Improved Ratio Estimators Using Some Robust Measures

Muhammad Abid,∗† Hafiz Zafar Nazir,‡ Muhammad Riaz,§ Zhengyan Lin¶ and Hafiz Muhammad Tahir∥

Abstract

Estimation of population mean is of prime concern in many studies and ratio estimators are popular choices for it. It is a common practice to use conventional measures of location to develop ratio estimators using information on auxiliary variables. In this article, we propose a class of ratio estimators for a finite population mean using information on two auxiliary variables with the help of some non-conventional location measures. We have incorporated tri-mean, Hodges-Lehmann, mid-range and decile mean of the two auxiliary variables to serve the purpose. The properties associated with the proposed class of ratio estimators are evaluated using mean square error. We have presented efficiency comparisons of the proposed class of ratio estimators with other existing estimators under the optimal conditions. An empirical study is also included for illustration and verification purposes. From theoretical and empirical study, we observed that the proposed estimators perform better as compared to the usual ratio and the existing estimators consider in this study.

Keywords: Auxiliary variables; Hodges-Lehmann estimator; Mean square error; Mid-range; Ratio Estimators; Tri-mean.

2000 AMS Classification: AMS

∗Department of Mathematics, Institute of Statistics, Zhejiang University, Hangzhou, 310027, China.
†Department of Statistics, Faculty of Science And Technology, Government College University, Faisalabad, Pakistan.
Email: mabid@zju.edu.cn
‡Department of Statistics, Faculty of Science, University of Sargodha, Pakistan.
Email: hafizzafaranazir@yahoo.com
§Department of Mathematics and Statistics, King Fahad University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia.
Email: riaz76@qu.edu.com
¶Department of Mathematics, Institute of Statistics, Zhejiang University, Hangzhou, 310027, China.
Email: zlin@zju.edu.cn
∥Department of Statistics, Faculty of Science And Technology, Government College University, Faisalabad, Pakistan.
Email: tahirquustat@yahoo.com
1. Introduction

In practice, we may come across different variables that offer information on every unit in the population. These variables are classified in two types namely variables of interest and auxiliary variables. The former are of direct interest in a study and are named study variables, while the later are instead employed to improve the sampling plan or to enhance estimation of the study variables. The auxiliary variables are generally associated with the study variables and we may use this information in different forms such as ratio, product and regression to mention a few etc. The auxiliary information may be available from different sources such as similar studies in past, economic reports, national census etc.

The ratio and regression estimators are used to improve the efficiency of the simple random sampling without replacement (SRSWOR) sample mean when there is a positive correlation exist between study variable (the variable of direct interest) and an auxiliary variable under certain conditions (see for example Cochran [7] and Murthy [18]). When the population parameters of the auxiliary variable, such as population mean, kurtosis, skewness, coefficient of variation, median, quartiles, correlation coefficient, deciles etc., are known, ratio estimators and their modifications are available in the literature which perform better than the usual sample mean under the SRSWOR. For further details on the modified ratio estimators, readers are referred to Abid et al. [1], [2], [3] and [4], Subramani and Kumarpandiyar [29], [30], [31], [32] and [33], Yan and Tian [37], Kadilar and Cingi [10] and [13], Upadhyaya and Singh [35], and Sisodia and Dwivedi [26] and reference therein.

Consider a finite population \(Z = \{Z_1, Z_2, Z_3, \ldots, Z_N\}\) of \(N\) distinct and identifiable units. Let \(Y, X_1, X_2\) be the study variable and the auxiliary variables with corresponding values \(Y_i, X_{1i}, X_{2i}\), respectively, for the \(i\)th unit \(i = \{1, 2, \ldots, N\}\) defined on a finite population \(Z\). Let \(X_{1(1)}, X_{1(2)}, \ldots, X_{1(N)}\) and \(X_{2(1)}, X_{2(2)}, \ldots, X_{2(N)}\) be the order statistics of two auxiliary variables, \(X_1\) and \(X_2\). The objective is to estimate population mean \(\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i\) on the basis of a random sample by using two auxiliary variables.

