

## $\alpha$ -separation axioms based on Łukasiewicz logic

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### Abstract

In the present paper, we introduce topological notions defined by means of  $\alpha$ -open sets when these are planted into the framework of Ying's fuzzifying topological spaces (by Łukasiewicz logic in  $[0, 1]$ ). We introduce  $T_0^\alpha$ -,  $T_1^\alpha$ -,  $T_2^\alpha$  ( $\alpha$ - Hausdorff)-,  $T_3^\alpha$  ( $\alpha$ -regular)- and  $T_4^\alpha$  ( $\alpha$ -normal)-separation axioms. Furthermore, the  $R_0^\alpha$ - and  $R_1^\alpha$ - separation axioms are studied and their relations with the  $T_1^\alpha$ - and  $T_2^\alpha$ - separation axioms are introduced. Moreover, we clarify the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms.

**Keywords:** Łukasiewicz logic, semantics, fuzzifying topology, fuzzifying separation axioms,  $\alpha$ -separation axioms.

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### 1. Introduction and Preliminaries

In the last few years fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [7-9, 14-15, 27]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [8], the kind of topologies defined by Chang [4] and Goguen [5] is called the topologies of fuzzy subsets, and further is naturally called  $L$ -topological spaces if a lattice  $L$  of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an  $L$ -topological space) is a family  $\tau$  of fuzzy subsets (resp.  $L$ -fuzzy subsets) of nonempty set  $X$ , and  $\tau$  satisfies the basic conditions of classical topologies [11]. On the other hand, Höhle in [6] proposed the terminology  $L$ -fuzzy topology to be an  $L$ -valued mapping on the traditional powerset  $P(X)$  of  $X$ . The authors in [10, 23] defined an  $L$ -fuzzy topology to be an  $L$ -valued mapping on the  $L$ -powerset  $L^X$  of  $X$ .

In 1952, Rosser and Turquette [25] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories? As an attempt to give a partial answer

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to this problem in the case of point set topology, Ying in 1991-1993 [28-30] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set  $X$  assigns each crisp subset of  $X$  to a certain degree of being open, other than being definitely open or not. In fact, fuzzifying topologies are a special case of the  $L$ -fuzzy topologies in [10, 23] since all the t-norms on  $I = [0, 1]$  are included as a special class of tensor products in these paper. Ying uses one particular tensor product, namely Lukasiewicz conjunction. Thus his fuzzifying topologies are a special class of all the  $I$ -fuzzy topologies considered in the categorical frameworks [10, 23]. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies [12-13, 20-21, 26]. For example, Shen [26] introduced and studied  $T_0$ -,  $T_1$ -,  $T_2$  (Hausdorff)-,  $T_3$  (regular)- and  $T_4$  (normal)- separation axioms in fuzzifying topology. In [13], the concepts of the  $R_0$ - and  $R_1$ - separation axioms in fuzzifying topology were added and their relations with the  $T_1$ - and  $T_2$ - separation axioms, were studied. Also, in [12] the concepts of fuzzifying  $\alpha$ -open set and fuzzifying  $\alpha$ -continuity were introduced and studied. In classical topology,  $\alpha$ -separation axioms have been studied in [2-3, 16-17, 19, 22]. As well as, they have been studied in fuzzy topology in [1, 18, 24]. In the present paper, we explore the problem proposed by Rosser and Turquette [25] in fuzzy  $\alpha$ -separation axioms.

A basic structure of the present paper is as follows. First, we offer some definitions and results which will be needed in this paper. Afterwards, in Section 2, in the framework of fuzzifying topology, the concept of  $\alpha$ -separation axioms  $T_0^\alpha$ -,  $T_1^\alpha$ -,  $T_2^\alpha$  ( $\alpha$ -Hausdorff)-,  $T_3^\alpha$  ( $\alpha$ -regular)- and  $T_4^\alpha$  ( $\alpha$ -normal) are discussed. In Section 3, on the bases of fuzzifying topology the  $R_0^\alpha$ - and  $R_1^\alpha$ - separation axioms are introduced and their relations with the  $T_1^\alpha$  and  $T_2^\alpha$ - separation axioms are studied. Furthermore, we give the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms. Finally, in a conclusion, we summarize the main results obtained and raise some related problems for further study. Thus we fill a gap in the existing literature on fuzzifying topology. We will use the terminologies and notations in [12-13, 26, 28, 29] without any explanation. We will use the symbol  $\otimes$  instead of the second "AND" operation  $\wedge$  as dot is hardly visible. This mean that  $[\alpha] \leq [\varphi \rightarrow \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi]$ .

A fuzzifying topology on a set  $X$  [6, 28] is a mapping  $\tau \in \mathfrak{S}(P(X))$  such that:

- (1)  $\tau(X) = 1, \tau(\emptyset) = 1$ ;
- (2) for any  $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ ;
- (3) for any  $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$ .

The family of all fuzzifying  $\alpha$ -open sets [12], denoted by  $\tau_\alpha \in \mathfrak{S}(P(X))$ , is defined as

$$A \in \tau_\alpha := \forall x(x \in A \rightarrow x \in \text{Int}(\text{Cl}(\text{Int}(A))))), \text{ i. e., } \tau_\alpha(A) = \bigwedge_{x \in A} \text{Int}(\text{Cl}(\text{Int}(A)))(x)$$

The family of all fuzzifying  $\alpha$ -closed sets [12], denoted by  $F_\alpha \in \mathfrak{S}(P(X))$ , is defined as  $A \in F_\alpha := X - A \in \tau_\alpha$ . The fuzzifying  $\alpha$ -neighborhood system of a point  $x \in X$

[12] is denoted by  $N_x^\alpha \in \mathfrak{S}(P(X))$  and defined as  $N_x^\alpha(A) = \bigvee_{x \in B \subseteq A} \tau_\alpha(B)$ . The fuzzifying  $\alpha$ -closure of a set  $A \subseteq X$  [12], denoted by  $Cl_\alpha \in \mathfrak{S}(X)$ , is defined as  $Cl_\alpha(A)(x) = 1 - N_x^\alpha(X - A)$ .

Let  $(X, \tau)$  be a fuzzifying topological space. The binary fuzzy predicates  $K, H, M \in \mathfrak{S}(X \times X)$ ,  $V \in \mathfrak{S}(X \times P(X))$  and  $W \in \mathfrak{S}(P(X) \times P(X))$  [13] are defined as follows:

- (1)  $K(x, y) := \exists A((A \in N_x \wedge y \notin A) \vee (A \in N_y \wedge x \notin A))$ ;
- (2)  $H(x, y) := \exists B \exists C((B \in N_x \wedge y \notin B) \wedge (C \in N_y \wedge x \notin C))$ ;
- (3)  $M(x, y) := \exists B \exists C(B \in N_x \wedge C \in N_y \wedge B \cap C \equiv \emptyset)$ ;
- (4)  $V(x, D) := \exists A \exists B(A \in N_x \wedge B \in \tau \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$ ;
- (5)  $W(A, B) := \exists G \exists H(G \in \tau \wedge H \in \tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$ .

