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RESULTS ON THE SUPREMUM OF FRACTIONAL BROWNIAN MOTION*

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Abstract

We show that the distribution of the square of the supremum of reflected fractional Brownian motion up to time a, with Hurst parameter-H greater than $1/2$, is related to the distribution of its hitting time to level 1, using the self similarity property of fractional Brownian motion. It is also proven that the second moment of supremum of reflected fractional Brownian motion up to time a is bounded above by a^{2H}. Similar relations are obtained for the supremum of fractional Brownian motion with Hurst parameter greater than $1/2$, and its hitting time to level 1. What is more, we obtain an upper bound on the complementary probability distribution of the supremum of fractional Brownian motion and reflected fractional Brownian motion up to time a, using Jensen's and Markov's inequalities. A sharper bound is observed on the distribution of the supremum of fractional Brownian motion by the properties of Gamma distribution. Finally, applications of the given results to financial markets are investigated, and partial results are provided.

Keywords: Fractional Brownian motion, Reflected fractional Brownian motion, Self similarity property, Hitting time, Gamma distribution, Hurst parameter, Markov's inequality, Jensen's inequality.

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1. Introduction

Stochastic Model. The price of a stock, a volatile asset, is often described by the Black-Scholes model, which is also known as the Geometric Brownian model.

The stochastic differential equation of this model is

$$dY_t = \bar{\mu} Y_t dt + \bar{\sigma} Y_t dW_t, \qquad 0 \le t \le T \tag{1.1}$$

where $\bar{\mu}$ is a constant mean rate of return and $\bar{\sigma}$ a constant volatility, W is the standard Brownian motion and T is the time of maturity of the stock. When we solve this equation explicitly, we obtain

$$Y_t = Y_0 \exp\left((\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)t + \bar{\sigma} W_t\right) \tag{1.2}$$

where Y_0 is the initial value of one share of stock. However, this model has well known deficiencies, such as increments of standard Brownian motion being independent. This is not what we are exposed to in real life. In order to overcome this problem various alternative models have been suggested, such as fractional Brownian motion (fBm).

Fractional Brownian motion also appears naturally in many other situations. Some examples are, the level of water in a river as a function of time, the characters of solar activity as a function of time, the values of the log returns of a stock, the prices of electricity in a liberated electricity market [3]. In the first three examples fBm with Hurst parameter $H > 1/2$ is used, which means that the process is persistent. And the last example is modeled by fBm with Hurst parameter $H < 1/2$. Throughout this article we are interested in the financial applications of fBm, specifically the model of the values of log returns of a stock. In other words we are interested in fBm with Hurst parameter $H > 1/2$. The Black-Scholes model for the values of the log returns of a stock using fBm is given as

$$Y_t = Y_0 \exp\left((r + \mu)t + \sigma B_t^H\right), \ 0 \le t \le T, \tag{1.3}$$

where Y_0 is the initial value, r the constant interest rate, μ the constant drift and σ the constant diffusion coefficient of fBm, which is denoted by $\left(B_t^H\right)_{t \ge 0}$.

The advantage of modeling with fBm over the other models is its capability of displaying the dependence between returns on different days. FBm on the other hand is not a semimartingale, and is not a Markov process and it allows arbitrage [11]. However, in order to display long range dependence and to have semimartingale property, other models can be constructed using fBm as a central model [11]. Due to the complex nature of fBm, the most useful and efficient classical mathematical techniques for stochastic calculus are not available for fBm. Therefore most of the results in the literature are given as bounds on the characteristics of fBm. In spite of this, fBm still possess some nice properties. It is a Gaussian process and the self similarity property holds. Most of the techniques developed for fBm are related to these properties. Now, there is no doubt that investors would be interested in having some information on the value of the supremum of the asset in order to manage the risk, or maybe to hedge financial assets and to construct portfolios. Therefore in this study, mainly two new results on the supremum of fBm and an application of these results are introduced. Specifically,

- An identity for the distribution of the square of the supremum of fBm,
- An identity for the distribution of the square of the supremum of reflected fBm,
- An upper bound on the second moment of reflected fBm,
- A lower bound on the distribution of the supremum of fBm up to time 1 or any fixed time t,
- A theoretical application of the above results, called “maximum drawdown,” which is commonly used in the literature to obtain a measure of risk in finance,

are obtained.