The notations used in this paper can be described as follows:

**NOMENCLATURE**

\begin{itemize}
  \item \(N\), Population size
  \item \(n\), Sample size
  \item \(f = n/N\), Sampling fraction
  \item \(Y\), Study variable
  \item \(X_1, X_2\), Auxiliary variables
  \item \(\bar{X}_1, \bar{X}_2, \bar{Y}\), Population means of auxiliary variables and study variable
  \item \(\bar{x}_1, \bar{x}_2, \bar{y}\), Sample means of auxiliary variables and study variable
  \item \(C_{x1}, C_{x2}, C_y\), Coefficient of variation of auxiliary variables and study variable
  \item \(MSE(\cdot)\), Mean square error of the estimator
  \item \(\hat{Y}_i\), Existing ratio estimators based on two auxiliary variables of \(\bar{Y}\)
  \item \(\hat{Y}_{pj}\), Proposed ratio estimators based on two auxiliary variables of \(\bar{Y}\)
\end{itemize}
Based on mentioned notations, the usual ratio multivariate estimator using information on two auxiliary variables \( X_1 \) and \( X_2 \) to estimate the population mean, \( \hat{Y} \), as follows:

\[
\hat{Y}_{MR} = \gamma_1 \bar{y} \frac{\bar{X}_1}{\bar{x}_1} + \gamma_2 \bar{y} \frac{\bar{X}_2}{\bar{x}_2}
\]

where \( \gamma_1 \) and \( \gamma_2 \) are the weights which satisfy the condition \( \gamma_1 + \gamma_2 = 1 \) and \( \bar{x}_1, \bar{x}_2 \) and \( \bar{X}_1, \bar{X}_2 \) are, respectively, the sample and populations means of auxiliary variables. The mean squared error (MSE) of the usual ratio estimator based on two auxiliary variables is given by

\[
MSE\left(\hat{Y}_{MR}\right) \approx \frac{1 - \frac{1}{n} \bar{Y}^2(C_y^2 + \gamma_1^* C_{x_1}^2 + \gamma_2^* C_{x_2}^2 - 2\gamma_1^* \rho_{yx_1} C_y C_{x_1})}{\bar{x}_1 C_{x_1} + C_{x_2}^2 - 2\rho_{x_1x_2} C_{x_1} C_{x_2} - \rho_{yx_2} C_y C_{x_2} + 2\gamma_1^* \rho_{x_1x_2} C_{x_1} C_{x_2}} \gamma_2^* = 1 - \gamma_1^*
\]

The optimum values of \( \gamma_1 \) and \( \gamma_2 \) are given by

\[
\gamma_1^* = \frac{C_{x_1}^2 + \rho_{yx_1} C_y C_{x_1} - \rho_{yx_1} C_y C_{x_2} - \rho_{x_1x_2} C_{x_1} C_{x_2}}{C_{x_1}^2 + C_{x_2}^2 - 2\rho_{x_1x_2} C_{x_1} C_{x_2}}, \gamma_2^* = 1 - \gamma_1^*
\]

So, the minimum MSE is

\[
MSE_{min}\left(\hat{Y}_{MR}\right) \approx \frac{1 - \frac{1}{n} \bar{Y}^2(C_y^2 + \gamma_1^* C_{x_1}^2 + \gamma_2^* C_{x_2}^2 - 2\gamma_1^* \rho_{yx_1} C_y C_{x_1})}{\bar{x}_1 C_{x_1} + C_{x_2}^2 - 2\rho_{x_1x_2} C_{x_1} C_{x_2} - \rho_{yx_2} C_y C_{x_2} + 2\gamma_1^* \rho_{x_1x_2} C_{x_1} C_{x_2}}
\]

Knowledge of two auxiliary variables in the framework of ratio estimators is used in this paper. Using information on two auxiliary variables several modified ratio estimators have been proposed by linking together ratio estimators, product estimators and regression estimators in order to find better results. For a more detailed discussion on the ratio estimator and its modifications using two auxiliary variables, one may refer to Lu and Yan [16], Lu et al. [17], Al-Hossain and Khan [6], Subramani and Prabavathy [34], Lu [15], Khare et al. [14], Perri [20], Kadilar and Cingi [11] and [12], Singh and Tailor [24] and [25], Abu-Dayyeh et al. [5], Srivastava and Jhajj [28], Srivastava [27], Raj [21] and Olkin [19].

The organization of the rest of the article is as follows: Section 2 provides a description of the existing estimators. The structure of proposed ratio estimator based on two auxiliary variables is given in Section 3. The efficiency comparisons of the proposed estimator with the existing estimator are presented in Section 4.
Section 5, consists of an empirical study of proposed estimators. Finally, Section 6 summarizes the findings of this study.

