u.na.c. (l.na.c.) multifunction, and $G : (Y, \vartheta) \to (Z, \sigma)$ is an u.na.c. (l.na.c.) multifunction, then GoF is an u.na.c. (l.na.c.) multifunction.* □

3.18. Lemma. [5] *Let $\{X_\lambda : \lambda \in D\}$ be a family of spaces and U_{λ_i} a subset of X_{λ_i}, $i = 1, 2, \dots, n$. Then $U = \prod_{i=1}^n U_{\lambda_i} \times \prod_{\lambda \neq \lambda_i} X_\lambda$ is δ-open (resp. α-open) in $\prod_{\lambda \in D} X_\lambda$ if and only if $U_{\lambda_i} \in \delta O(X_{\lambda_i})$ (resp. $U_{\lambda_i} \in \alpha O(X_{\lambda_i})$) for each $i = 1, 2, \dots, n$.* □

3.19. Theorem. *Let $F_\lambda : (X_\lambda, \tau_\lambda) \to (Y_\lambda, \vartheta_\lambda)$ be a multifunction for each $\lambda \in D$ and $F : \prod X_\lambda \to \prod Y_\lambda$ the multifunction defined by $F((x_\lambda)) = \prod F_\lambda^+(x_\lambda)$. If F is u.na.c. (l.na.c.) then F_λ is u.na.c. (l.na.c.) for each $\lambda \in D$.*

Proof. Let $V_\lambda \in \alpha O(Y_\lambda)$. Then by Lemma 3.18, $V = V_\lambda \times \prod_{\lambda \neq \beta} Y_\beta$ is α-open in $\prod Y_\lambda$ and

$$F^+(V) = F^+\left(V_\lambda \times \prod_{\lambda \neq \beta} Y_\beta\right) = F^+(V_\lambda) \times F^+\left(\prod_{\lambda \neq \beta} Y_\beta\right) = F^+(V_\lambda) \times \prod_{\lambda \neq \beta} X_\beta$$

is δ-open in $\prod X_\lambda$. From Lemma 3.18, $F^+(V_\lambda) \in \delta O(X)$. Therefore F_λ is u.na.c.

For l.na.c. the proof is similar. □

4. Preservation properties

Recall that for a multifunction $F : X \to Y$, the *graph multifunction* $G_F : X \to X \times Y$ of F is defined by $G_F(x) = \{x\} \times F(x)$ for every $x \in X$. The subset $\bigcup\{\{x\} \times F(x) : x \in X\}$ of $X \times Y$ is called the *multigraph* of F, and is denoted by $G(F)$.

4.1. Lemma. [11] *For a multifunction $F : X \to Y$, the following hold:*

(1) $G_F^+(A \times B) = A \cap F^+(B)$,
(2) $G_F^-(A \times B) = A \cap F^-(B)$,

for any $A \subset X$ and $B \subset Y$. □

4.2. Theorem. . *Let $F : (X, \tau) \to (Y, \vartheta)$ be a multifunction. Then F is u.na.c. if the graph multifunction G_F is u.na.c.*

Proof. Let $x \in X$ and V be any α-open set in Y such that $F(x) \subset V$. Lemma 3.18, $X \times V$ is an α-open set in $X \times Y$. Since $\{x\} \times F(x) \subset X \times V$, $G_F(x) \subset X \times V$. Since G_F is u.na.c, there exists $U \in \delta O(X, x)$ such that $U \subset G_F^+(X \times V)$. By using Lemma 4.1, we obtain $U \subset F^+(V)$. This shows that F is u.na.c. □

4.3. Theorem. *Let $F : (X, \tau) \to (Y, \vartheta)$ be a multifunction. Then F is l.na.c. if the graph multifunction G_F is l.na.c.*

Proof. Let $x \in X$ and V be any α-open set in Y such that $F(x) \cap V \neq \emptyset$. By Lemma 3.18, $X \times V$ is an α-open set in $X \times Y$. We now obtain,

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset.$$

Since G_F is l.na.c, there exists $U \in \delta O(X,x)$ such that $U \subset G_F^-(X \times V)$. By using Lemma 4.1, we obtain $U \subset F^-(V)$. This shows that F is l.na.c. □

4.4. Definition. For a multifunction $F : (X,\tau) \to (Y,\vartheta)$ the multigraph $G(F) = \{(x,y) : x \in X, y \in F(x)\}$ is said to be δ-α-closed in $X \times Y$ if for each $(x,y) \in (X \times Y) \setminus G(F)$, there exist $U \in \delta O(X,x)$ and $V \in \alpha O(Y,y)$ such that $(U \times V) \cap G(F) = \emptyset$.

4.5. Lemma. *A multifunction $F : (X,\tau) \to (Y,\vartheta)$ has a δ-α-closed multigraph if and only if for each $(x,y) \in (X \times Y) \setminus G(F)$, there exist $U \in \delta O(X,x)$ and $V \in \alpha O(Y,y)$ such that $F(U) \cap V = \emptyset$.* □

4.6. Definition. A space X is said to be *α-compact* [6] (*nearly compact* [14]) if every α-open (regular open) cover of X has a finite subcover.

4.7. Definition. [7] A topological space X is said to be α-Hausdorff if for any pair of distinct points x and y in X, there exist disjoint α-open sets U and V in X such that $x \in U$ and $y \in V$.