Fractional Brownian Motion. Let us start by introducing fBm. FBm was first introduced within the Hilbert space framework by Kolmogorov [7]. It was named due to the stochastic integral representation in terms of standard Brownian motion, which was given by Mandelbrot and Van Ness [9].

Let H be a constant in $(0,1)$. FBm $(B_t^H)_{t\geq 0}$ with Hurst parameter H is a continuous and centered Gaussian process with covariance function

$$\mathrm{E}\big[B_t^H B_s^H\big] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

For $H = 1/2$, fBm corresponds to standard Brownian motion. A standard fBm, $(B_t^H)_{t\geq 0}$, has the following properties:

- $B_0^H = 0$ and $\mathrm{E}\big[B_t^H\big] = 0$ for all $t \geq 0$.
- B^H has homogenous increments, that is $B_{t+s}^H - B_s^H$ has the same law as B_t^H for $s, t \geq 0$.
- B^H is a Gaussian process and $\mathrm{E}[(B_t^H)^2] = t^{2H},\ t \geq 0$, for all $H \in (0,1)$.
- B^H has continuous trajectories.

Stochastic integral representation. Several stochastic integral representations have been developed for the fBm. For example, it is known that the following process is a fBm with Hurst parameter $H \in (0,1)$,

$$\begin{aligned} B_t^H &= \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-s)_+^{H-1/2})\, dW_s \\ &= \frac{1}{\Gamma(H+1/2)} \bigg(\int_{-\infty}^{0} \big((t-s)^{H-1/2} - (-s)^{H-1/2}\big)\, dW_s \\ &\qquad + \int_0^t (t-s)^{H-1/2}\, dW_s \bigg), \end{aligned} \tag{1.4}$$

where W_t is a standard Brownian motion with $W_0 = 0$ considered on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The paths of W are continuous, the increments of W over disjoint intervals are independent Gaussian random variables with zero-mean and with variance equal to the length of the interval. And Γ is the gamma function. Now let us omit the constant $1/\Gamma(H+1/2)$ for simplicity and use the change of variable $s = tu$, then

$$\begin{aligned} \mathrm{E}[(B_t^H)^2] &= \int_{\mathbb{R}} \big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\big)^2\, ds \\ &= t^{2H} \int_{\mathbb{R}} \big((1-u)_+^{H-1/2} - (-u)_+^{H-1/2}\big)^2\, du \\ &= C(H) t^{2H} \end{aligned} \tag{1.5}$$

Also

$$\begin{aligned} \mathrm{E}[(B_t^H - B_s^H)^2] &= \int_{\mathbb{R}} \big((t-u)_+^{H-1/2} - (s-u)_+^{H-1/2}\big)^2\, ds \\ &= t^{2H} \int_{\mathbb{R}} \big((t-s-u)_+^{H-1/2} - (-u)_+^{H-1/2}\big)^2\, du \\ &= C(H)\big|t-s\big|^{2H}. \end{aligned} \tag{1.6}$$

Hence

$$\begin{aligned} \mathrm{E}\big[B_t^H B_s^H\big] &= -\frac{1}{2}\mathrm{E}\big[(B_t^H - B_s^H)^2\big] - \mathrm{E}\big[(B_t^H)^2\big] - \mathrm{E}\big[(B_s^H)^2\big] \\ &= \frac{1}{2}\big(t^{2H} + s^{2H} - \big|t-s\big|^{2H}\big). \end{aligned} \tag{1.7}$$

Correlation between two increments. For $H = \frac{1}{2}$, the process $(B_t^H)_{t\geq 0}$ corresponds to a standard Brownian motion, in which the increments are independent. For $H \neq 1/2$ the increments are not independent. By the definition of fBm, we know the covariance between $B^H(t+h) - B^H(t)$ and $B^H(s+h) - B^H(s)$ with $s + h \leq t$ and $t - s = nh$ is

$$\rho_H(n) = \frac{1}{2}h^{2H}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right].$$

We observe that two increments of the form $B^H(t+h)-B^H(t)$ and $B^H(t+2h)-B^H(t+h)$ are positively correlated for $H > 1/2$, and they are negatively correlated for $H < 1/2$.