2. Existing ratio estimators

Singh [23] proposed a ratio estimator using information on two auxiliary variables for estimating the population mean $\bar{Y}$ as follows:

$$\hat{Y}_S = \bar{y} \left( \frac{\bar{X}_1}{x_1} \right) \left( \frac{\bar{X}_2}{x_2} \right)$$

The MSE of the estimator suggested by Singh [23] is given below:

$$(2.1) \quad MSE(\hat{Y}_S) \approx \frac{1}{n} \bar{Y}^2 \left( \frac{C_y^2}{C_{x_1}} + \frac{C_{x_2}^2}{C_{x_1}^2} + 2 \rho_{yx_1} C_y C_{x_1} + 2 \rho_{x_2} C_x C_{x_2} \right)$$

Using the known value of correlation coefficient of two auxiliary variables Singh and Tailor [25] suggested the following modified ratio cum product estimator

$$\hat{Y}_{ST} = \bar{y} \left( \frac{\bar{X}_1 + \rho_{x_1 x_2}}{x_1 + \rho_{x_1 x_2}} \right) \left( \frac{\bar{X}_2 + \rho_{x_1 x_2}}{x_2 + \rho_{x_1 x_2}} \right)$$

The MSE of Singh and Tailor [25] proposed estimator is given as

$$(2.2) \quad MSE(\hat{Y}_{ST}) \approx \frac{1}{n} \bar{Y}^2 \left( \frac{C_y^2}{C_{x_1}} + \frac{C_{x_2}^2}{C_{x_1}^2} + \frac{\delta_1^*}{\delta_2^*} \right)$$

where, $k_{yx} = \rho_{yx} \frac{C_y}{C_{x_1}}$, $k_{x_1} = \rho_{x_1 x_2} \frac{C_{x_1}}{C_{x_2}}$, $k_{x_2} = \rho_{x_1 x_2} \frac{C_{x_1}}{C_{x_2}}$, $\delta_1^* = \frac{\bar{Y}_1}{x_1 + \rho_{x_1 x_2}}$ and $\delta_2^* = \frac{\bar{Y}_2}{x_2 + \rho_{x_1 x_2}}$.

Lu and Yan [16] proposed the ratio estimators by using the known values of correlation coefficient, coefficient of variation and coefficient of kurtosis of two auxiliary variables. They showed that their proposed estimator performs efficiently as compared to the estimators suggested by Abu-Dayeh et al. [5] and usual ratio estimator based on two auxiliary variables. The Lu and Yan [16] proposed the following estimators

$$\hat{Y}_1 = a_1 \bar{y} \left( \frac{\bar{X}_1 + C_{x_1}}{x_1 + C_{x_1}} \right) + a_2 \bar{y} \left( \frac{\bar{X}_2 + C_{x_2}}{x_2 + C_{x_2}} \right)$$

$$\hat{Y}_2 = a_1 \bar{y} \left( \frac{\bar{X}_1 + \beta_{2(x_1)}}{x_1 + \beta_{2(x_1)}} \right) + a_2 \bar{y} \left( \frac{\bar{X}_2 + \beta_{2(x_2)}}{x_2 + \beta_{2(x_2)}} \right)$$

$$\hat{Y}_3 = a_1 \bar{y} \left( \frac{\bar{X}_1 \beta_{2(x_1)} + C_{x_1}}{x_1 \beta_{2(x_1)} + C_{x_1}} \right) + a_2 \bar{y} \left( \frac{\bar{X}_2 \beta_{2(x_2)} + C_{x_2}}{x_2 \beta_{2(x_2)} + C_{x_2}} \right)$$

$$\hat{Y}_4 = a_1 \bar{y} \left( \frac{\bar{X}_1 C_{x_1} + \beta_{2(x_1)}}{x_1 C_{x_1} + \beta_{2(x_1)}} \right) + a_2 \bar{y} \left( \frac{\bar{X}_2 C_{x_2} + \beta_{2(x_2)}}{x_2 C_{x_2} + \beta_{2(x_2)}} \right)$$

$$\hat{Y}_5 = a_1 \bar{y} \left( \frac{\bar{X}_1 + \rho_{yx_1}}{x_1 + \rho_{yx_1}} \right) + a_2 \bar{y} \left( \frac{\bar{X}_2 + \rho_{yx_2}}{x_2 + \rho_{yx_2}} \right)$$

