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SUMS OF PRODUCTS OF THE TERMS OF THE GENERALIZED LUCAS SEQUENCE $\{\mathbf{V_{kn}}\}$

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Abstract

In this study we consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression. We also compute the sums of products of the terms of the Lucas sequence $\{V_{kn}\}$ for positive odd integers k.

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1. Introduction

The binary linear recurrence $W_n = W_n(a, b; p, q)$ is defined as follows for $n > 1$,

$$W_n = pW_{n-1} + qW_{n-2},$$

where $W_0 = a, W_1 = b$.

The Binet formula for $\{W_n\}$ is

$$W_n = A\alpha^n + B\beta^n, \tag{1.1}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$, $B = \frac{a\alpha-b}{\alpha-\beta}$ and $\alpha, \beta = \left(p \pm \sqrt{p^2+4q}\right)/2$.

For $n > 1$ and a fixed positive integer k, the terms of $\{W_{kn}\}$ satisfy the recursion [6, 7]:

$$W_{kn} = V_k W_{k(n-1)} - (-q)^k W_{k(n-2)},$$

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where $V_k = \alpha^k + \beta^k$. Specifically, define the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences as $U_n = W_n(0,1;p,1)$, $V_n = W_n(2,p;p,1)$, respectively. Thus:

(1.2) $$U_{kn} = V_k U_{k(n-1)} - (-1)^k U_{k(n-2)},$$

(1.3) $$V_{kn} = V_k V_{k(n-1)} - (-1)^k V_{k(n-2)}.$$

The Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_F$ are defined by the relation

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1 F_2 \cdots F_n}{(F_1 F_2 \cdots F_{n-m})(F_1 F_2 \cdots F_m)},$$

for $n \geq m \geq 1$, with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = \begin{bmatrix} n \\ n \end{bmatrix}_F = 1$, where F_n is the n^{th} Fibonacci number. These coefficients satisfy the relation:

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.$$

Hoggatt [4] defined the following generalization by taking F_{kn} instead of F_n for a fixed positive integer k:

$$\begin{bmatrix} n \\ m \end{bmatrix}_{F_k} = \frac{F_k F_{2k} \cdots F_{kn}}{\left(F_k F_{2k} \cdots F_{k(n-m)}\right)(F_k F_{2k} \cdots F_{km})}.$$

Jarden and Motzkin were the first to study the generalized Fibonomial coefficients formed by terms of the sequence $\{U_n\}$ as follows: for $n \geq m \geq 1$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_U = \frac{U_1 U_2 \cdots U_n}{(U_1 U_2 \cdots U_{n-m})(U_1 U_2 \cdots U_m)},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_U = \begin{bmatrix} n \\ n \end{bmatrix}_U = 1$.

When $p = 1$, the generalized Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_U$ are reduced to the Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_F$.

By taking U_{kn} instead of U_n for a fixed positive integer k, one can get

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = \frac{U_k U_{2k} \cdots U_{kn}}{\left(U_k U_{2k} \cdots U_{k(n-m)}\right)(U_k U_{2k} \cdots U_{km})}.$$

These coefficients satisfy the relations

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = U_{km+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{U_k} + U_{k(n-m)-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_{U_k}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = U_{km-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{U_k} + U_{k(n-m)+1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_{U_k}.$$

Golomb [3] found the generating function for the numbers F_n^2, and this result started the effort to find a recurrences or closed form for the generating function

$$f_m(x) = \sum_{n=0}^{\infty} F_n^m x^n$$

of the m^{th} powers of the Fibonacci numbers.

In [8], Riordan found the general recurrence relation for $f_m(x)$ (see also [2]). Carlitz [1] and Horadam [5] generalized the result of Riordan and found similar recurrences for the generating functions of different types of generalized Fibonacci numbers. They also found a closed form for the polynomial $N_m(x)$ in the numerator, and the polynomial $D_m(x)$ in the denominator of the generating functions. As a special case of Horadam's result, it is possible to get the following formula for the generating function of odd integer powers of the Fibonacci numbers:

$$f_m(x) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{\frac{j(j+1)}{2}}\begin{bmatrix}m+1\\j\end{bmatrix}_F F_{i-j}^m x^i}{\sum_{i=0}^{m+1}(-1)^{\frac{i(i+1)}{2}}\begin{bmatrix}m+1\\i\end{bmatrix}_F x^i}. \tag{1.4}$$

In [13], applying Carlitz's approach, Shannon obtained some special results for the numerator and the denominator in the expression of the generating function $f_m(x)$. Using the q-analogue of the terminating binomial theorem, he obtained the relation

$$\prod_{i=0}^{m}\left(1-q^i x\right) = \sum_{i=0}^{m+1}(-1)^i q^{\frac{i}{2}(i-1)}\begin{Bmatrix}m+1\\i\end{Bmatrix}x^i,$$

where $\begin{Bmatrix}m\\i\end{Bmatrix} = \frac{(1-q^m)\left(1-q^{m-1}\right)\cdots\left(1-q^{m-i+1}\right)}{(1-q)\left(1-q^2\right)\cdots\left(1-q^i\right)}$ is the Gaussian q-binomial coefficient for $i \geq 1$, any complex numbers q, x and any positive integer m with $\begin{Bmatrix}m\\0\end{Bmatrix} = 1$. Replacing q by β/α and x by $\alpha^m x$, one can get

$$\prod_{i=0}^{m}\left(1-\alpha^{m-i}\beta^i x\right) = \sum_{i=0}^{m+1}(-1)^{\frac{i}{2}(i+1)}\begin{bmatrix}m+1\\i\end{bmatrix}_F x^i.$$