Let  $\Omega$  be the class of all fuzzifying topological spaces. The unary fuzzy predicates  $T_i \in \mathfrak{S}(\Omega)$ ,  $i = 0, 1, 2, 3, 4$  [26] (see the rewritten form in [13]) and  $R_i \in \mathfrak{S}(\Omega)$ ,  $i = 0, 1$  [13] are defined as follows:

- (1)  $(X, \tau) \in T_0 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K(x, y)$ ;
- (2)  $(X, \tau) \in T_1 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H(x, y)$ ;
- (3)  $(X, \tau) \in T_2 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M(x, y)$ ;
- (4)  $(X, \tau) \in T_3 := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V(x, D)$ ;
- (5)  $(X, \tau) \in T_4 := \forall A \forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W(A, B)$ ;
- (6)  $(X, \tau) \in R_0 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K(x, y) \longrightarrow H(x, y))$ ;
- (7)  $(X, \tau) \in R_1 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K(x, y) \longrightarrow M(x, y))$ .

## 2. Fuzzifying $\alpha$ - separation axioms and their equivalents

For simplicity we give the following definition.

**2.1. Definition.** Let  $(X, \tau)$  be a fuzzifying topological space. The binary fuzzy predicates  $K^\alpha, H^\alpha, M^\alpha \in \mathfrak{S}(X \times X)$ ,  $V^\alpha \in \mathfrak{S}(X \times P(X))$  and  $W^\alpha \in \mathfrak{S}(P(X) \times P(X))$  are defined as follows:

- (1)  $K^\alpha(x, y) := \exists A((A \in N_x^\alpha \wedge y \notin A) \vee (A \in N_y^\alpha \wedge x \notin A))$ ;
- (2)  $H^\alpha(x, y) := \exists B \exists C((B \in N_x^\alpha \wedge y \notin B) \wedge (C \in N_y^\alpha \wedge x \notin C))$ ;
- (3)  $M^\alpha(x, y) := \exists B \exists C(B \in N_x^\alpha \wedge C \in N_y^\alpha \wedge B \cap C \equiv \emptyset)$ ;
- (4)  $V^\alpha(x, D) := \exists A \exists B(A \in N_x^\alpha \wedge B \in \tau_\alpha \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$ ;
- (5)  $W^\alpha(A, B) := \exists G \exists H(G \in \tau_\alpha \wedge H \in \tau_\alpha \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$ .

**2.2. Definition.** Let  $\Omega$  be the class of all fuzzifying topological spaces. The unary fuzzy predicates  $T_i^\alpha \in \mathfrak{S}(\Omega)$ ,  $i = 0, 1, 2, 3, 4$  and  $R_i^\alpha \in \mathfrak{S}(\Omega)$ ,  $i = 0, 1$  are defined as follows:

- (1)  $(X, \tau) \in T_0^\alpha := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K^\alpha(x, y)$ ;
- (2)  $(X, \tau) \in T_1^\alpha := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H^\alpha(x, y)$ ;
- (3)  $(X, \tau) \in T_2^\alpha := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M^\alpha(x, y)$ ;
- (4)  $(X, \tau) \in T_3^\alpha := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V^\alpha(x, D)$ ;
- (5)  $(X, \tau) \in T_4^\alpha := \forall A \forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W^\alpha(A, B)$ ;
- (6)  $(X, \tau) \in T_3^{\alpha'} := \forall x \forall D(x \in X \wedge D \in F_\alpha \wedge x \notin D) \longrightarrow V(x, D)$ ;
- (7)  $(X, \tau) \in T_4^{\alpha'} := \forall A \forall B(A \in F_\alpha \wedge B \in F_\alpha \wedge A \cap B = \emptyset) \longrightarrow W(A, B)$ ;
- (8)  $(X, \tau) \in R_0^\alpha := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^\alpha(x, y) \longrightarrow H^\alpha(x, y))$ ;
- (9)  $(X, \tau) \in R_1^\alpha := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^\alpha(x, y) \longrightarrow M^\alpha(x, y))$ .

**2.3. Theorem.** Let  $(X, \tau)$  be a fuzzifying topological space. Then we have

$\models (X, \tau) \in T_0^\alpha \iff \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow (\neg(x \in Cl_\alpha(\{y\})) \vee \neg(y \in Cl_\alpha(\{x\})))$ .

*Proof.* Since for any  $x, A, B$ ,  $\models A \subseteq B \rightarrow (A \in N_x^\alpha \rightarrow B \in N_x^\alpha)$  (see [12, Theorem 4.2 (2)]), we have

$$\begin{aligned} [(X, \tau) \in T_0^\alpha] &= \bigwedge_{x \neq y} \max\left(\bigvee_{y \notin A} N_x^\alpha(A), \bigvee_{x \notin A} N_y^\alpha(A)\right) \\ &= \bigwedge_{x \neq y} \max(N_x^\alpha(X - \{y\}), N_y^\alpha(X - \{x\})) \\ &= \bigwedge_{x \neq y} \max(1 - Cl_\alpha(\{y\})(x), 1 - Cl_\alpha(\{x\})(y)) \\ &= \bigwedge_{x \neq y} (\neg(Cl_\alpha(\{y\})(x)) \vee \neg(Cl_\alpha(\{x\})(y))) \\ &= [\forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow (\neg(x \in Cl_\alpha(\{y\})) \vee \neg(y \in Cl_\alpha(\{x\})))]. \end{aligned}$$

□

**2.4. Theorem.** For any fuzzifying topological space  $(X, \tau)$  we have  
 $\models \forall x (\{x\} \in F_\alpha) \leftrightarrow (X, \tau) \in T_1^\alpha$ .