$$\hat{Y}_6 = a_1 \bar{y} \left( \frac{\bar{X}_1 C_{x_1} + \rho_{yx_1}}{x_1 C_{x_1} + \rho_{yx_1}} \right) + a_2 \bar{y} \left( \frac{\bar{X}_2 C_{x_2} + \rho_{yx_2}}{x_2 C_{x_2} + \rho_{yx_2}} \right)$$
\[
\hat{Y}_7 = a_1 \hat{y} \left( \frac{\bar{X}_1 \rho_{xy_1} + C_{x_1}}{\bar{x}_1 \rho_{xy_1} + C_{x_1}} \right) + a_2 \hat{y} \left( \frac{\bar{X}_2 \rho_{xy_2} + C_{x_2}}{\bar{x}_2 \rho_{xy_2} + C_{x_2}} \right)
\]
\[
\hat{Y}_8 = a_1 \hat{y} \left( \frac{\bar{X}_1 \beta_{2(x_1)} + \rho_{xy_1}}{\bar{x}_1 \beta_{2(x_1)} + \rho_{xy_1}} \right) + a_2 \hat{y} \left( \frac{\bar{X}_2 \beta_{2(x_2)} + \rho_{xy_2}}{\bar{x}_2 \beta_{2(x_2)} + \rho_{xy_2}} \right)
\]
\[
\hat{Y}_9 = a_1 \hat{y} \left( \frac{\bar{X}_1 \rho_{xy_1} + \beta_{2(x_1)}}{\bar{x}_1 \rho_{xy_1} + \beta_{2(x_1)}} \right) + a_2 \hat{y} \left( \frac{\bar{X}_2 \rho_{xy_2} + \beta_{2(x_2)}}{\bar{x}_2 \rho_{xy_2} + \beta_{2(x_2)}} \right)
\]

The MSE of Lu and Yan [16] proposed estimators are given as

\[
MSE \left( \hat{Y}_i \right) \approx \frac{1}{n} \bar{Y}^2 \left( C_y^2 + a_1^2 R_{11}^2 C_x^2 + a_2^2 R_{22}^2 C_x^2 \right)
-2a_1 R_{111} \rho_{xy_1} C_y C_x - 2a_2 R_{221} \rho_{xy_2} C_y C_x + 2a_1 a_2 R_{11} R_{22} \rho_{x_1x_2} C_{x_1} C_{x_2}
\]

where, \( i = 1, 2, \ldots, 9 \).

The optimum values of \( a_1 \) and \( a_2 \) can easily be found by differentiating equation (2.3) with respect to \( a_1 \) and \( a_2 \) and equating it equal to zero. The optimum values of \( a_1 \) and \( a_2 \) are

\[
a_1^* = \frac{R_{22}^2 C_x^2 + R_{111} \rho_{xy_1} C_y C_x - R_{11} R_{22} \rho_{x_1x_2} C_{x_1} C_{x_2} - R_{222} \rho_{xy_2} C_y C_x}{R_{11}^2 C_x^2 - 2 R_{11} R_{22} \rho_{x_1x_2} C_{x_1} C_{x_2} + R_{22}^2 C_x^2},
\]
\[
a_2^* = 1 - k_1^*.
\]

Hence, the minimum MSE of Lu and Yan [16] estimators are given by

\[
MSE_{\text{min}} \left( \hat{Y}_i \right) \approx \frac{1}{n} \bar{Y}^2 \left( C_y^2 + a_1^* R_{11}^2 C_x^2 + a_2^* R_{22}^2 C_x^2 \right)
-2a_1^* R_{111} \rho_{xy_1} C_y C_x - 2a_2^* R_{222} \rho_{xy_2} C_y C_x + 2a_1^* a_2^* R_{11} R_{22} \rho_{x_1x_2} C_{x_1} C_{x_2}
\]

where \( i = 1, 2, \ldots, 9 \) and the values of constants \( R_{11} \) and \( R_{22} \) are,