4.8. Theorem. *If $F : (X,\tau) \to (Y,\vartheta)$ is an u.na.c. multifunction such that $F(x)$ is α-compact for each $x \in X$, and Y is an α-Hausdorff space, then $G(F)$ is δ-α-closed.*

Proof. Let $(x,y) \in (X \times Y) \setminus G(F)$. Then $y \in Y \setminus F(x)$. Since Y is an α-Hausdorff space, for each $p \in F(x)$, there exist disjoint α-open sets U_p and V_p of Y such that $p \in U_p$ and $y \in V_p$. Then $\{U_p : p \in F(x)\}$ is an α-open cover of $F(x)$ and since $F(x)$ is α-compact for each $x \in X$, there exist a finite number of points $p_1, p_2, \ldots, p_n$ in $F(x)$ such that $F(x) \subset \bigcup\{U_{p_i} : i = 1,2,\ldots,n\}$. Put $U = \bigcup\{U_{p_i} : i = 1,2,\ldots,n\}$ and $V = \bigcap\{V_{p_i} : i = 1,2,\ldots,n\}$.

Then U and V are disjoint α-open sets in Y such that $F(x) \subset U$ and $y \in V$. Hence we have $F(F^+(U)) \cap V = \emptyset$, and since F is an u.na.c. multifunction, $x \in F^+(U) \in \delta O(X,x)$ by Theorem 3.2 (3). This shows that $G(F)$ is δ-α-closed. □

4.9. Theorem. *Let $F : (X,\tau) \to (Y,\vartheta)$ be an u.na.c. surjective multifunction such that $F(x)$ is α-compact for each $x \in X$. If X is a nearly compact space then Y is α-compact.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be an α-open cover of Y. Since $F(x)$ is α-compact for each $x \in X$, there exists a finite subset Λ_x of Λ such that $F(x) \subset \bigcup_{\lambda \in \Lambda_x} V_\lambda$. Put

$$V_x = \bigcup_{\lambda \in \Lambda_x} V_\lambda.$$

Since F is an u.na.c. multifunction, there exists $U_x \in RO(X,x)$ such that $F(U_x) \subset V_x$. The family $\{U_x : x \in X\}$ is a regular open cover of X, and since X is a nearly compact space there exist a finite number of points $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^n U_{x_i}$. Hence we have,

$$Y = F(X) = F\left(\bigcup_{i=1}^n U_{x_i}\right) = \bigcup_{i=1}^n F(U_{x_i}) \subset \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n \bigcup_{\lambda \in \Lambda_{x_i}} V_{\lambda_i}$$

This shows that Y is α-compact. □

4.10. Definition. A topological space X is said to be an *α-Normal* space if for any disjoint closed subsets K and F of X there exist two α-open sets U and V such that $K \subset U, F \subset V$ and $U \cap V = \emptyset$.

4.11. Remark. Every normal space is an α-Normal space.

Recall that a multifunction $F : (X, \tau) \to (Y, \vartheta)$ is said to be *point closed* if for each $x \in X$, $F(x)$ is closed.

4.12. Theorem. *Let F and G be u.na.c point closed multifunctions from a topological space (X, τ) to an α-Normal space (Y, ϑ). Then the set $K = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.*

Proof. Let $x \in X \setminus K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions, $F(x)$ and $G(x)$ are closed sets, and Y is an α-Normal space, so there exist disjoint α-open sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are u.na.c. multifunctions, $F^+(U)$ and $G^+(V)$ are δ-open by Theorem 3.2 (3), and so open sets containing x. Put

$$H = F^+(U) \cap G^+(V).$$

Then H is an open set containing x, and $H \cap K = \emptyset$. Hence K is closed in X. □

4.13. Definition. A topological space X is said to be *δ-Hausdorff* if for any pair of distinct points x and y in X, there exist disjoint δ-open sets U and V in X such that $x \in U$ and $y \in V$.

4.14. Remark. Every δ-Hausdorff space is a Hausdorff space.

4.15. Theorem. *Let $F : (X, \tau) \to (Y, \vartheta)$ be an u.na.c. point closed multifunction from a topological space X to an α-Normal space Y, and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a δ-Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then, $F(x) \cap F(y) = \emptyset$. Since F is point closed, $F(x)$ and $F(y)$ are closed sets, and since Y is an α-Normal space, there exist disjoint α-open sets U and V containing $F(x)$ and $F(y)$, respectively. Since F is u.na.c, $F^+(U)$ and $F^+(V)$ are disjoint δ-open sets containing x, y, respectively. This shows that X is a δ-Hausdorff space. □

4.16. Definition. A topological space X is said to be *δ-connected* provided that X is not the union of two disjoint nonempty δ-open sets.

4.17. Remark. Every connected space is a δ-connected space.

Recall that $F : (X, \tau) \to (Y, \vartheta)$ is *punctually connected* if for each $x \in X$, $F(x)$ is connected.

4.18. Theorem. *Let $F : (X, \tau) \to (Y, \vartheta)$ be an u.na.c. surjective multifunction. If X is δ-connected and F is punctually connected, then Y is connected.*

Proof. Suppose that Y is not connected. Then there exist nonempty open sets U and V of Y such that $Y = U \cup V$ and $U \cap V = \emptyset$. Since $F(x)$ is connected for each $x \in X$, we have either $F(x) \subset U$ or $F(x) \subset V$. This implies $x \in F^+(U) \cup F^+(V)$, so $F^+(U) \cup F^+(V) = X$. Since $U \neq \emptyset$ we may choose $u \in U$, and since F is surjective there exists $x \in X$ with $u \in F(x)$, so $F(x) \subset U$ and $x \in F^+(U) \neq \emptyset$. In the same way, $V \neq \emptyset$ implies $F+(V) \neq \emptyset$. Finally $F^+(U) \cap F^+(V) = \emptyset$. But $F^+(U)$ and $F^+(V)$ are δ-open sets since F is an u.na.c. multifunction, which is a contradiction since X is δ-connected. Hence we obtain that Y is connected. □

Acknowledgment.

The authors thank the referee for his valuable comments.

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