It is easy to obtain for any odd integer m that

$$f_m(x) = 5^{-\frac{m-1}{2}}\sum_{i=0}^{\frac{m-1}{2}}\binom{m}{j}\frac{F_{m-2j}x}{1-(-1)^j L_{m-2j}x - x^2}, \tag{1.5}$$

after simplifications of one of Shannon's results. Seibert and Trojovsky [11] gave certain generalizations of the well-known formulas for the Fibonacci and Lucas numbers. For example,

$$\sum_{i=0}^{n}(-1)^i L_{n-2i} = 2F_{n+1}.$$

For odd positive integer m, the authors concentrated on the sums

$$\sum_{i_n=0}^{\frac{m-1}{2}}\sum_{i_{n-1}=i_n+1}^{\frac{m-1}{2}}\cdots\sum_{i_1=i_2+1}^{\frac{m-1}{2}}(-1)^{i_1+i_2+\cdots+i_n}\prod_{j=1}^{n}L_{m-2i_j}.$$

Combining (1.4) and (1.5), they gave some new results about these sums with the help of the Fibonomial coefficients.

In [12], for arbitrary positive integer m, taking the sums

$$\sum_{i_n=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\sum_{i_{n-1}=i_n+1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\cdots\sum_{i_1=i_2+1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(-1)^{i_1+i_2+\cdots+i_n}\prod_{j=1}^{n}L_{m-2i_j}, \tag{1.6}$$

the authors gave analogous new results for an even integer m using the method from [11].

We consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression, and then compute the sums of products of terms of the sequence $\{V_{kn}\}$ for a positive odd integer k.

2. Some identities including the terms of $\{U_{kn}\}$ and $\{V_{kn}\}$

We shall give some results for later use. Throughout this study, we will denote $W_n(a,b,p,1)$ by H_n.

2.1. Lemma. *For $m, n > 0$,*

$$V_{k(m+n)} + V_{k(m-n)} = \begin{cases} \frac{V_k^2+4}{U_k^2} U_{km} U_{kn} & \text{if } n \text{ is odd,} \\ V_{km} V_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.1}$$

$$V_{k(m+n)} - V_{k(m-n)} = \begin{cases} V_{km} V_{kn} & \text{if } n \text{ is odd,} \\ \frac{V_k^2+4}{U_k^2} U_{km} U_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.2}$$

$$U_{k(m+n)} + U_{k(m-n)} = \begin{cases} V_{km} U_{kn} & \text{if } n \text{ is odd,} \\ U_{km} V_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.3}$$

$$U_{k(m+n)} - U_{k(m-n)} = \begin{cases} U_{km} V_{kn} & \text{if } n \text{ is odd,} \\ V_{km} U_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.4}$$

$$V_{k(m-2n)} V_{k(2(m-2n)-3)} = \begin{cases} V_{3k(m-2n-1)} - V_{k(m-2n-3)} & \text{if } m \text{ is odd,} \\ V_{3k(m-2n-1)} + V_{k(m-2n-3)} & \text{if } m \text{ is even,} \end{cases} \tag{2.5}$$

$$V_{k(m-2n)} U_{k(m-2n-1)} = \begin{cases} U_{k(2m-4n-1)} - U_k & \text{if } m \text{ is odd,} \\ U_{k(2m-4n-1)} + U_k & \text{if } m \text{ is even,} \end{cases} \tag{2.6}$$

$$V_{3k(m+1)} + V_{3k(m-1)} = \left(V_k^2 + 4\right) U_{3km} U_{3k}, \tag{2.7}$$

$$V_{kn} U_{k(n-1)} - U_{k(2n-1)} = (-1)^{kn} U_k, \tag{2.8}$$

$$U_{k(m+n)} = U_{km} V_{kn} + (-1)^{n+1} U_{k(m-n)}, \tag{2.9}$$

$$U_{k(m+n)} U_{k(m+t)} - U_{km} U_{k(m+t+n)} = (-1)^m U_{kn} U_{kt}. \tag{2.10}$$

Proof. The proof follows by the Binet formulas for $\{U_{kn}\}$ and $\{V_{kn}\}$. □

2.2. Theorem. *For any integers r, c, d with $c \neq 0$ and $n \geq 0$,*

i)
$$\sum_{i=r}^{n} H_{k(ci+d)} = [H_{k(cr+d)} - H_{k(c(n+1)+d)} - (-1)^c H_{k(c(r-1)+d)} + (-1)^c H_{k(cn+d)}]/1 - V_{kc} + (-1)^c, \tag{2.11}$$

ii)
$$\sum_{i=r}^{n} (-1)^i H_{k(ci+d)} = [(-1)^r H_{k(cr+d)} + (-1)^n H_{k(c(n+1)+d)} + (-1)^{c+r} H_{k(c(r-1)+d)} + (-1)^{c+n} H_{k(cn+d)}]/1 + V_{kc} + (-1)^c. \tag{2.12}$$

Proof. i) By the Binet formula for $\{H_n\}$, we have

$$\sum_{i=r}^{n} H_{k(ci+d)} = H_{k(cr+d)} + H_{k(c(r+1)+d)} + \cdots + H_{k(cn+d)}$$

$$\begin{aligned}
&= A\alpha^{k(cr+d)} + B\beta^{k(cr+d)} + A\alpha^{k(c(r+1)+d)} + B\beta^{k(c(r+1)+d)} \\
&\qquad + \cdots + A\alpha^{k(cn+d)} + B\beta^{k(cn+d)} \\
&= A\alpha^{k(cr+d)}\left[1 + \alpha^{kc} + \alpha^{2kc} + \cdots + \alpha^{kc(n-r)}\right] \\
&\qquad + B\beta^{k(cr+d)}\left[1 + \beta^{kc} + \beta^{2kc} + \cdots + \beta^{kc(n-r)}\right] \\
&= A\alpha^{k(cr+d)}\frac{1-\alpha^{kc(n-r+1)}}{1-\alpha^{kc}} + B\beta^{k(cr+d)}\frac{1-\beta^{kc(n-r+1)}}{1-\beta^{kc}},
\end{aligned}$$