Self similarity property. Since the covariance function of fBm is homogenous of order $2H$, fBm possess the self-similarity property, that is for any constant $c > 0$,

$$(1.8) \qquad \left(B_{ct}^H\right)_{t\geq 0} \stackrel{law}{=} \left(c^H B_t^H\right)_{t\geq 0}.$$

Note that, by taking $H = \frac{1}{2}$ we obtain the self similarity property of standard Brownian motion $(W_t)_{t\geq 0}$, that is

$$(1.9) \qquad \left(W_{at}\right)_{t\geq 0} \stackrel{law}{=} \left(a^{1/2} W_t^H\right)_{t\geq 0}.$$

Using equation (1.9), it was shown in [6] that the supremum of reflected standard Brownian motion up to time 1 is identical in law with the the reciprocal of the square root of the hitting time of level 1. From there on, the first moment of this supremum was calculated using the properties of the Normal distribution.

2. Main results

Summary of the main results. In this section, we prove that the second moment of reflected fBm up to time a is bounded above by a^{2H}. In our proof, as a new approach we apply the properties of the Gamma distribution to fBm combined with the result given in [5], and use equation (1.8).

Another new result given in this section is to provide a lower bound on the distribution of the supremum of fBm up to time 1 and up to fixed time t. In this proof, we consider fBm up to a random exponentially distributed time T, which is independent of the process, and using the self similarity property we obtain an upper bound on the expected value of the supremum of fBm up to time 1 and then combine this result with Markov's inequality to find the lower bound on the distribution of the supremum.

Notation. Let $\left(B_t^H\right)_{t\geq 0}$ be a fBm defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Also, let us define the reflected fBm around 0, denoted by $(|B_t^H|)_{t\geq 0}$, as

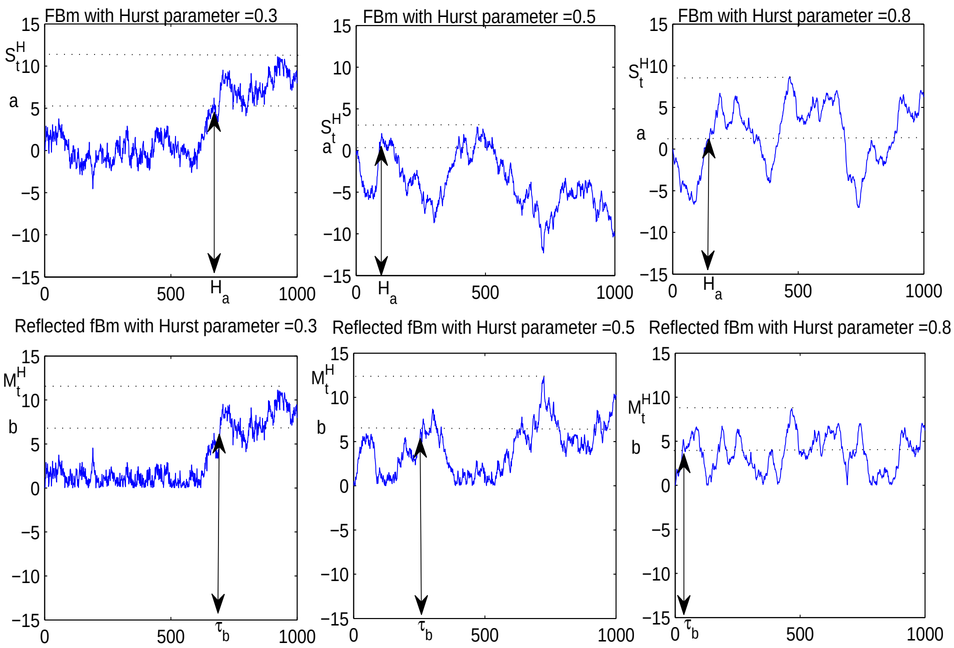
$$(2.1) \qquad \left|B_t^H\right|_{t\geq 0} := \begin{cases} B_t^H & \text{if } B_t^H \geq 0 \\ -B_t^H, & \text{if } B_t^H < 0. \end{cases}$$