*Proof.* Since  $\tau_\alpha(A) = \bigwedge_{x \in A} N_x^\alpha(A)$  (Corollary 4.1 in [12]), for any  $x_1, x_2$  with  $x_1 \neq x_2$ , we have

$$\begin{aligned} [\forall x (\{x\} \in F_\alpha)] &= \bigwedge_{x \in X} F_\alpha(\{x\}) = \bigwedge_{x \in X} \tau_\alpha(X - \{x\}) \leq \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} N_y^\alpha(X - \{x\}) \\ &\leq \bigwedge_{y \in X - \{x_2\}} N_y^\alpha(X - \{x_2\}) \leq N_{x_1}^\alpha(X - \{x_2\}) = \bigvee_{x_2 \notin A} N_{x_1}^\alpha(A). \end{aligned}$$

Similarly, we have,  $[\forall x (\{x\} \in F_\alpha)] \leq \bigvee_{x_1 \notin B} N_{x_2}^\alpha(B)$ . Then

$$\begin{aligned} [\forall x (\{x\} \in F_\alpha)] &\leq \bigwedge_{x_1 \neq x_2} \min\left(\bigvee_{x_2 \notin A} N_{x_1}^\alpha(A), \bigvee_{x_1 \notin B} N_{x_2}^\alpha(B)\right) \\ &= \bigwedge_{x_1 \neq x_2} \bigvee_{x_1 \notin B, x_2 \notin A} \min(N_{x_1}^\alpha(A), N_{x_2}^\alpha(B)) \\ &= [(X, \tau) \in T_1^\alpha]. \end{aligned}$$

On the other hand

$$\begin{aligned}
 [(X, \tau) \in T_1^\alpha] &= \bigwedge_{x_1 \neq x_2} \min\left(\bigvee_{x_2 \notin A} N_{x_1}^\alpha(A), \bigvee_{x_1 \notin B} N_{x_2}^\alpha(B)\right) \\
 &= \bigwedge_{x_1 \neq x_2} \min(N_{x_1}^\alpha(X - \{x_2\}), N_{x_2}^\alpha(X - \{x_1\})) \\
 &\leq \bigwedge_{x_1 \neq x_2} N_{x_1}^\alpha(X - \{x_2\}) = \bigwedge_{x_2 \in X} \bigwedge_{x_1 \in X - \{x_2\}} N_{x_1}^\alpha(X - \{x_2\}) \\
 &= \bigwedge_{x_2 \in X} \tau_\alpha(X - \{x_2\}) = \bigwedge_{x \in X} \tau_\alpha(X - \{x\}) \\
 &= [\forall x(\{x\} \in F_\alpha)].
 \end{aligned}$$

Therefore  $[\forall x(\{x\} \in F_\alpha)] = [(X, \tau) \in T_1^\alpha]$ . □

**2.5. Definition.** Let  $(X, \tau)$  be a fuzzifying topological space. The fuzzifying  $\alpha$ -derived set  $D_\alpha(A)$  of  $A$  is defined as follows:  $x \in D_\alpha(A) := \forall B(B \in N_x^\alpha \rightarrow B \cap (A - \{x\}) \neq \phi)$ .

**2.6. Lemma.**  $D_\alpha(A)(x) = 1 - N_x^\alpha((X - A) \cup \{x\})$ .

*Proof.* From Theorem 4.2 (2) [12] we have

$$D_\alpha(A)(x) = 1 - \bigvee_{B \cap (A - \{x\}) = \phi} N_x^\alpha(B) = 1 - N_x^\alpha((X - A) \cup \{x\}).$$

□

**2.7. Theorem.** For any finite set  $A \subseteq X$ ,  $\models T_1^\alpha(X, \tau) \rightarrow D_\alpha(A) \equiv \phi$ .

*Proof.* From Theorem 4.2 (2) [12] we have

$$\begin{aligned}
 \bigwedge_{y \in X - A} N_y^\alpha((X - A) \cup \{y\}) &\geq \bigwedge_{y \in X - A} N_y^\alpha(X - A) = \bigwedge_{y \in X - A} N_y^\alpha\left(\bigcap_{x \in A} (X - \{x\})\right) \\
 &\geq \bigwedge_{y \in X - A} \bigwedge_{x \in A} N_y^\alpha(X - \{x\}) \geq \bigwedge_{x \neq y} N_y^\alpha(X - \{x\}).
 \end{aligned}$$

Also

$$\begin{aligned}
 \bigwedge_{y \in A} N_y^\alpha((X - A) \cup \{y\}) &= \bigwedge_{y \in A} N_y^\alpha(X - (A - \{y\})) = \bigwedge_{y \in A} N_y^\alpha\left(\bigcap_{x \in A - \{y\}} (X - \{x\})\right) \\
 &\geq \bigwedge_{y \in A} \bigwedge_{x \in A - \{y\}} N_y^\alpha(X - \{x\}) \geq \bigwedge_{x \neq y} N_y^\alpha(X - \{x\}).
 \end{aligned}$$

Therefore

$$\begin{aligned}
[D_\alpha(A) \equiv \phi] &= \bigwedge_{x \in X} N_x^\alpha((X - A) \cup \{x\}) \\
&= \min\left(\bigwedge_{y \in X - A} N_y^\alpha((X - A) \cup \{y\}), \bigwedge_{y \in A} N_y^\alpha((X - A) \cup \{y\})\right) \\
&\geq \bigwedge_{x \neq y} N_y^\alpha(X - \{x\}) = \bigwedge_{x \in X} \bigwedge_{x \in X - \{y\}} N_y^\alpha(X - \{x\}) \\
&= \bigwedge_{x \in X} \tau_\alpha(X - \{x\}) = \bigwedge_{x \in X} F_\alpha(\{x\}) = T_1^\alpha(X, \tau).
\end{aligned}$$

□

**2.8. Definition.** The fuzzifying  $\alpha$ -local basis  $\beta_x^\alpha$  of  $x$  is a function from  $P(X)$  into  $I = [0, 1]$  satisfying the following conditions:

$$(1) \models \beta_x^\alpha \subseteq N_x^\alpha, \text{ and } (2) \models A \in N_x^\alpha \longrightarrow \exists B(B \in \beta_x^\alpha \wedge x \in B \subseteq A).$$

**2.9. Lemma.**  $\models A \in N_x^\alpha \longleftrightarrow \exists B(B \in \beta_x^\alpha \wedge x \in B \subseteq A)$ .

*Proof.* From condition (1) in Definition 2.8 and Theorem 4.2 (2) in [12] we have  $N_x^\alpha(A) \geq N_x^\alpha(B) \geq \beta_x^\alpha(B)$  for each  $B \in P(X)$  such that  $x \in B \subseteq A$ . So  $N_x^\alpha(A) \geq \bigvee_{x \in B \subseteq A} \beta_x^\alpha(B)$ . From condition (2) in Definition 2.8 we have  $N_x^\alpha(A) \leq \bigvee_{x \in B \subseteq A} \beta_x^\alpha(B)$ . Hence  $N_x^\alpha(A) = \bigvee_{x \in B \subseteq A} \beta_x^\alpha(B)$ . □

**2.10. Theorem.** If  $\beta_x^\alpha$  is a fuzzifying  $\alpha$ -local basis of  $x$ , then

$$\models (X, \tau) \in T_1^\alpha \longleftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow \exists A (A \in \beta_x^\alpha \wedge y \notin A)).$$

*Proof.* For any  $x, y$  with  $x \neq y$ ,  $\bigvee_{y \notin A} \beta_x^\alpha(A) \leq \bigvee_{y \notin A} N_x^\alpha(A)$ ,  $\bigvee_{x \notin B} \beta_y^\alpha(B) \leq \bigvee_{x \notin B} N_y^\alpha(B)$ .