which, by $(\alpha\beta)^k = -1$ gives us

$$\begin{aligned}
\sum_{i=r}^{n} H_{k(ci+d)} &= \Big[A\alpha^{k(cr+d)} + B\beta^{k(cr+d)} - \left(A\alpha^{k(c(n+1)+d)} + B\beta^{k(c(n+1)+d)}\right) \\
&\quad - (-1)^c\left(A\alpha^{k(c(r-1)+d)} + B\beta^{k(c(r-1)+d)}\right) \\
&\quad + (-1)^c\left(A\alpha^{k(cn+d)} + B\beta^{k(cn+d)}\right)\Big] / 1 - \alpha^{kc} - \beta^{kc} + (-1)^c \\
&= (H_{k(cr+d)} - H_{k(c(n+1)+d)} - (-1)^c H_{k(c(r-1)+d)} \\
&\quad + (-1)^c H_{k(cn+d)})/1 - V_{kc} + (-1)^c.
\end{aligned}$$

Thus the proof of (i) is complete. The equation (2.12) can be similarly proven. □

2.3. Theorem. *For any integers $r, c\,(c \neq 0)$, d and $n \geq 0$,*

i) (2.13)

$$\begin{aligned}
&\sum_{i=r}^{n} iH_{k(ci+d)} \\
&= \Big(nH_{k(c(n+2)+d)} - \left(n+1+2n(-1)^c\right)H_{k(c(n+1)+d)} \\
&\quad + \left(n + 2(n+1)(-1)^c\right)H_{k(cn+d)} - (n+1)H_{k(c(n-1)+d)} \\
&\quad - \left(r+1+2r(-1)^c\right)H_{k(c(r-1)+d)} + rH_{k(c(r-2)+d)} \\
&\quad + \left(r + 2(r-1)(-1)^c\right)H_{k(cr+d)} \\
&\quad - (r-1)H_{k(c(r+1)+d)}\Big) \Big/ \left(1 - V_{kc} + (-1)^c\right)^2,
\end{aligned}$$

ii) (2.14)

$$\begin{aligned}
&\sum_{i=r}^{n} (-1)^{i-1} iH_{k(ci+d)} \\
&= \Big[\left(n(-1)^{n+1} - 2(n+1)(-1)^{c+n}\right)H_{k(cn+d)} \\
&\quad - (n+1)(-1)^n H_{k\left(c(n-1)+d\right)} \\
&\quad + \left(2n(-1)^{c+n+1} - (n+1)(-1)^n\right)H_{k\left(c(n+1)+d\right)} \\
&\quad + n(-1)^{n+1} H_{k\left(c(n+2)+d\right)} \\
&\quad - \left((r-1)(-1)^r - 2r(-1)^{c+r-1}\right)H_{k\left(c(r-1)+d\right)} \\
&\quad + \left(r(-1)^{r-1} - 2(r-1)(-1)^{c+r}\right)H_{kcr+d} \\
&\quad + \left(r(-1)^{r-1}\right)H_{k(c(r-2)+d)} \\
&\quad - (r-1)(-1)^r H_{k\left(c(r+1)+d\right)}\Big] \Big/ \left(1 + V_{kc} + (-1)^c\right)^2.
\end{aligned}$$

Proof. Theorem 2.3 is proved by considering

$$\sum_{i=r}^{n} ix^{i-1} = \frac{nx^{n+1} - (n+1)x^n - (r-1)x^r + rx^{r-1}}{(x-1)^2},$$

and using the Binet formula. □

2.4. Corollary. *For an odd positive integer m,*

$$\sum_{i=j+1}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{U_k}{V_k\left(V_k^2+4\right)}\left[V_{k(2m-4j-3)} + V_k\right], \tag{2.15}$$

$$\sum_{i=r}^{\frac{m-1}{2}} (-1)^i V_{k(m-2i)} = (-1)^r \frac{U_{k(m-2r+1)}}{U_k}, \tag{2.16}$$

$$\sum_{i=0}^{\frac{m-1}{2}} (-1)^i V_{k(3(m-2i-1))} = (-1)^{\frac{m-1}{2}} + \frac{U_{3km}U_{3k}}{\left(V_k^2+1\right)^2}, \tag{2.17}$$

$$\sum_{i=0}^{\frac{m-1}{2}} (-1)^{i-1} iV_{k(m-2i)} = \frac{(-1)^{\frac{m-1}{2}} V_k - V_{km}}{V_k^2+4}, \tag{2.18}$$

Proof. Substituting $n = \frac{m-1}{2}$, $c = -4$, $d = 2m-1$ and $r = j+1$ in (2.11) gives us

$$\sum_{i=j+1}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{-U_{k(2m-4j-5)} + U_{k(2m-4j-1)} + U_{3k} - U_k}{V_k^2\left(V_k^2+4\right)},$$

which, by (2.4), is equal to

$$\frac{U_{2k}V_{k(2m-4j-3)} + U_kV_k^2}{V_k^2\left(V_k^2+4\right)} = \frac{U_k}{V_k\left(V_k^2+4\right)}\left[V_{k(2m-4j-3)} + V_k\right],$$

as claimed in (2.15). Using Lemma 2.1, the identities (2.16)-(2.18) can be proved in a similar way as in the proof of (2.15). □