Let $\tau_1 := \inf\left\{t \geq 0 : |B_t^H| = 1\right\}$. In other words, τ_1 is the first hitting time of level 1 for the reflected fBm.

Let $H_a := \inf\left\{t \geq 0 : B_t^H = a\right\}$ be the first hitting time of level a, where $a \in \mathbb{R}_+$. Let $M_t^H := \sup_{0\leq v\leq t}\left|B_v^H\right|$, that is the supremum of the reflected fBm. Similarly, let $S_t^H := \sup_{0\leq v\leq t} B_v^H$ be the supremum of the fBm.

In Figure 2., we display sample paths of fBm and reflected fBm with Hurst parameters $H = 0.3$, $H = 0.5$ and $H = 0.8$ from left to right. These sample paths are generated using the Matlab code and the algorithm proposed by Abry and Sellan given in [1]. This algorithm is a fast implementation of a method proposed by Sellan [13], which is using the sum through a fractional wavelet basis. In this method, final paths carry both the short-term and long-term correlation information. For the details of simulations of fBm, one can also see a study by Caglar, given in [4].

Figure 1. Fractional Brownian motions and their reflections around zero with Hurst parameters $H = 0.3,\ H = 0.5,\ H = 0.8$



2.1. Theorem. *For fBm with Hurst parameter $H > \frac{1}{2}$ and $a \in \mathbb{R}_+$, we have*

$$\left(M_a^H\right)^2 \overset{law}{=} \left(\frac{a}{\tau_1}\right)^{2H} \text{ and } \left(S_a^H\right)^2 \overset{law}{=} \left(\frac{a}{H_1}\right)^{2H}$$

The second moment satisfies

$$\mathrm{E}\left(M_a^H\right)^2 \leq a^{2H}$$

Proof. For $x > 0$, by equation (1.8) we obtain

$$\begin{aligned} P\left(\left(\frac{a}{\tau_1}\right)^{2H} \leq x\right) &= P\left(\tau_1 \geq \frac{a}{x^{1/2H}}\right) = P\left(\sup_{0\leq t\leq \frac{a}{x^{1/2H}}} \left|B_t^H\right| \leq 1\right) \\ &= P\left(\sup_{0\leq u\leq a} \left|B^H_{\frac{u}{x^{1/2H}}}\right| \leq 1\right) = P\left(\sup_{0\leq u\leq a} \left|B_u^H\right| \leq \sqrt{x}\right) \\ &= P\left(\left(M_a^H\right)^2 \leq x\right) \end{aligned} \tag{2.2}$$

Hence, it is seen that

$$\left(M_a^H\right)^2 \overset{law}{=} \left(\frac{a}{\tau_1}\right)^{2H}.$$

Now note that,

$$P\Big(\sup_{0\le u\le a}\left|B_u^H\right| \le \sqrt{x}\Big) = P\Big(\sup_{0\le u\le a} B_u^H \le \sqrt{x},\ \inf_{0\le u\le a} B_u^H \ge \sqrt{x}\Big)$$
$$\le P\Big(\sup_{0\le u\le a} B_u^H \le \sqrt{x}\Big) = P\big((S_a^H)^2 \le x\big) \tag{2.3}$$

And by following the same argument given in (2.2) it is also observed that

$$\big(S_a^H\big)^2 \overset{law}{=} \Big(\frac{a}{H_1}\Big)^{2H}.$$

Hence, combining equations (2.2) and (2.3) the following inequality can be obtained

$$P\big((M_a^H)^2 \le x\big) \le P\big((S_a^H)^2 \le x\big) = P\Big(\Big(\frac{a}{H_1}\Big)^{2H} \le x\Big)$$