So  $\min(\bigvee_{y \notin A} \beta_x^\alpha(A), \bigvee_{x \notin B} \beta_y^\alpha(B)) \leq \min(\bigvee_{y \notin A} N_x^\alpha(A), \bigvee_{x \notin B} N_y^\alpha(B)) = \bigvee_{y \notin A, x \notin B} \min(N_x^\alpha(A), N_y^\alpha(B))$ ,

i.e.,  $\bigwedge_{x \neq y} \bigvee_{y \notin A} \beta_x^\alpha(A) \leq \bigwedge_{x \neq y} \bigvee_{y \notin A, x \notin B} \min(N_x^\alpha(A), N_y^\alpha(B)) = [(X, \tau) \in T_1^\alpha]$ . On the

other hand, for any  $B$  with  $x \in B \subseteq X - \{y\}$  we have  $y \notin B$ . So  $\bigvee_{y \notin A} \beta_x^\alpha(A) \geq \beta_x^\alpha(B)$ . According to Definition 2.8 we have  $\bigvee_{y \notin A} \beta_x^\alpha(A) \geq \bigvee_{x \in B \subseteq X - \{y\}} \beta_x^\alpha(B) = N_x^\alpha(X - \{y\})$ . Furthermore, from Corollary 4.1 [12] we have  $\bigwedge_{x \neq y} \bigvee_{y \notin A} \beta_x^\alpha(A) \geq \bigwedge_{x \neq y} N_x(X - \{y\}) = \bigwedge_{y \in X} \bigwedge_{x \in X - \{y\}} N_x(X - \{y\}) = \bigwedge_{y \in X} \tau_\alpha(X - \{y\}) = \bigwedge_{y \in X} F_\alpha(\{y\}) = [(X, \tau) \in T_1^\alpha]$ . □

**2.11. Theorem.** If  $\beta_x^\alpha$  is a fuzzifying  $\alpha$ -local basis of  $x$ , then

$$\models (X, \tau) \in T_2^\alpha \longleftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow \exists B (B \in \beta_x^\alpha \wedge y \in \neg(Cl_\alpha(B)))).$$

*Proof.*

$$\begin{aligned}
 & [\forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow \exists B (B \in \beta_x^\alpha \wedge y \in \neg(Cl_\alpha(B))))] \\
 &= \bigwedge_{x \neq y} \bigvee_{B \in P(X)} \min(\beta_x^\alpha(B), \neg(1 - N_y^\alpha(X - B))) \\
 &= \bigwedge_{x \neq y} \bigvee_{B \in P(X)} \min(\beta_x^\alpha(B), N_y^\alpha(X - B)) \\
 &= \bigwedge_{x \neq y} \bigvee_{B \in P(X)} \bigvee_{y \in C \subseteq X - B} \min(\beta_x^\alpha(B), \beta_y^\alpha(C)) \\
 &= \bigwedge_{x \neq y} \bigvee_{B \cap C = \emptyset} \bigvee_{x \in D \subseteq B, y \in E \subseteq C} \min(\beta_x^\alpha(D), \beta_y^\alpha(E)) \\
 &= \bigwedge_{x \neq y} \bigvee_{B \cap C = \emptyset} \min(\bigvee_{x \in D \subseteq B} \beta_x^\alpha(D), \bigvee_{y \in E \subseteq C} \beta_y^\alpha(E)) \\
 &= \bigwedge_{x \neq y} \bigvee_{B \cap C = \emptyset} \min(N_x^\alpha(B), N_y^\alpha(C)) = [(X, \tau) \in T_2^\alpha].
 \end{aligned}$$

□

**2.12. Definition.** The binary fuzzy predicate  $\triangleright^\alpha \in \mathfrak{S}(N(X) \times X)$ , is defined as  $S \triangleright^\alpha x := \forall A (A \in N_x^\alpha \longrightarrow S \lesssim A)$ , where  $N(X)$  is the set of all nets of  $X$ ,  $[S \triangleright^\alpha x]$  stands for the degree to which  $S$   $\alpha$ -converges to  $x$  and " $\lesssim$ " is the binary crisp predicates "almost in".

**2.13. Theorem.** Let  $(X, \tau)$  be a fuzzifying topological space and  $S \in N(X)$ .  
 $\models (X, \tau) \in T_2^\alpha \iff \forall S \forall x \forall y ((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \triangleright^\alpha x) \wedge (S \triangleright^\alpha y) \longrightarrow x = y)$ .

*Proof.*  $[(X, \tau) \in T_2^\alpha] = \bigwedge_{x \neq y} \bigvee_{A \cap B = \emptyset} (N_x^\alpha(A) \wedge N_y^\alpha(B))$ ,  
 $[\forall S \forall x \forall y ((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \triangleright^\alpha x) \wedge (S \triangleright^\alpha y) \longrightarrow x = y)]$   
 $= \bigwedge_{x \neq y} \bigwedge_{S \subseteq X} ( \bigvee_{S \not\lesssim A} N_x^\alpha(A) \vee \bigvee_{S \not\lesssim B} N_y^\alpha(B) )$   
 $= \bigwedge_{x \neq y} \bigwedge_{S \subseteq X} \bigvee_{S \not\lesssim A} \bigvee_{S \not\lesssim B} (N_x^\alpha(A) \vee N_y^\alpha(B))$ .

(1) If  $A \cap B = \emptyset$ , then for any  $S \in N(X)$ , we have  $S \not\lesssim A$  or  $S \not\lesssim B$ . Therefore, we obtain  $N_x^\alpha(A) \wedge N_y^\alpha(B) \leq \bigvee_{S \not\lesssim A} N_x^\alpha(A)$  or  $N_x^\alpha(A) \wedge N_y^\alpha(B) \leq \bigvee_{S \not\lesssim B} N_x^\alpha(B)$ .

Consequently,  $\bigvee_{A \cap B = \emptyset} (N_x^\alpha(A) \wedge N_y^\alpha(B)) \leq \bigwedge_{S \subseteq X} ( \bigvee_{S \not\lesssim A} N_x^\alpha(A) \vee \bigvee_{S \not\lesssim B} N_y^\alpha(B) )$ ,

and  $[(X, \tau) \in T_2^\alpha] \leq [\forall S \forall x \forall y ((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \triangleright^\alpha x) \wedge (S \triangleright^\alpha y) \longrightarrow x = y)]$ .