2.5. Corollary. *For an even positive integer m,*

$$\sum_{i=j+1}^{\frac{m-2}{2}} U_{k(2m-4i-1)} = \frac{U_k}{V_k\left(V_k^2+4\right)}\left[V_{k(2m-4j-3)} - V_k\right],$$

$$\sum_{i=r}^{\frac{m-2}{2}} (-1)^i V_{k(m-2i)} = (-1)^r \frac{U_{k(m-2r+1)}}{U_k} - (-1)^{\frac{m}{2}},$$

$$\sum_{i=0}^{\frac{m-2}{2}} (-1)^i V_{k(3(m-2i-1))} = \frac{U_{3km}}{U_k\left(V_k^2+1\right)},$$

$$\sum_{i=0}^{\frac{m-2}{2}} (-1)^{i-1} iV_{k(m-2i)} = (-1)^{\frac{m}{2}} \frac{m}{2} + \frac{V_{km} - 2(-1)^{\frac{m}{2}}}{V_k^2+4}.$$

□

3. Sums of products of the terms of $\{V_{kn}\}$

Define the sequence $\left\{S_n^k(m)\right\}_{n=0}^{\infty}$ in the following way: for $m>0$

$$S_0^k(m)=U_k,\quad S_1^k(m)=\sum_{i_1=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(-1)^{i_1}V_{k(m-2i_1)} \tag{3.1}$$

and for $n>1$,

$$S_n^k(m)=\sum_{i_n=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\ \sum_{i_{n-1}=i_n+1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\cdots\sum_{i_1=i_2+1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(-1)^{i_1+i_2+\cdots+i_n}\prod_{j=1}^{n}V_{k\left(m-2i_j\right)}. \tag{3.2}$$

Throughout this section we shall frequently follow the organization of the work [11] while giving our results.

3.1. Theorem. *For an odd positive integer m,*

$$S_1^k(m)=\sum_{i_1=0}^{\frac{m-1}{2}}(-1)^{i_1}V_{k(m-2i_1)}=\frac{U_{k(m+1)}}{U_k},$$

$$S_2^k(m)=\sum_{i_2=0}^{\frac{m-1}{2}}\sum_{i_1=i_2+1}^{\frac{m-1}{2}}(-1)^{i_1+i_2}V_{k(m-2i_2)}V_{k(m-2i_1)}$$

$$=\frac{m+1}{2}-\frac{1}{U_kU_{2k}}U_{km}U_{k(m+1)},$$

and

$$S_3^k(m)=\sum_{i_3=0}^{\frac{m-1}{2}}\sum_{i_2=i_3+1}^{\frac{m-1}{2}}\sum_{i_1=i_2+1}^{\frac{m-1}{2}}(-1)^{i_1+i_2+i_3}V_{k(m-2i_3)}V_{k(m-2i_2)}V_{k(m-2i_1)}$$

$$=\left(\frac{m-1}{2}\right)\frac{U_{k(m+1)}}{U_k}-\frac{1}{U_kU_{2k}U_{3k}}U_{k(m+1)}U_{km}U_{k(m-1)}.$$

Proof. i) Substituting $n=\frac{m-1}{2}$, $c=-2$, $d=m$ and $r=0$ in (2.12), we get

$$S_1^k(m)=\sum_{i_1=0}^{\frac{m-1}{2}}(-1)^{i_1}V_{k(m-2i_1)}$$

$$=\frac{V_{km}+(-1)^{\frac{m-1}{2}}V_k+V_{k(m+2)}+(-1)^{\frac{m-1}{2}}V_k}{(V_k^2+4)}.$$

Since $V_{-k}=V_k$ and by (2.1), we get

$$S_1^k(m)=\frac{V_{km}+V_{k(m+2)}}{(V_k^2+4)}=\sum_{i_1=0}^{\frac{m-1}{2}}(-1)^{i_1}V_{k(m-2i_1)}=\frac{U_{k(m+1)}}{U_k}.$$

ii) Using (2.6) and (2.16), we have

$$S_2^k(m) = \sum_{i_2=0}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2} V_{k(m-2i_2)} V_{k(m-2i_1)}$$
$$= -\frac{1}{U_k} \sum_{i_2=0}^{\frac{m-1}{2}} V_{k(m-2i_2)} U_{k(m-2i_2-1)}$$
$$= \frac{1}{U_k} \sum_{i_2=0}^{\frac{m-1}{2}} \left(U_k - U_{k(2m-4i_2-1)}\right).$$

Taking $j=-1$ in (2.15), we get

$$\sum_{i=0}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{U_{k(m+1)} U_{km}}{U_{2k}}.$$

Then

$$S_2^k(m) = \frac{m+1}{2} - \frac{1}{U_k U_{2k}} U_{km} U_{k(m+1)}.$$

iii) Using (2.16) and (2.6), we write

$$S_3^k(m) = \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)}$$
$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} V_{k(m-2i_2)} U_{k(m-2i_2-1)}$$
$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} \left(U_{k(2m-4i_2-1)} - U_k\right)$$
$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \left(U_{k(2m-4i_2-1)} - U_k\right).$$

From (2.15), we have

$$S_3^k(m) = \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)}$$
$$\times \left(\frac{U_k}{V_k\left(V_k^2+4\right)} \left(V_{k(2m-4i_3-3)} + V_k\right) + \Big(i_3 - \frac{m-1}{2}\Big) U_k \right)$$
$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} V_{k(m-2i_3)}$$
$$\times \left(-\frac{U_k}{V_k\left(V_k^2+4\right)} \left(V_{k(2m-4i_3-3)} + V_k\right) + \Big(\frac{m-1}{2} - i_3\Big) U_k \right)$$

$$= -\frac{1}{V_k\left(V_k^2+4\right)} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} V_{k(m-2i_3)} V_{k(2m-4i_3-3)}$$
$$+ \left(\frac{m-1}{2} - \frac{1}{V_k^2+4}\right) \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} V_{k(m-2i_3)}$$
$$- \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} i_3 V_{k(m-2i_3)}.$$