Now, by applying the properties of the Gamma distribution, for $x > 0$, $k > 0$ and $\theta > 0$ we have

$$\int_0^\infty \frac{x^{k-1}e^{-\frac{x}{\theta}}}{\Gamma(k)}\,dx = \theta^k. \tag{2.4}$$

Therefore

$$\mathrm{E}\big((M_a^H)^2\big) \le \mathrm{E}\big((S_a^H)^2\big) = a^{2H}\mathrm{E}\Big(\Big(\frac{1}{H_1}\Big)^{2H}\Big) = a^{2H}\int_0^\infty \mathrm{E}\Big(e^{-xH_1^{2H}}\Big)\,dx$$

For a standard Brownian motion W it is well known that $\mathrm{E}\big(e^{-\lambda T_a}\big) = e^{-a\sqrt{2\lambda}}$, where T_a denotes the first hitting time of level a for standard Brownian motion. It was shown in [5] that for fBm with Hurst parameter $H > \frac{1}{2}$, the following inequality holds

$$\mathrm{E}\Big(e^{-\lambda H_a^{2H}}\Big) \le e^{-a\sqrt{2\lambda}} \tag{2.5}$$

for all $\lambda,\ a > 0$. Therefore, by equation (2.5)

$$a^{2H}\int_0^\infty \mathrm{E}\Big(e^{-x\tau_1^{2H}}\Big)dx \le a^{2H}\int_0^\infty e^{-\sqrt{2x}}\,dx = a^{2H} \tag{2.6}$$

As a result, $\mathrm{E}\big((M_a^H)^2\big) \le a^{2H}$. □

2.2. Corollary. *For fBm with Hurst parameter $H > \frac{1}{2}$ and $a \in \mathbb{R}_+$,*

$$P\big(S_a^H > x\big) \le \frac{a^H}{x} \ \textit{and}\ P\big(M_a^H > x\big) \le \frac{a^H}{x}.$$

Proof. Note that, by Jensen's inequality,

$$\mathrm{E}^2\big(S_a^H\big) \le \mathrm{E}\big((S_a^H)^2\big) \le a^{2H} \text{ and } \mathrm{E}^2\big(M_a^H\big) \le \mathrm{E}\big((M_a^H)^2\big) \le a^{2H}.$$

Hence, by Markov's inequality,

$$P\big(S_a^H > x\big) \le \frac{a^H}{x} \text{ and } P\big(M_a^H > x\big) \le \frac{a^H}{x}.$$ □

The bounds given above already provide very useful information for the investor because it tells the investor the probability of the values of the supremum up to a fixed time.

In the next theorem a closer upper bound is obtained for the distribution of the supremum of fBm, which actually provides better information for investors.

2.3. Theorem. *Consider fBm up to time a, with Hurst parameter $H > \frac{1}{2}$. Then*

$$P\big(S_a^H \ge x\big) \le \frac{\sqrt{2}a^H}{x\sqrt{\pi}}$$

Proof. Consider taking fBm up to time T, where T is an exponentially distributed random variable with mean $1/\lambda$, independent of the underlying fBm, B. Then using the self similarity property of fBm,

$$P\big(H_a^{2H} \le T\big) = P\Big(S_{T^{\frac{1}{2H}}} \ge a\Big) = \mathrm{E}\big(e^{-\lambda H_a^{2H}}\big) \le e^{-a\sqrt{2\lambda}}, \tag{2.7}$$

and hence

$$\mathrm{E}\big(S_{T^{\frac{1}{2H}}}\big) = \int_0^\infty P\Big(S_{T^{\frac{1}{2H}}} \ge a\Big) \le \frac{1}{\sqrt{2\lambda}}. \tag{2.8}$$

Again by the self similarity property we have,

$$\mathrm{E}\left(\sup_{0\le u\le T^{\frac{1}{2H}}} B_u\right) = \mathrm{E}\left(T^{1/2} \sup_{0\le u\le 1} B_s\right) = \mathrm{E}\big(\sqrt{T}\big)\mathrm{E}\big(S_1\big) \tag{2.9}$$

where $\mathrm{E}\big(T^{p/2}\big) = \frac{\Gamma\left(\frac{2+p}{2}\right)}{\lambda^{p/2}}$.