(2) First, for any  $x, y$  with  $x \neq y$ , if  $\bigvee_{A \cap B = \emptyset} (N_x^\alpha(A) \wedge N_y^\alpha(B)) < t$ , then  $N_x^\alpha(A) < t$  or  $N_y^\alpha(B) < t$  provided  $A \cap B = \emptyset$ , i.e.,  $A \cap B \neq \emptyset$  when  $A \in (N_x^\alpha)_t$  and  $B \in (N_y^\alpha)_t$ . Now, set a net  $S^* : (N_x^\alpha)_t \times (N_y^\alpha)_t \longrightarrow X$ ,  $(A, B) \mapsto x_{(A,B)} \in A \cap B$ . Then for any  $A \in (N_x^\alpha)_t$ ,  $B \in (N_y^\alpha)_t$ , we have  $S^* \lesssim A$  and  $S^* \lesssim B$ . Therefore, if  $S^* \not\lesssim A$  and

$S^* \not\lesssim B$ , then  $A \notin (N_x^\alpha)_t$ ,  $B \notin (N_y^\alpha)_t$ , i.e.,  $N_x^\alpha(A) \vee N_y^\alpha(B) < t$ . Consequently  $\bigvee_{S^* \not\lesssim A} \bigvee_{S^* \not\lesssim B} (N_x^\alpha(A) \vee N_y^\alpha(B)) \leq t$ . Moreover  $\bigwedge_{S \subseteq X} \bigvee_{S \not\lesssim A} \bigvee_{S \not\lesssim B} (N_x^\alpha(A) \vee N_y^\alpha(B)) \leq t$ .

Second, for any positive integer  $i$ , there exists  $x_i, y_i$  with  $x_i \neq y_i$ , and

$$\bigvee_{A \cap B = \emptyset} (N_{x_i}^\alpha(A) \wedge N_{y_i}^\alpha(B)) < [(X, \tau) \in T_2^\alpha] + 1/i,$$

and hence

$$\bigwedge_{S \subseteq X} \bigvee_{S \not\lesssim A} \bigvee_{S \not\lesssim B} (N_{x_i}^\alpha(A) \vee N_{y_i}^\alpha(B)) < [(X, \tau) \in T_2^\alpha] + 1/i.$$

So we have

$$\begin{aligned} & [\forall S \forall x \forall y ((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \triangleright^\alpha x) \wedge (S \triangleright^\alpha y) \longrightarrow x = y)] \\ &= \bigwedge_{x \neq y} \bigwedge_{S \subseteq X} \bigvee_{S \not\lesssim A} \bigvee_{S \not\lesssim B} (N_x^\alpha(A) \vee N_y^\alpha(B)) \leq [(X, \tau) \in T_2^\alpha]. \end{aligned}$$

□

**2.14. Lemma.** Let  $(X, \tau)$  be a fuzzifying topological space.

- (1) If  $D \subseteq B$ , then  $\bigvee_{A \cap B = \emptyset} N_x^\alpha(A) = \bigvee_{A \cap B = \emptyset, D \subseteq B} N_x^\alpha(A)$ ,
- (2)  $\bigvee_{A \cap B = \emptyset} \bigwedge_{y \in D} N_y^\alpha(X - A) = \bigvee_{A \cap B = \emptyset, D \subseteq B} \tau_\alpha(B)$ .

*Proof.* (1) Since  $D \subseteq B$  then

$$\bigvee_{A \cap B = \emptyset} N_x^\alpha(A) = \bigvee_{A \cap B = \emptyset} N_x^\alpha(A) \wedge [D \subseteq B] = \bigvee_{A \cap B = \emptyset, D \subseteq B} N_x^\alpha(A).$$

(2) Let  $y \in D$  and  $A \cap B = \emptyset$ . Then

$$\begin{aligned} \bigvee_{A \cap B = \emptyset, D \subseteq B} \tau_\alpha(B) &= \bigvee_{A \cap B = \emptyset, D \subseteq B} \tau_\alpha(B) \wedge [y \in D] \\ &= \bigvee_{y \in D \subseteq B \subseteq X - A} \tau_\alpha(B) = \bigvee_{y \in B \subseteq X - A} \tau_\alpha(B) \\ &= N_y^\alpha(X - A) = \bigwedge_{y \in D} N_y^\alpha(X - A) \\ &= \bigvee_{A \cap B = \emptyset} \bigwedge_{y \in D} N_y^\alpha(X - A). \end{aligned}$$

□

**2.15. Definition.** Let  $(X, \tau)$  be a fuzzifying topological space.

$$\alpha T_3^{(1)}(X, \tau) := \forall x \forall D (x \in X \wedge D \in F \wedge x \notin D \longrightarrow \exists A (A \in N_x^\alpha \wedge (D \subseteq X - Cl_\alpha(A)))).$$

**2.16. Theorem.**  $\models (X, \tau) \in T_3^\alpha \iff (X, \tau) \in \alpha T_3^{(1)}$ .



*Proof.*

$$\begin{aligned} \alpha T_3^{(1)}(X, \tau) &= \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in D} (1 - Cl_\alpha(A)(y)))) \\ &= \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in D} N_y^\alpha(X - A))) \end{aligned}$$

$$\text{and } T_3^\alpha(X, \tau) = \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \cap B = \emptyset, D \subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B))).$$

So, the result holds if we prove that

$$\bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in D} N_y^\alpha(X - A)) = \bigvee_{A \cap B = \emptyset, D \subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B)) \quad (*)$$

It is clear that, on the left-hand side of (\*) in the case of  $A \cap D \neq \emptyset$  there exists  $y \in X$  such that  $y \in D$  and  $y \notin X - A$ . So,  $\bigwedge_{y \in D} N_y^\alpha(X - A) = 0$  and thus (\*) becomes

$$\bigvee_{A \in P(X), A \cap B = \emptyset} \min(N_x^\alpha(A), \bigwedge_{y \in D} N_y^\alpha(X - A)) = \bigvee_{A \cap B = \emptyset, D \subseteq B} \min(N_x^\alpha(A), \tau_\alpha(B)),$$

which is obtained from Lemma 2.14. □

**2.17. Definition.** Let  $(X, \tau)$  be a fuzzifying topological space.

$$\alpha T_3^{(2)}(X, \tau) := \forall x \forall B (x \in B \wedge B \in \tau \longrightarrow \exists A (A \in N_x^\alpha \wedge Cl_\alpha(A) \subseteq B)).$$

**2.18. Theorem.**  $\models (X, \tau) \in T_3^\alpha \iff (X, \tau) \in \alpha T_3^{(2)}$ .

*Proof.* From Theorem 2.16 we have

$$T_3^\alpha(X, \tau) = \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in D} N_y^\alpha(X - A))).$$