\[
R_{11} = \left( \frac{\bar{X}_1}{\bar{X}_1 + C_{x_1}} \right), \quad R_{12} = \left( \frac{\bar{X}_1}{\bar{X}_1 + \beta_{2(x_1)}} \right);
\]
\[
R_{13} = \left( \frac{\bar{X}_1 \beta_{2(x_1)}}{\bar{x}_1 \beta_{2(x_1)} + C_{x_1}} \right), \quad R_{14} = \left( \frac{\bar{X}_1 C_{x_1}}{\bar{X}_1 C_{x_1} + \beta_{2(x_1)}} \right);
\]
\[
R_{15} = \left( \frac{\bar{X}_1}{\bar{x}_1 + \rho_{xy_1}} \right), \quad R_{16} = \left( \frac{\bar{X}_1 C_{x_1}}{\bar{X}_1 C_{x_1} + \rho_{xy_1}} \right);
\]
\[
R_{17} = \left( \frac{\bar{X}_1 \rho_{xy_1}}{\bar{x}_1 \rho_{xy_1} + C_{x_1}} \right), \quad R_{18} = \left( \frac{\bar{X}_1 \beta_{2(x_1)}}{\bar{X}_1 \beta_{2(x_1)} + \rho_{xy_1}} \right);
\]
\[
R_{19} = \left( \frac{\bar{X}_1 \rho_{xy_1}}{\bar{x}_1 \rho_{xy_1} + \beta_{2(x_1)}} \right), \quad R_{21} = \left( \frac{\bar{X}_2}{\bar{X}_2 + C_{x_2}} \right);
\]
\[
R_{22} = \left( \frac{\bar{X}_2}{\bar{X}_2 + \beta_{2(x_2)}} \right), \quad R_{23} = \left( \frac{\bar{X}_2 \beta_{2(x_2)}}{\bar{X}_2 \beta_{2(x_2)} + C_{x_2}} \right);
\]
\[
R_{24} = \left( \frac{\bar{X}_2 C_{x_2}}{\bar{X}_2 C_{x_2} + \beta_{2(x_2)}} \right), \quad R_{25} = \left( \frac{\bar{X}_2}{\bar{X}_2 + \rho_{xy_2}} \right);
\]
\[
R_{26} = \left( \frac{\bar{X}_2 C_{x_2}}{\bar{X}_2 C_{x_2} + \rho_{xy_2}} \right), \quad R_{27} = \left( \frac{\bar{X}_2 \rho_{xy_2}}{\bar{X}_2 \rho_{xy_2} + C_{x_2}} \right);
\]
The estimator is defined as:

\[ R_{28} = \left( \frac{\bar{X}_2 \beta_{2(x_2)}}{\bar{X}_2 \beta_{2(x_2)} + \rho_{yx_2}} \right), \quad R_{29} = \left( \frac{\bar{X}_2 \rho_{yx_2} + \beta_{2(x_2)}}{\bar{X}_2 \rho_{yx_2} + \beta_{2(x_2)}} \right). \]

3. Proposed class of ratio estimators

In this section, we propose different ratio type estimators using the known information on population tri-mean, mid-range, Hodges-Lehmann, decile mean, coefficient of variation, coefficient of kurtosis and correlation coefficient of two auxiliary variables. The mid-range defined as: \( MR_1 = \frac{X_{1(1)} + X_{1(N)}}{2} \), and \( MR_2 = \frac{X_{2(1)} + X_{2(N)}}{2} \), where \( X_{1(1)} \) and \( X_{1(N)} \) are the lowest and highest order statistics in a population of size \( N \) for \( X_1 \) and \( X_{2(1)} \) and \( X_{2(N)} \) are the lowest and highest order statistics in a population of size \( N \) for \( X_2 \). It is highly sensitive to outliers as its design structure is based on only extreme values of data (cf. Ferrell [8] for more details). We also include the measure based on the median of the pairwise Walsh averages which is known as Hodges-Lehmann (HL) estimator. The HL estimator is defined as: \( HL_1 = \text{median} \left( \left( X_{1(l)} + X_{1(k)} \right) / 2, 1 \leq l \leq k \leq N \right) \), and \( HL_2 = \text{median} \left( \left( X_{2(l)} + X_{2(k)} \right) / 2, 1 \leq l \leq k \leq N \right) \) for two auxiliary variables. The main advantage of the HL is that it is robust against outliers. For more properties of HL (see Hettmansperger and McKean [9]). The next measure included in this study is the tri-mean (TM), which is the weighted average of the population median and two quartiles and is defined as: \( TM_1 = \frac{Q_{1(1)} + 2Q_{1(2)} + Q_{1(3)}}{4} \), and \( TM_2 = \frac{Q_{2(1)} + 2Q_{2(2)} + Q_{2(3)}}{4} \), where \( Q_{1(p)} \) \((p = 1, 2, 3)\) denote one of the three quartiles in a population for \( X_1 \) and \( Q_{2(p)} \) \((p = 1, 2, 3)\) denote one of the three quartiles in a population for \( X_2 \). For detailed properties of TM (see Wang et al. [36] and Abid et al. [1]). The last measure include in this study is the decile mean (DM) which is defined as: \( DM_1 = \frac{D_{1(1)} + D_{1(2)} + \cdots + D_{1(9)}}{9} \), and \( DM_2 = \frac{D_{2(1)} + D_{2(2)} + \cdots + D_{2(9)}}{9} \), where \( D_{1(1)} + D_{1(2)} + \cdots + D_{1(9)} \) and \( D_{2(1)} + D_{2(2)} + \cdots + D_{2(9)} \) are the deciles for \( X_1 \) and \( X_2 \), respectively. The main advantage of the DM is that it is also less sensitive to extreme values than any other existing measures as well as it depends on the eighty percent of a sample, a population, or a probability distribution. So, it is also referred as a robust measure in this regard (see Abid et al. [2] and [4] for more detail).