From (2.16)- (2.18) and (2.1), we get

$$\begin{aligned}
&S_3^k(m) \\
&= \frac{1}{V_k\left(V_k^2+4\right)}\left(\frac{U_{k(m-2)}}{U_k} + (-1)^{\frac{m-1}{2}}\left(V_k^2+1\right) - \frac{U_{3km}U_{3k}}{U_k^2\left(V_k^2+1\right)^2} - (-1)^{\frac{m-1}{2}}\right) \\
&\quad + \left(\frac{m-1}{2} - \frac{1}{V_k^2+4}\right)\left(\frac{V_{k(m+2)}+V_{km}}{V_k^2+4}\right) - \frac{1}{V_k^2+4}\left((-1)^{\frac{m-1}{2}} V_k - V_{km}\right) \\
&= -\frac{U_{3km}}{U_{2k}\left(V_k^2+4\right)\left(V_k^2+1\right)} + \left(\frac{m-1}{2} - \frac{1}{V_k^2+4}\right)\frac{U_{k(m+1)}}{U_k} \\
&\qquad + \frac{1}{V_k^2+4}\left(\frac{U_{k(m-2)}}{U_kV_k} + \frac{V_{km}U_{2k}}{U_kV_k}\right) \\
&= -\frac{U_{3km}}{U_{2k}\left(V_k^2+4\right)\left(V_k^2+1\right)} + \left(\frac{m-1}{2} - \frac{1}{V_k^2+4}\right)\frac{U_{k(m+1)}}{U_k} + \frac{U_{k(m+2)}}{\left(V_k^2+4\right)U_{2k}} \\
&= \left(\frac{m-1}{2}\right)\frac{U_{k(m+1)}}{U_k} - \frac{U_{3km}}{U_{2k}\left(V_k^2+4\right)\left(V_k^2+1\right)} + \frac{U_{km}}{\left(V_k^2+4\right)U_{2k}} \\
&= \left(\frac{m-1}{2}\right)\frac{U_{k(m+1)}}{U_k} - \frac{1}{U_kU_{2k}U_{3k}} U_{k(m+1)}U_{km}U_{k(m-1)}.
\end{aligned}$$

Thus we have the conclusion. □

3.2. Theorem. *For even positive integer m,*

$$S_1^k(m) = \sum_{i_1=0}^{\frac{m-2}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k} - (-1)^{\frac{m}{2}},$$

$$\begin{aligned}
S_2^k(m) &= \sum_{i_2=0}^{\frac{m-2}{2}} \sum_{i_1=i_2+1}^{\frac{m-2}{2}} (-1)^{i_1+i_2} V_{k(m-2i_2)} V_{k(m-2i_1)} \\
&= \frac{m-2}{2} + (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} + \frac{U_{km}U_{k(m+1)}}{U_kU_{2k}},
\end{aligned}$$

$$\begin{aligned}
S_3^k(m) &= \sum_{i_3=0}^{\frac{m-2}{2}} \sum_{i_2=i_3+1}^{\frac{m-2}{2}} \sum_{i_1=i_2+1}^{\frac{m-2}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)} \\
&= \left(\frac{m-4}{2}\right)\left((-1)^{\frac{m}{2}} - \frac{U_{k(m+1)}}{U_k}\right) \\
&\quad + U_{km}U_{k(m+1)}\left((-1)^{\frac{m}{2}} - \frac{U_{k(m-1)}}{U_kU_{2k}U_{3k}}\right).
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.1. □

In [14], Stanica gave the generating function for powers of the terms of the sequence $\{W_n\}$, $W(m,x)=\sum_{i=0}^{\infty} W_i^m x^i$ as follows:

3.3. Theorem. *For $n \geq 0$ and odd positive integer m,*

$$W(m,x)=\sum_{i=0}^{\frac{m-1}{2}} (-AB)^i \binom{m}{i} \times \frac{A^{m-2i}-B^{m-2i}+(-q)^i\left(B^{m-2i}\alpha^{m-2i}-A^{m-2i}\beta^{m-2i}\right)x}{1-(-q)^i V_{m-2i}x-q^m x^2}, \tag{3.3}$$

and for even positive integer m,

$$W(m,x)=\sum_{i=0}^{\frac{m}{2}-1} (-AB)^i \binom{m}{i} \times \frac{B^{m-2i}+A^{m-2i}-(-q)^i\left(B^{m-2i}\alpha^{m-2i}+A^{m-2i}\beta^{m-2i}\right)x}{1-(-q)^i V_{m-2i}x+q^m x^2} + \binom{m}{\frac{m}{2}} \frac{(-AB)^{\frac{m}{2}}}{1-(-q)^{\frac{m}{2}}x}. \quad \square \tag{3.4}$$

3.4. Theorem. *For $n \geq 0$ and an odd positive integer m,*

$$S_n^k(m)=\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i}\theta(i,m,n)\begin{bmatrix} m+1 \\ n-2i \end{bmatrix}_{U_k}, \tag{3.5}$$

and for an even positive integer m,

$$S_n^k(m)=\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=0}^{n-2i} (-1)^{i+n\left(\frac{m}{2}+1\right)+\frac{j}{2}(j+m+1)}\theta(i,m,n)\begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k},$$

where $\theta(i,m,n)=\binom{\lfloor \frac{m+1}{2} \rfloor -n+i}{i}+\binom{\lfloor \frac{m+1}{2} \rfloor -n+i-1}{i-1}$.