Now,

$$\begin{aligned} \alpha T_3^{(2)}(X, \tau) &= \bigwedge_{x \in B} \min(1, 1 - \tau(B) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in X - B} (1 - Cl_\alpha(A)(y)))) \\ &= \bigwedge_{x \in B} \min(1, 1 - \tau(B) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in X - B} (1 - (1 - N_y^\alpha(X - A)))))) \\ &= \bigwedge_{x \in B} \min(1, 1 - \tau(B) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in X - B} N_y^\alpha(X - A))). \end{aligned}$$

Put  $B = X - D$  we have

$$\begin{aligned} \alpha T_3^{(2)}(X, \tau) &= \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \in P(X)} \min(N_x^\alpha(A), \bigwedge_{y \in D} N_y^\alpha(X - A))) \\ &= T_3^\alpha(X, \tau). \end{aligned}$$

□

**2.19. Definition.** Let  $(X, \tau)$  be a fuzzifying topological space and  $\varphi$  be a subbase of  $\tau$  then

$$\alpha T_3^{(3)}(X, \tau) := \forall x \forall D(x \in D \wedge D \in \varphi \longrightarrow \exists B(B \in N_x^\alpha \wedge Cl_\alpha(B) \subseteq D)).$$

**2.20. Theorem.**  $\models (X, \tau) \in T_3^\alpha \longleftrightarrow (X, \tau) \in \alpha T_3^{(3)}$ .

*Proof.* Since  $[\varphi \subseteq \tau] = 1$ , from Theorems 2.16 we have

$$\alpha T_3^{(3)}(X, \tau) \geq \alpha T_3^{(2)}(X, \tau) = T_3^\alpha(X, \tau).$$

So, it suffices to prove that  $\alpha T_3^{(3)}(X, \tau) \leq \alpha T_3^{(2)}(X, \tau)$  and this is obtained if we prove for any  $x \in A$ ,

$$\min(1, 1 - \tau(A) + \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X-A} N_y^\alpha(X-B))) \geq \alpha T_3^{(3)}(X, \tau).$$

Set  $\alpha T_3^{(3)}(X, \tau) = \delta$ . Then for any  $x \in X$  and any  $D_{\lambda_i} \in P(X), x \in D_{\lambda_i}, \lambda_i \in I_\lambda$  ( $I_\lambda$  denotes a finite index set),  $\lambda \in \Lambda, \bigcup_{\lambda \in \Lambda} \bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = A$  we have

$$1 - \varphi(D_{\lambda_i}) + \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X-D_{\lambda_i}} N_y^\alpha(X-B)) \geq \delta > \delta - \epsilon,$$

where  $\epsilon$  is any positive number. Thus

$$\bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X-D_{\lambda_i}} N_y^\alpha(X-B)) > \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$

Set  $\gamma_{\lambda_i} = \{B : B \subseteq D_{\lambda_i}\}$ . From the completely distributive law we have

$$\begin{aligned} & \bigwedge_{\lambda_i \in I_\lambda} \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X-D_{\lambda_i}} N_y^\alpha(X-B)) \\ &= \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \bigwedge_{\lambda_i \in I_\lambda} \min(N_x^\alpha(f(\lambda_i)), \bigwedge_{y \in X-D_{\lambda_i}} N_y^\alpha(X-f(\lambda_i))) \\ &= \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\alpha(f(\lambda_i)), \bigwedge_{\lambda_i \in I_\lambda} \bigwedge_{y \in X-D_{\lambda_i}} N_y^\alpha(X-f(\lambda_i))) \\ &= \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\alpha(f(\lambda_i)), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X-D_{\lambda_i}} N_y^\alpha(X-f(\lambda_i))) \\ &= \bigvee_{B \in P(X)} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\alpha(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X-D_{\lambda_i}} N_y^\alpha(X-B)) \\ &= \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X-D_{\lambda_i}} N_y^\alpha(X-B)), \end{aligned}$$

where  $B = f(\lambda_i)$ .

Similarly, we can prove

$$\begin{aligned}
 & \bigwedge_{\lambda \in \Lambda} \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\alpha(X - B)) \\
 &= \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in \bigcup_{\lambda \in \Lambda} \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\alpha(X - B)) \\
 &\leq \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in \bigcap_{\lambda \in \Lambda} \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\alpha(X - B)) \\
 &\leq \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - A} N_y^\alpha(X - B)),
 \end{aligned}$$

so we have

$$\begin{aligned}
 & \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - A} N_y^\alpha(X - B)) \\
 &\geq \bigwedge_{\lambda \in \Lambda} \bigwedge_{\lambda_i \in I_\lambda} \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\alpha(X - B)) \\
 &\geq \bigwedge_{\lambda \in \Lambda} \bigwedge_{\lambda_i \in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.
 \end{aligned}$$

For any  $I_\lambda$  and  $\Lambda$  that satisfy  $\bigcup_{\lambda \in \Lambda} \bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = A$  the above inequality is true. So,

$$\begin{aligned}
 & \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - A} N_y^\alpha(X - B)) \\
 &\geq \bigvee_{\bigcup_{\lambda \in \Lambda} D_{\lambda} = A} \bigwedge_{\lambda \in \Lambda} \bigvee_{\bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = D_{\lambda}} \bigwedge_{\lambda_i \in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon \\
 &= \tau(A) - 1 + \delta - \epsilon.
 \end{aligned}$$

$$\text{i.e., } \min(1, 1 - \tau(A) + \bigvee_{B \in P(X)} \min(N_x^\alpha(B), \bigwedge_{y \in X - A} N_y^\alpha(X - B))) \geq \delta - \epsilon.$$

Because  $\epsilon$  is any arbitrary positive number, when  $\epsilon \rightarrow 0$  we have

$$\alpha T_3^{(2)}(X, \tau) \geq \delta = \alpha T_3^{(3)}(X, \tau). \text{ So, } \models (X, \tau) \in T_3^\alpha \iff (X, \tau) \in \alpha T_3^{(3)}. \quad \square$$

**2.21. Definition.** Let  $(X, \tau)$  be any fuzzifying topological space.

- (1)  $\alpha' T_3^{(1)}(X, \tau) := \forall x \forall D (x \in X \wedge D \in F_\alpha \wedge x \notin D \rightarrow \exists A (A \in N_x \wedge (D \subseteq X - Cl(A))))$ ;
- (2)  $\alpha' T_3^{(2)}(X, \tau) := \forall x \forall B (x \in B \wedge B \in \tau_\alpha \rightarrow \exists A (A \in N_x \wedge Cl(A) \subseteq B))$ ;
- (3)  $\alpha T_4^{(1)}(X, \tau) := \forall A \forall B (A \in \tau \wedge B \in F \wedge A \cap B \equiv \emptyset \rightarrow \exists G (G \in \tau \wedge A \subseteq G \wedge Cl_\alpha(G) \cap B \equiv \emptyset))$ ;
- (4)  $\alpha T_4^{(2)}(X, \tau) := \forall A \forall B (A \in F \wedge B \in \tau \wedge A \subseteq B \rightarrow \exists G (G \in \tau \wedge A \subseteq G \wedge Cl_\alpha(G) \subseteq B))$ ;
- (5)  $\alpha' T_4^{(1)}(X, \tau) := \forall A \forall B (A \in \tau \wedge B \in F_\alpha \wedge A \cap B \equiv \emptyset \rightarrow \exists G (G \in \tau \wedge A \subseteq G \wedge Cl(G) \cap B \equiv \emptyset))$ ;

(6)  $\alpha'T_4^{(2)}(X, \tau) := \forall A \forall B (A \in F \wedge B \in \tau_\alpha \wedge A \subseteq B \rightarrow \exists G (G \in \tau \wedge A \subseteq G \wedge Cl(G) \subseteq B))$ .