The reader is refereed to [12] for details related to taking a random process up to a random time which is independent of the underlying process.

Therefore we obtain

$$\mathrm{E}(S_1) \le \frac{\sqrt{2}}{\sqrt{\pi}},$$

and by Markov's inequality, for any $x > 0$ we can write,

$$P(S_1 \ge x) \le \frac{\mathrm{E}(S_1)}{x} = \frac{\sqrt{2}}{x\sqrt{\pi}}. \tag{2.10}$$

Now, by using the scaling property, for any fixed time a we observe

$$\begin{aligned} P(\sup_{0\le u\le a} B_u^H \le x) &= P(a^H \sup_{0\le u\le 1} B_u^H \le x) = P(a^H S_1^H \le x) \\ &= P(S_1^H \le \frac{x}{a^H}) \end{aligned} \tag{2.11}$$

□

3. Application of the results to financial mathematics

In finance, depending on the investor's demand, risk can be defined in many ways. One of the commonly used definitions of risk is known as *"downfall"* or *"maximum drawdown"*. Maximum drawdown is the highest possible loss in the price of one share of the risky asset. One of the fundamental results related to standard Brownian motion is the Lévy Theorem (see [10]). This theorem leads us to the result that "maximum drawdown," the highest possible loss in the trajectories of the standard Brownian motion, is identical in law with the supremum of the reflected Brownian motion. In other words, the "maximum drawdown", before time t is defined as

$$D_t^- := \sup_{0\le u\le v\le t} (W_u - W_v) = \sup_{0\le v\le t} (\sup_{0\le u\le v} W_u - W_v)$$

and it satisfies

$$\Big(\sup_{0\le u\le v} W_u - W_v\Big) \overset{law}{=} \Big(\sup_{0\le v\le t} |W_v|\Big)$$

by the Lévy isomorphism.

In the above result, we would like to replace standard Brownian motion with fBm, because it is obvious that fBm is a more realistic model than standard Brownian motion for risky assets. In the literature, due to the complex nature of fBm, there are not many exact results on the distributions related to the trajectories. For this reason, we also expect to obtain not exact results but some bounds.

In the fBm notation let,

$$D_t^{-,H} := \sup_{0\le a'\le a\le t}(B_{a'} - B_a) = \sup_{0\le a\le t}\big(\sup_{0\le a'\le a} B_{a'} - B_a\big)$$

be the "maximum drawdown". Our goal is to find the exact distribution, or alternatively to find bounds for the distribution of "maximum drawdown", which is directly applicable to finance. However, we have not been able to attain this goal yet. In this study, we would like to present a lower bound for the distribution of the difference between the supremum of fBm and the value of fBm at time a, which is the first step in finding a lower bound for the distribution of "maximum drawdown" of fBm up to time t. And we hope to extend this result to finding the distribution of "maximum drawdown" in later studies.

Now, let

$$Y_a^H := S_a^H - B_a^H \tag{3.1}$$

be the difference between the supremum of the fBm up to time a and the value of fBm at time a.

3.1. Theorem. *For $y \ge 0$,*

$$P\big(Y_a^H \le y\big) \ge 1 - \frac{\sqrt{2}a^H}{y\sqrt{\pi}}$$

Proof. For $a \in \mathbb{R}_+$,

$$\begin{aligned} P\big(S_a^H \ge b\big) &= P(S_a^H \ge b, B_a^H \ge b) + P\big(S_a^H \ge b, B_a^H < b\big) \\ &= P\big(B_a^H \ge b\big) + P\big(S_a^H \ge b, B_a^H < b\big) \le \frac{\sqrt{2}a^H}{x\sqrt{\pi}} \end{aligned} \tag{3.2}$$

by Theorem (2.3). Thus we have,

$$0 \le P\big(S_a^H \ge b, B_a^H < b\big) \le \frac{\sqrt{2}a^H}{b\sqrt{\pi}} - P\big(B_a^H \ge b\big) \tag{3.3}$$