Proof. We give the proof for an odd integer m. From (3.3), we write

$$U_{kn}(m,x)=\left(\frac{U_k}{\sqrt{V_k^2+4}}\right)^{\frac{m-1}{2}}\sum_{j=0}^{\frac{m-1}{2}}\binom{m}{j}\frac{U_{k(m-2j)}x}{1-(-1)^j V_{k(m-2j)}x-x^2}. \tag{3.6}$$

Relation (3.6), which hold for odd m, leads to

$$D_{m+1}^k(x)=\prod_{j=0}^{\frac{m-1}{2}}\left(1-(-1)^j V_{k(m-2j)}x-x^2\right)=\sum_{i=0}^{m+1} d_{m+1,i}x^i,$$

where $d_{m+1,i}=(-1)^{\frac{i(i+1)}{2}}\begin{bmatrix} m+1 \\ i \end{bmatrix}_{U_k}$. After multiplication of all the factors in $D_{m+1}^k(x)$, we obtain

$$d_{m+1,0}=S_0^k(m),\ d_{m+1,i}=\sum_{l=0}^{\lfloor \frac{i}{2} \rfloor}\binom{\frac{m+1}{2}-(i-2l)}{l}(-1)^{i+l}S_{i-2l}^k(m),$$

where $i = 1, 2, \ldots, m+1$. Since $\binom{n}{m} = (-1)^m \binom{m-n-1}{m}$, we rewrite the last identity for $n > 0$ as follows:

$$d_{m+1,2n-1} = -\sum_{i=1}^{n} \binom{n+i-\frac{m+5}{2}}{n-i} S_{2i-1}^{k}(m)$$

$$d_{m+1,2(n-1)} = \sum_{i=1}^{n} \binom{n+i-\frac{m+7}{2}}{n-i} S_{2(i-1)}^{k}(m).$$

By the binomial inversion theorem (for more details, see [10]),

$$a_n = \sum_{i=1}^{n} \binom{n+i+r}{n-i} b_i \tag{3.7}$$

holds if and only if

$$b_n = \sum_{i=1}^{n} (-1)^{i+n} \left(\binom{2n+r}{n-i} - \binom{2n+r}{n-i-1} \right) a_i,$$

where r is any integer. After this, by taking $a_n = d_{m+1,2n-1}$, $b_i = -S_{2i-1}^{k}$ and $r = -\frac{m+5}{2}$ in (3.7), we obtain

$$\begin{aligned} S_{2n-1}^{k}(m) &= \sum_{i=1}^{n} (-1)^{-i+n+1} \left[\binom{2n-\frac{m+5}{2}}{n-i} - \binom{2n-\frac{m+5}{2}}{n-i-1} \right] d_{m+1,2i-1} \\ &= \sum_{i=1}^{n} (-1) \left[\binom{-n-i+\frac{m+3}{2}}{n-i} - \binom{-n-i+\frac{m+1}{2}}{n-i-1} \right] d_{m+1,2i-1} \\ &= \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[\binom{-n-i+\frac{m+3}{2}}{n-i} - \binom{-n-i+\frac{m+1}{2}}{n-i-1} \right] \\ &\qquad \times \begin{bmatrix} m+2 \\ 2i-1 \end{bmatrix}_{U_k}. \end{aligned} \tag{3.8}$$

Similarly if we take $a_n = d_{m+1,2(n-1)}$, $b_i = -S_{2(i-1)}^{k}$ and $r = -\frac{m+7}{2}$ in (3.7), then we get

$$\begin{aligned} S_{2(n-1)}^{k}(m) &= \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[\binom{-n-i+\frac{m+5}{2}}{n-i} - \binom{-n-i+\frac{m+5}{2}}{n-i-1} \right] \\ &\qquad \times \begin{bmatrix} m+1 \\ 2(i-1) \end{bmatrix}_{U_k}. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain (3.5).

For the case when m is even, the proof is completed by considering (3.4) as for the calculation of $S_n^k(m)$ for odd m. □

3.5. Lemma. *For a positive integer s and a positive even integer m, it holds that*

$$\begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+s} \begin{bmatrix} m+1 \\ s-1 \end{bmatrix}_{U_k} = \frac{U_{k(\frac{m}{2}+1-s)}}{U_{k(\frac{m}{2}+1)}} \begin{bmatrix} m+2 \\ s \end{bmatrix}_{U_k}. \tag{3.10}$$

For a positive integer s and a positive odd integer m, it holds that

$$\begin{bmatrix} m \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m-1}{2}+s} \begin{bmatrix} m \\ s-1 \end{bmatrix}_{U_k} = \frac{U_{k(\frac{m+1}{2}-s)}}{U_{k(\frac{m+1}{2})}} \begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k}.$$

Proof. For an even positive integer m and a positive integer s, substituting $m = \frac{m}{2} - s + 1$, $n = \frac{m}{2} + 1$ in (2.9), we get

$$U_{k\left(\frac{m}{2}-s+1\right)} V_{k\left(\frac{m}{2}+1\right)} = U_{k(m-s+2)} + (-1)^{\frac{m}{2}+s} U_{ks}. \tag{3.11}$$