By a similar proof of Theorem 2.16 and 2.18 we have the following theorem.

**2.22. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space.*

- (1)  $\models (X, \tau) \in T_3^{\alpha'} \longleftrightarrow (X, \tau) \in \alpha'T_3^{(i)}$ ;
- (2)  $\models (X, \tau) \in T_4^\alpha \longleftrightarrow (X, \tau) \in \alpha T_4^{(i)}$ ;
- (3)  $\models (X, \tau) \in T_4^{\alpha'} \longleftrightarrow (X, \tau) \in \alpha'T_4^{(i)}$ , where  $i = 1, 2$ .

### 3. Relation among fuzzifying separation axioms

**3.1. Lemma.** (1)  $\models K(x, y) \rightarrow K^\alpha(x, y)$ ,

- (2)  $\models H(x, y) \rightarrow H^\alpha(x, y)$ ,
- (3)  $\models M(x, y) \rightarrow M^\alpha(x, y)$ ,
- (4)  $\models V(x, D) \rightarrow V^\alpha(x, D)$ ,
- (5)  $\models W(A, B) \rightarrow W^\alpha(A, B)$ .

*Proof.* Since  $\models \tau \subseteq \tau_\alpha$ ,  $N_x(A) \leq N_x^\alpha(A)$  for any  $A \in P(X)$ . Then the proof is immediate.  $\square$

**3.2. Theorem.**  $\models (X, \tau) \in T_i \longrightarrow (X, \tau) \in T_i^\alpha$ , where  $i = 0, 1, 2, 3, 4$ .

*Proof.* It is obtained from Lemma 3.1.  $\square$

**3.3. Theorem.** *If  $T_0(X, \tau) = 1$ , then*

- (1)  $\models (X, \tau) \in R_0 \longrightarrow (X, \tau) \in R_0^\alpha$ ,
- (2)  $\models (X, \tau) \in R_1 \longrightarrow (X, \tau) \in R_1^\alpha$ ,

*Proof.* Since  $T_0(X, \tau) = 1$ , for each  $x, y \in X$  and  $x \neq y$ , we have  $[K(x, y)] = 1$  and so  $[K^\alpha(x, y)] = 1$ .

(1) Using Lemma 3.1 (1) and (2) we obtain

$$\begin{aligned} [(X, \tau) \in R_0] &= \bigwedge_{x \neq y} [K(x, y) \rightarrow H(x, y)] \leq \bigwedge_{x \neq y} [K(x, y) \rightarrow H^\alpha(x, y)] \\ &\leq \bigwedge_{x \neq y} [K^\alpha(x, y) \rightarrow H^\alpha(x, y)] = R_0^\alpha(X, \tau). \end{aligned}$$

(2) Using Lemma 3.1 (1) and (3) the proof is similar to (1).  $\square$

**3.4. Lemma.** (1)  $\models M^\alpha(x, y) \longrightarrow H^\alpha(x, y)$ ;

- (2)  $\models H^\alpha(x, y) \longrightarrow K^\alpha(x, y)$ ;
- (3)  $\models M^\alpha(x, y) \longrightarrow K^\alpha(x, y)$ .

*Proof.* (1) Since  $\{B, C \in P(X) : B \cap C \equiv \emptyset\} \subseteq \{B, C \in P(X) : y \notin B \text{ and } x \notin C\}$ , then

$$[M^\alpha(x, y)] = \bigvee_{B \cap C = \emptyset} \min(N_x^\alpha(B), N_y^\alpha(C)) \leq \bigvee_{y \notin B, x \notin C} \min(N_x^\alpha(B), N_y^\alpha(C)) = [H^\alpha(x, y)].$$

$$(2) [K^\alpha(x, y)] = \max(\bigvee_{y \notin A} N_x^\alpha(A), \bigvee_{x \notin A} N_y^\alpha(A)) \geq \bigvee_{y \notin A} N_x^\alpha(A) \geq \bigvee_{y \notin A, x \notin B} (N_x^\alpha(A) \wedge$$

$$N_y^\alpha(B)) = [H^\alpha(x, y)].$$

(3) From (1) and (2) it is obvious.  $\square$

**3.5. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space. Then we have*

- (1)  $\models (X, \tau) \in T_1^\alpha \longrightarrow (X, \tau) \in T_0^\alpha$ ;
- (2)  $\models (X, \tau) \in T_2^\alpha \longrightarrow (X, \tau) \in T_1^\alpha$ ;
- (3)  $\models (X, \tau) \in T_2^\alpha \longrightarrow (X, \tau) \in T_0^\alpha$ .

*Proof.* The proof of (1) and (2) are obtained from Lemma 3.4 (2) and (1), respectively.

(3) From (1) and (2) above the result is obtained. □

**3.6. Theorem.**  $\models (X, \tau) \in R_1^\alpha \longrightarrow (X, \tau) \in R_0^\alpha$ .

*Proof.* From Lemma 3.4 (2), the proof is immediate. □

**3.7. Theorem.** *For any fuzzifying topological space  $(X, \tau)$  we have*

- (1)  $\models (X, \tau) \in T_1^\alpha \longrightarrow (X, \tau) \in R_0^\alpha$ ;
- (2)  $\models (X, \tau) \in T_1^\alpha \longrightarrow (X, \tau) \in R_0^\alpha \wedge (X, \tau) \in T_0^\alpha$ ;
- (3) *If  $T_0^\alpha(X, \tau) = 1$ , then  $\models (X, \tau) \in T_1^\alpha \longleftrightarrow (X, \tau) \in R_0^\alpha \wedge (X, \tau) \in T_0^\alpha$ .*

*Proof.* (1)  $T_1^\alpha(X, \tau) = \bigwedge_{x \neq y} [H^\alpha(x, y)] \leq \bigwedge_{x \neq y} [K^\alpha(x, y) \longrightarrow H^\alpha(x, y)] = R_0^\alpha(X, \tau)$ .