The proposed ratio estimators based on two auxiliary variables are given below;

\[
\hat{Y}_{p1} = k_1 \bar{y} \left( \frac{\bar{x}_1 + MR_1}{\bar{x}_1 + MR_1} \right) + k_2 \bar{y} \left( \frac{\bar{x}_2 + MR_2}{\bar{x}_2 + MR_2} \right),
\]

\[
\hat{Y}_{p2} = k_1 \bar{y} \left( \frac{\bar{x}_1 + TM_1}{\bar{x}_1 + TM_1} \right) + k_2 \bar{y} \left( \frac{\bar{x}_2 + TM_2}{\bar{x}_2 + TM_2} \right),
\]

\[
\hat{Y}_{p3} = k_1 \bar{y} \left( \frac{\bar{x}_1 + HL_1}{\bar{x}_1 + HL_1} \right) + k_2 \bar{y} \left( \frac{\bar{x}_2 + HL_2}{\bar{x}_2 + HL_2} \right),
\]

\[
\hat{Y}_{p4} = k_1 \bar{y} \left( \frac{\bar{x}_1 + DM_1}{\bar{x}_1 + DM_1} \right) + k_2 \bar{y} \left( \frac{\bar{x}_2 + DM_2}{\bar{x}_2 + DM_2} \right),
\]

\[
\hat{Y}_{p5} = k_1 \bar{y} \left( \frac{\bar{x}_1 \beta_{2(x_1)} + MR_1}{\bar{x}_1 \beta_{2(x_1)} + MR_1} \right) + k_2 \bar{y} \left( \frac{\bar{x}_2 \beta_{2(x_2)} + MR_2}{\bar{x}_2 \beta_{2(x_2)} + MR_2} \right).
\]
\[ \hat{Y}_{p6} = k_1 \left( \frac{X_1 \beta_2(x_1) + TM_1}{x_1 \beta_2(x_1) + TM_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + TM_2}{x_2 \beta_2(x_2) + TM_2} \right) \]
\[ \hat{Y}_{p7} = k_1 \left( \frac{X_1 \beta_2(x_1) + HL_1}{x_1 \beta_2(x_1) + HL_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + HL_2}{x_2 \beta_2(x_2) + HL_2} \right) \]
\[ \hat{Y}_{p8} = k_1 \left( \frac{X_1 \beta_2(x_1) + DM_1}{x_1 \beta_2(x_1) + DM_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + DM_2}{x_2 \beta_2(x_2) + DM_2} \right) \]
\[ \hat{Y}_{p9} = k_1 \left( \frac{X_1 \beta_2(x_1) + MR_1}{x_1 \beta_2(x_1) + MR_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + MR_2}{x_2 \beta_2(x_2) + MR_2} \right) \]
\[ \hat{Y}_{p10} = k_1 \left( \frac{X_1 \beta_2(x_1) + TM_1}{x_1 \beta_2(x_1) + TM_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + TM_2}{x_2 \beta_2(x_2) + TM_2} \right) \]
\[ \hat{Y}_{p11} = k_1 \left( \frac{X_1 \beta_2(x_1) + HL_1}{x_1 \beta_2(x_1) + HL_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + HL_2}{x_2 \beta_2(x_2) + HL_2} \right) \]
\[ \hat{Y}_{p12} = k_1 \left( \frac{X_1 \beta_2(x_1) + DM_1}{x_1 \beta_2(x_1) + DM_1} \right) + k_2 \left( \frac{X_2 \beta_2(x_2) + DM_2}{x_2 \beta_2(x_2) + DM_2} \right) \]
\[ \hat{Y}_{p13} = k_1 \left( \frac{X_1 \rho_{yx_1} + MR_1}{x_1 \rho_{yx_1} + MR_1} \right) + k_2 \left( \frac{X_2 \rho_{yx_2} + MR_2}{x_2 \rho_{yx_2} + MR_2} \right) \]
\[ \hat{Y}_{p14} = k_1 \left( \frac{X_1 \rho_{yx_1} + TM_1}{x_1 \rho_{yx_1} + TM_1} \right) + k_2 \left( \frac{X_2 \rho_{yx_2} + TM_2}{x_2 \rho_{yx_2} + TM_2} \right) \]
\[ \hat{Y}_{p15} = k_1 \left( \frac{X_1 \rho_{yx_1} + HL_1}{x_1 \rho_{yx_1} + HL_1} \right) + k_2 \left( \frac{X_2 \rho_{yx_2} + HL_2}{x_2 \rho_{yx_2} + HL_2} \right) \]
\[ \hat{Y}_{p16} = k_1 \left( \frac{X_1 \rho_{yx_1} + DM_1}{x_1 \rho_{yx_1} + DM_1} \right) + k_2 \left( \frac{X_2 \rho_{yx_2} + DM_2}{x_2 \rho_{yx_2} + DM_2} \right) \]