From (3.11), we get

$$\begin{aligned}
&\begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+s} \begin{bmatrix} m+1 \\ s-1 \end{bmatrix}_{U_k} \\
&\quad = \frac{U_{k(m+1)} U_{km} \ldots U_{k(m-s+2)}}{(U_k U_{2k} \cdots U_{ks})} + (-1)^{\frac{m}{2}+s} \frac{U_{k(m+1)} U_{km} \cdots U_{k(m-s+3)}}{\left(U_k U_{2k} \cdots U_{k(s-1)}\right)} \\
&\quad = \frac{U_{k(m+2)} U_{k(m+1)} \cdots U_{k(m-s+3)}}{(U_k U_{2k} \cdots U_{ks})} \left[\frac{U_{k(m-s+2)} + (-1)^{\frac{m}{2}+s} U_{ks}}{U_{k(m+2)}} \right] \\
&\quad = \frac{U_{k(m+2)} U_{k(m+1)} \cdots U_{k(m-s+3)}}{(U_k U_{2k} \cdots U_{ks})} \left[\frac{U_{k\left(\frac{m}{2}-s+1\right)} V_{k\left(\frac{m}{2}+1\right)}}{U_{k(m+2)}} \right] \\
&\quad = \frac{U_{k(m+2)} U_{k(m+1)} \cdots U_{k(m-s+3)}}{(U_k U_{2k} \cdots U_{ks})} \frac{U_{k\left(\frac{m}{2}-s+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \\
&\quad = \frac{U_{k\left(\frac{m}{2}+1-s\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \begin{bmatrix} m+2 \\ s \end{bmatrix}_{U_k}.
\end{aligned}$$

Thus the proof is complete. □

Define

$$\sigma_m^k(t) := \sum_{j=0}^{m-t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k}, \tag{3.12}$$

where m is an even positive integer and t is any integer.

3.6. Lemma. *For an even positive integer m and any integer t,*

i) $\sigma_m^k(t) = 0$, *for* $t \leq -1$ *or* $t \geq m+1$,
ii) $\sigma_m^k(m-t) = \sigma^k(t)$,
iii) $\sigma_m^k(0) = 1$, $\sigma_m^k(1) = 1 + \frac{1}{U_k} (-1)^{\frac{m-2}{2}} U_{k(m+1)}$,
$\sigma_m^k(2) = 1 - \frac{1}{U_k U_{2k}} V_{k\left(\frac{m+2}{2}\right)} U_{k(m+1)} U_{k\left(\frac{m-2}{2}\right)}$,
$\sigma_m^k(3) = 1 - \frac{1}{U_k U_{2k} U_{3k}} (-1)^{\frac{m}{2}} U_{k(m+1)} \left(U_{2k} U_{3k} - V_{k\left(\frac{m+2}{2}\right)} U_{km} U_{k\left(\frac{m-4}{2}\right)} \right)$.

Proof. In order to get the proof of i) and ii), one can follow the method of proof of in [12, Lemma 16 i) and ii)], since they are similar statements.

iii) The identities for $\sigma_m^k(0)$ and $\sigma_m^k(1)$ are directly implied by $\sigma_m^k(-1) = 0$. Using ii) and (3.12), we have

$$\begin{aligned}
\sigma_m^k(2) &= \sum_{j=0}^{2} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} \\
&= 1 + (-1)^{\frac{(m+2)}{2}} \frac{U_{k(m+1)}}{U_k} - (-1)^m \frac{U_{k(m+1)} U_{km}}{U_k U_{2k}} \\
&= 1 - \frac{U_{k(m+1)}}{U_k U_{2k}} \left(U_{km} + (-1)^{\frac{m}{2}} \right) \\
&= 1 - \frac{U_{k(m+1)}}{U_k U_{2k}} \left(U_{k\left(\frac{m}{2}-1\right)} V_{k\left(\frac{m}{2}+1\right)} \right),
\end{aligned}$$

and

$$\begin{aligned}\sigma_m^k(3) &= \sigma_m^k(2) - (-1)^{\frac{(m-2)}{2}} \frac{U_{k(m-1)}U_{km}U_{k(m+1)}}{U_kU_{2k}U_{3k}} \\ &= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} - \frac{U_{k(m+1)}U_{km}}{U_kU_{2k}} + (-1)^{\frac{m}{2}} \frac{U_{k(m-1)}U_{km}U_{k(m+1)}}{U_kU_{2k}U_{3k}} \\ &= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k}\left(U_{2k}U_{3k} - U_{k(m-1)}U_{km} + (-1)^{\frac{m}{2}} U_{3k}U_{km}\right) \\ &= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k}\left(U_{2k}U_{3k} - U_{km}\left(U_{k(m-1)} - (-1)^{\frac{m}{2}} U_{3k}\right)\right) \\ &= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k}\left(U_{2k}U_{3k} - U_{km}U_{k\left(\frac{m-4}{2}\right)}V_{k\left(\frac{m+2}{2}\right)}\right),\end{aligned}$$

as desired. □

3.7. Lemma. *For an even positive integer m and any integer t,*

$$\sigma_m^k(t) - \sigma_m^k(t-2) = (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}-t+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}}.$$

Proof. For $t < 2$, the claim follows from the definition of Fibonomial coefficients and Lemma 3.6. For $m \geq 2$, we have

$$\begin{aligned}&\sigma_m^k(t) - \sigma_m^k(t-2) \\ &\qquad = \sigma_m^k(m-t) - \sigma_m^k(m-t+2) \\ &\qquad = \sum_{j=0}^{t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} - \sum_{j=0}^{t-2} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} \\ &\qquad = (-1)^{\frac{t}{2}(t+m+1)} \left(\begin{bmatrix} m+1 \\ t \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+t} \begin{bmatrix} m+1 \\ t-1 \end{bmatrix}_{U_k} \right).\end{aligned}$$

By Lemma 3.5, the proof is complete. □

3.8. Lemma. *For an even positive integer m and any integer t,*

$$\begin{aligned}\sigma_m^k(t) - \sigma_m^k(t-4) = (-1)^{\frac{t}{2}(t+m+1)} &\begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k} \\ &\times \frac{U_{k\left(\frac{m}{2}-t+2\right)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}}\omega(t,m),\end{aligned}$$

where $\omega(t,m) = U_{k\left(\frac{m}{2}+1-t\right)}V_{k\left(\frac{m}{2}+2-t\right)}U_{k(m+3)} - V_kU_{km}U_{k(m-1)}$.