For $b \ge x$, $x \in \mathbb{R}_+$,

$$\begin{aligned} &P\big(B_a^H < x\big) - P\big(S_a^H < b, B_a^H < x\big) \\ &= P\big(S_a^H \ge x, B_a^H < x\big) \le P(S_a^H \ge b, B_a^H < b) \le \frac{\sqrt{2}a^H}{b\sqrt{\pi}} - P\big(B_a^H \ge b\big), \end{aligned} \tag{3.4}$$

by (3.3) and since $x \le b$. Therefore, we have

$$P\big(S_a^H < b, B_a^H < x\big) \ge P\big(B_a^H < x\big) + P\big(B_a^H \ge b\big) - \frac{\sqrt{2}a^H}{b\sqrt{\pi}} \tag{3.5}$$

Now, as a result we get

$$P\big(Y_a^H \le y, B_a^H \le x\big) \ge P\big(B_a^H < x\big) + P\big(B_a^H \ge y + x\big) - \frac{\sqrt{2}a^H}{(y+x)\sqrt{\pi}} \tag{3.6}$$

by (3.1), where $y = b - x$ and

$$P\big(Y_a^H \le y, B_a^H \in dx\big) \ge P\big(B_a^H \in dx\big) + \frac{dP\big(B_a^H \ge y + x\big)}{dx} - \frac{\sqrt{2}a^H}{(y+x)^2\sqrt{\pi}}$$

finally

$$P\big(Y_a^H \ge y\big) \ge \int_0^\infty P\big(B_a^H \in dx\big) + \int_0^\infty \frac{dP(B_a^H \ge y + x)}{dx}\,dx - \int_0^\infty \frac{\sqrt{2}a^H}{(y+x)^2\sqrt{\pi}}\,dx.$$

Hence, for $y \geq 0$,

$$
\begin{aligned}
P\big(Y_a^H \leq y\big) &\geq \frac{1}{2} + \int_0^\infty dP(B_a^H \geq y + x) - \int_0^\infty \frac{\sqrt{2}}{(y+x)^2\sqrt{\pi}}\,dx \\
&= \frac{1}{2} - \frac{\sqrt{2}a^H}{\sqrt{\pi}y} + \int_0^\infty dP(B_a^H \geq y + x),
\end{aligned} \tag{3.7}
$$

and by the Fundamental Theorem of the Calculus,

$$
P\big(Y_a^H \leq y\big) \geq \frac{1}{2} - \frac{\sqrt{2}a^H}{\sqrt{\pi}y} + \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-(x+y)^2}{2}}\,dx = 1 - \frac{\sqrt{2}a^H}{\sqrt{\pi}y}. \tag{3.8}
$$

□

4. Conclusion

The Black-Scholes model for the values of the log returns of a risky stock using fBm is a more realistic model than using standard Brownian motion, due the advantage afforded by its capability of displaying the dependence of increments. Investors in risky stocks or their derivatives would naturally be interested in the highest value of the asset, in a certain time period. Towards that end, we have provided an upper bound on the second moment of the supremum of reflected fBm as a first result, and based on this, we have given an upper bound on the distribution of the supremum of reflected fBm.

Later, we have observed a lower bound on the distribution of the supremum of fBm up to time 1, as well as up to fixed time a. As an application of the second result, we have given the joint distribution of the supremum of fBm up to time a, and the value of fBm at time a, and obtained a lower bound on the distribution of the difference between them. Obviously, this application is already very useful for the investors. Based on the given results, we will investigate the distribution of "maximum drawdown" because of its important use as a measure of risk, in finance, as future work. Our conjecture on this distribution is that it is related both to the distribution of the supremum of the reflected fBm and the distribution of the difference between the supremum of fBm and the value of fBm at the maturity time.

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