The MSE of the proposed estimators are given as

\[ \text{MSE} \left( \hat{Y}_{pj} \right) \equiv 1 - \frac{1}{n} \sum_{i=1}^{n} \left( \hat{Y}_{pj} - Y_i \right)^2 + k_1^2 R_{p1j}^2 C_{x1}^2 + k_2^2 R_{p2j}^2 C_{x2}^2 \]
\[ -2k_1 R_{p1j} \rho_{yx_1} C_y C_{x1} - 2k_2 R_{p2j} \rho_{yx_2} C_y C_{x2} + 2k_1 k_2 R_{p1j} R_{p2j} \rho_{x1,x2} C_{y,x1} C_{x1,x2} \]

where, \( j = 1, 2, \ldots, 16 \). The optimum values of \( k_1 \) and \( k_2 \) for the proposed ratio estimators can easily be found by differentiating equation (3.1) with respect to \( k_1 \) and \( k_2 \) and equating it equal to zero. The optimum values of \( k_1 \) and \( k_2 \) are

\[ k_1^* = \frac{R_{p2j}^2 C_{x2}^2 + R_{p2j} \rho_{yx_1} C_y C_{x1} - R_{p1j} R_{p2j} \rho_{x1,x2} C_{x1} C_{x2} - R_{p2j} \rho_{yx_2} C_y C_{x2}}{R_{p1j}^2 C_{x1}^2 + 2 R_{p1j} R_{p2j} \rho_{x1,x2} C_{x1} C_{x2} + R_{p2j}^2 C_{x2}^2}, \]
\[ k_2^* = 1 - k_1^*. \]

Hence, the minimum MSE of the proposed ratio estimators are given by

\[ \text{MSE}_{\min} \left( \hat{Y}_{pj} \right) \equiv 1 - \frac{1}{n} \sum_{i=1}^{n} \left( \hat{Y}_{pj} - Y_i \right)^2 + k_1^* R_{p1j}^2 C_{x1}^2 + k_2^* R_{p2j}^2 C_{x2}^2 \]
\[ -2k_1^* R_{p1j} \rho_{yx_1} C_y C_{x1} - 2k_2^* R_{p2j} \rho_{yx_2} C_y C_{x2} + 2k_1^* k_2^* R_{p1j} R_{p2j} \rho_{x1,x2} C_{y,x1} C_{x1,x2} \]
where, \( j = 1, 2, \ldots, 16 \) and the values of constant \( R_{p1j} \) and \( R_{p2j} \) are,