Proof. By Lemma 3.7, we have for any integer t that

$$\begin{aligned}
&\sigma_m^k(t)-\sigma_m^k(t-4)\\
&\quad=\sigma_m^k(t)-\sigma_m^k(t-2)+\sigma_m^k(t-2)-\sigma_m^k(t-4)\\
&\quad=(-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix}m+2\\t\end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}-t+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}}\\
&\qquad\qquad+(-1)^{\frac{t-2}{2}(t+m-1)}\begin{bmatrix}m+2\\t-2\end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}-t+3\right)}}{U_{k\left(\frac{m}{2}+1\right)}}\\
&\quad=(-1)^{\frac{t}{2}(t+m+1)}\frac{1}{U_{k\left(\frac{m}{2}+1\right)}}\left(U_{k\left(\frac{m}{2}-t+1\right)}\begin{bmatrix}m+2\\t\end{bmatrix}_{U_k}-U_{k\left(\frac{m}{2}-t+3\right)}\begin{bmatrix}m+2\\t-2\end{bmatrix}_{U_k}\right)\\
&\quad=(-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix}m+4\\t\end{bmatrix}_{U_k}\frac{1}{U_{k(m+3)}U_{k(m+4)}}\\
&\qquad\qquad\times\left(U_{k\left(\frac{m}{2}-t+1\right)}U_{k(m+3-t)}-U_{k\left(\frac{m}{2}-t+3\right)}U_{kt}U_{k(t-1)}\right).
\end{aligned}$$

From (2.10), we get

$$\begin{aligned}
&\sigma_m^k(t)-\sigma_m^k(t-4)\\
&\quad=(-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix}m+4\\t\end{bmatrix}_{U_k}\frac{1}{U_{k(m+3)}U_{k(m+4)}}\\
&\quad\times\left(U_{k\left(\frac{m}{2}-t+1\right)}\left(U_{k(m+4-2t)}U_{k(m+3)}+U_{kt}U_{k(t-1)}\right)-U_{k\left(\frac{m+6}{2}-t\right)}U_{kt}U_{k(t-1)}\right)\\
&\quad=(-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix}m+4\\t\end{bmatrix}_{U_k}\frac{1}{U_{k(m+3)}U_{k(m+4)}}\\
&\qquad\qquad\times\left(U_{k\left(\frac{m}{2}-t+1\right)}U_{k(m+4-2t)}U_{k(m+3)}-V_kU_{k\left(\frac{m}{2}+2-t\right)}U_{kt}U_{k(t-1)}\right)\\
&\quad=(-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix}m+4\\t\end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}-t+2\right)}}{U_{k(m+3)}U_{k(m+4)}}\\
&\qquad\qquad\times\left(V_{k\left(\frac{m}{2}+2-t\right)}U_{k(m+4-2t)}U_{k(m+3)}-V_kU_{kt}U_{k(t-1)}\right),
\end{aligned}$$

as claimed. □

3.9. Theorem. *For any integer m,*

$$\begin{aligned}
&\sum_{j=0}^{t}(-1)^{\frac{j}{2}(j+m+1)}\begin{bmatrix}m+1\\j\end{bmatrix}_{U_k}\\
&\quad=\frac{(-1)^{\frac{t}{2}(t+m+1)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}}\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}\begin{bmatrix}m+4\\t-4i\end{bmatrix}_{U_k}U_{k\left(\frac{m}{2}+2-t+4i\right)}\\
&\qquad\times\left(U_{k\left(\frac{m}{2}+1-t+4i\right)}V_{k\left(\frac{m}{2}+2-t+4i\right)}U_{k(m+3)}-V_kU_{k(t-4i)}U_{k(t-4i-1)}\right).
\end{aligned}$$

Proof. By Lemma 3.8, we can write

$$\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}\left[\sigma_m^k(t-4i)-\sigma_m^k(t-4(i+1))\right]=\sigma_m^k(t)-\sigma_m^k\left(t-4\left(\left\lfloor\frac{t}{4}\right\rfloor+1\right)\right).$$

From Lemma 3.6, we have

$$\sigma_m^k(t) = \sum_{i=0}^{\lfloor \frac{t}{4} \rfloor} \left[\sigma_m^k(t-4i) - \sigma_m^k(t-4(i+1)) \right].$$

Again by Lemma 3.8, we get

$$\begin{aligned} \sigma_m^k(t) &= \sum_{i=0}^{\lfloor \frac{t}{4} \rfloor} (-1)^{\frac{t-4i}{2}(t-4i+m+1)} \begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}+2-t+4i\right)}}{U_{k\left(\frac{m}{2}+1\right)} U_{k(m+3)} U_{k(m+4)}} \\ &\quad \times \left(U_{k\left(\frac{m}{2}+1-t+4i\right)} V_{k\left(\frac{m}{2}+2-t+4i\right)} U_{k(m+3)} - V_k U_{k(t-4i)} U_{k(t-4i-1)} \right) \\ &= \frac{(-1)^{\frac{t}{2}(t+m+1)}}{U_{k\left(\frac{m}{2}+1\right)} U_{k(m+3)} U_{k(m+4)}} \sum_{i=0}^{\lfloor \frac{t}{4} \rfloor} \begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} U_{k\left(\frac{m}{2}+2-t+4i\right)} \\ &\quad \times \left(U_{k\left(\frac{m}{2}+1-t+4i\right)} V_{k\left(\frac{m}{2}+2-t+4i\right)} U_{k(m+3)} - V_k U_{k(t-4i)} U_{k(t-4i-1)} \right), \end{aligned}$$

as claimed. □

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