(2) It is obtained from (1) and from Theorem 3.5 (1).

(3) Since  $T_0^\alpha(X, \tau) = 1$ , for every  $x, y \in X$  such that  $x \neq y$ , then we have  $[K^\alpha(x, y)] = 1$ . Therefore

$$\begin{aligned} [(X, \tau) \in R_0^\alpha \wedge (X, \tau) \in T_0^\alpha] &= [(X, \tau) \in R_0^\alpha] \\ &= \bigwedge_{x \neq y} \min(1, 1 - [K^\alpha(x, y)] + [H^\alpha(x, y)]) \\ &= \bigwedge_{x \neq y} [H^\alpha(x, y)] = T_1^\alpha(X, \tau). \end{aligned}$$

□

**3.8. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space.*

- (1)  $\models (X, \tau) \in R_0^\alpha \otimes (X, \tau) \in T_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha$ , and
- (2) *If  $T_0^\alpha(X, \tau) = 1$ , then  $\models (X, \tau) \in R_0^\alpha \otimes (X, \tau) \in T_0^\alpha \longleftrightarrow (X, \tau) \in T_1^\alpha$ .*

*Proof.* (1)

$$\begin{aligned} [(X, \tau) \in R_0^\alpha \otimes (X, \tau) \in T_0^\alpha] &= \max(0, R_0^\alpha(X, \tau) + T_0^\alpha(X, \tau) - 1) \\ &= \max(0, \bigwedge_{x \neq y} \min(1, 1 - [K^\alpha(x, y)] + [H^\alpha(x, y)]) + \bigwedge_{x \neq y} [K^\alpha(x, y)] - 1) \\ &\leq \max(0, \bigwedge_{x \neq y} \{\min(1, 1 - [K^\alpha(x, y)] + [H^\alpha(x, y)]) + [K^\alpha(x, y)]\} - 1) \\ &= \bigwedge_{x \neq y} [H^\alpha(x, y)] = T_1^\alpha(X, \tau). \end{aligned}$$

(2)

$$\begin{aligned}
((X, \tau) \in R_0^\alpha \otimes (X, \tau) \in T_0^\alpha) &= [(X, \tau) \in R_0^\alpha] \\
&= \bigwedge_{x \neq y} \min(1, 1 - [K^\alpha(x, y)] + [H^\alpha(x, y)]) \\
&= \bigwedge_{x \neq y} [H^\alpha(x, y)] = T_1^\alpha(X, \tau),
\end{aligned}$$

because  $T_0^\alpha(X, \tau) = 1$ , implies that for each  $x, y$  such that  $x \neq y$  we have  $[K^\alpha(x, y)] = 1$ . □

**3.9. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space.*

- (1)  $\models (X, \tau) \in T_0^\alpha \longrightarrow ((X, \tau) \in R_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha)$ , and  
(2)  $\models (X, \tau) \in R_0^\alpha \longrightarrow ((X, \tau) \in T_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha)$ .

*Proof.* It obtained From Theorems 3.7 (1) and 3.8 (1) and the fact that  $[\alpha] \leq [\varphi \rightarrow \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi]$ . □

**3.10. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space.*

- (1)  $\models (X, \tau) \in T_2^\alpha \longrightarrow (X, \tau) \in R_1^\alpha$ ;  
(2)  $\models (X, \tau) \in T_2^\alpha \longrightarrow (X, \tau) \in R_i^\alpha \wedge (X, \tau) \in T_i^\alpha$ , where  $i = 0, 1$ ;  
(3) If  $T_0^\alpha(X, \tau) = 1$ , then  
(i)  $\models (X, \tau) \in T_2^\alpha \longleftrightarrow (X, \tau) \in R_1^\alpha \wedge (X, \tau) \in T_0^\alpha$ .  
(ii)  $\models (X, \tau) \in T_2^\alpha \longleftrightarrow (X, \tau) \in R_1^\alpha \wedge (X, \tau) \in T_1^\alpha$ .

*Proof.* It is similar to the proof of Theorem 3.7. □

**3.11. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space.*

- (1)  $\models (X, \tau) \in R_1^\alpha \otimes (X, \tau) \in T_0^\alpha \longrightarrow (X, \tau) \in T_2^\alpha$ , and  
(2) If  $T_0^\alpha(X, \tau) = 1$ , then  $\models (X, \tau) \in R_1^\alpha \otimes (X, \tau) \in T_0^\alpha \longleftrightarrow (X, \tau) \in T_2^\alpha$ .

*Proof.* It is similar to the proof of Theorem 3.8. □

**3.12. Theorem.** *Let  $(X, \tau)$  be a fuzzifying topological space.*

- (1)  $\models (X, \tau) \in T_0^\alpha \longrightarrow ((X, \tau) \in R_1^\alpha \longrightarrow (X, \tau) \in T_2^\alpha)$ , and  
(2)  $\models (X, \tau) \in R_1^\alpha \longrightarrow ((X, \tau) \in T_0^\alpha \longrightarrow (X, \tau) \in T_2^\alpha)$ .

*Proof.* It is similar to the proof of Theorem 3.9. □

**3.13. Theorem.** *If  $T_0^\alpha(X, \tau) = 1$ , then*

- (1)  $\models ((X, \tau) \in T_0^\alpha \longrightarrow ((X, \tau) \in R_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha)) \wedge ((X, \tau) \in T_1^\alpha \longrightarrow \neg((X, \tau) \in T_0^\alpha \longrightarrow \neg((X, \tau) \in \alpha_0^\alpha)))$ ;  
(2)  $\models ((X, \tau) \in R_0^\alpha \longrightarrow ((X, \tau) \in T_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha)) \wedge ((X, \tau) \in T_1^\alpha \longrightarrow \neg((X, \tau) \in T_0^\alpha \longrightarrow \neg((X, \tau) \in \alpha_0^\alpha)))$ ;  
(3)  $\models ((X, \tau) \in T_0^\alpha \longrightarrow ((X, \tau) \in R_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha)) \wedge ((X, \tau) \in T_1^\alpha \longrightarrow \neg((X, \tau) \in R_0^\alpha \longrightarrow \neg((X, \tau) \in T_0^\alpha)))$ ;  
(4)  $\models ((X, \tau) \in R_0^\alpha \longrightarrow ((X, \tau) \in T_0^\alpha \longrightarrow (X, \tau) \in T_1^\alpha)) \wedge ((X, \tau) \in T_1^\alpha \longrightarrow \neg((X, \tau) \in R_0^\alpha \longrightarrow \neg((X, \tau) \in T_0^\alpha)))$ .