\[
R_{p11} = \left( \frac{\bar{X}_1}{X_1 + MR_1} \right), R_{p12} = \left( \frac{\bar{X}_1}{X_1 + TM_1} \right),
\]
\[
R_{p13} = \left( \frac{\bar{X}_1}{X_1 + HL_1} \right), R_{p14} = \left( \frac{\bar{X}_1}{X_1 + DM_1} \right),
\]
\[
R_{p15} = \left( \frac{\bar{X}_1 \beta_{2(x_1)}}{X_1 \beta_{2(x_1)} + MR_1} \right), R_{p16} = \left( \frac{\bar{X}_1 \beta_{2(x_1)}}{X_1 \beta_{2(x_1)} + TM_1} \right),
\]
\[
R_{p17} = \left( \frac{\bar{X}_1 \beta_{2(x_1)}}{X_1 \beta_{2(x_1)} + HL_1} \right), R_{p18} = \left( \frac{\bar{X}_1 \beta_{2(x_1)}}{X_1 \beta_{2(x_1)} + DM_1} \right),
\]
\[
R_{p19} = \left( \frac{\bar{X}_1 C_{x1}}{X_1 C_{x1} + MR_1} \right), R_{p110} = \left( \frac{\bar{X}_1 C_{x1}}{X_1 C_{x1} + TM_1} \right),
\]
\[
R_{p111} = \left( \frac{\bar{X}_1 C_{x1}}{X_1 C_{x1} + HL_1} \right), R_{p112} = \left( \frac{\bar{X}_1 C_{x1}}{X_1 C_{x1} + DM_1} \right),
\]
\[
R_{p113} = \left( \frac{\bar{X}_1 \rho_{y|x_1}}{X_1 \rho_{y|x_1} + MR_1} \right), R_{p114} = \left( \frac{\bar{X}_1 \rho_{y|x_1}}{X_1 \rho_{y|x_1} + TM_1} \right),
\]
\[
R_{p115} = \left( \frac{\bar{X}_1 \rho_{y|x_1}}{X_1 \rho_{y|x_1} + HL_1} \right), R_{p116} = \left( \frac{\bar{X}_1 \rho_{y|x_1}}{X_1 \rho_{y|x_1} + DM_1} \right),
\]
\[
R_{p21} = \left( \frac{\bar{X}_2}{X_2 + MR_2} \right), R_{p22} = \left( \frac{\bar{X}_2}{X_2 + TM_2} \right),
\]
\[
R_{p23} = \left( \frac{\bar{X}_2}{X_2 + HL_2} \right), R_{p24} = \left( \frac{\bar{X}_2}{X_2 + DM_2} \right),
\]
\[
R_{p25} = \left( \frac{\bar{X}_2 \beta_{2(x_2)}}{X_2 \beta_{2(x_2)} + MR_2} \right), R_{p26} = \left( \frac{\bar{X}_2 \beta_{2(x_2)}}{X_2 \beta_{2(x_2)} + TM_2} \right),
\]
\[
R_{p27} = \left( \frac{\bar{X}_2 \beta_{2(x_2)}}{X_2 \beta_{2(x_2)} + HL_2} \right), R_{p28} = \left( \frac{\bar{X}_2 \beta_{2(x_2)}}{X_2 \beta_{2(x_2)} + DM_2} \right),
\]
\[
R_{p29} = \left( \frac{\bar{X}_2 C_{x2}}{X_2 C_{x2} + MR_2} \right), R_{p210} = \left( \frac{\bar{X}_2 C_{x2}}{X_2 C_{x2} + TM_2} \right),
\]
\[
R_{p211} = \left( \frac{\bar{X}_2 C_{x2}}{X_2 C_{x2} + HL_2} \right), R_{p212} = \left( \frac{\bar{X}_2 C_{x2}}{X_2 C_{x2} + DM_2} \right),
\]
\[
R_{p213} = \left( \frac{\bar{X}_2 \rho_{y|x_2}}{X_2 \rho_{y|x_2} + MR_2} \right), R_{p214} = \left( \frac{\bar{X}_2 \rho_{y|x_2}}{X_2 \rho_{y|x_2} + TM_2} \right),
\]
\[
R_{p215} = \left( \frac{\bar{X}_2 \rho_{y|x_2}}{X_2 \rho_{y|x_2} + HL_2} \right), R_{p216} = \left( \frac{\bar{X}_2 \rho_{y|x_2}}{X_2 \rho_{y|x_2} + DM_2} \right).
\]

It is to be noted that Lu and Yan [16] and the proposed estimators using information of two auxiliary variables are belongs to the following general class of ratio estimators for \( \hat{Y} \) defined as (cf. Lu and Yan [16])

\[
\hat{Y}_{ge} = K_1 \left( \frac{T_1 \bar{X}_1 + P_1}{T_1 \bar{x}_1 + P_1} \right) + K_2 \left( \frac{T_2 \bar{X}_2 + P_2}{T_2 \bar{x}_2 + P_2} \right)
\]
where $(K_1, K_2)$ are weights that satisfy the condition, $K_1 + K_2 = 1$, $T_1 (\neq 0)$, $T_2 (\neq 0)$, $P_1$, $P_2$ are either constant or function of known parameters of the population.

To the first degree of approximation the MSE of general class of ratio estimators for $\bar{Y}$ can be obtained as follows:

Let us define, $e_0 = \frac{\bar{Y} - \bar{Y}}{\bar{Y}}$, $e_1 = \frac{x_1 - \bar{x}_1}{\bar{x}_1}$, $e_2 = \frac{x_2 - \bar{x}_2}{\bar{x}_2}$, then $\bar{y} = \bar{Y}(1 + e_0)$, $\bar{x}_1 = \bar{X}_1(1 + e_1)$, and $\bar{x}_2 = \bar{X}_2(1 + e_2)$. From the definition of $e_0$, $e_1$ and $e_2$, we get $E(e_0) = E(e_1) = E(e_2) = 0$, where $E(e_0) = \frac{(1-\hat{C}_y^2)}{n}C_y$, $E(e_1^2) = \frac{(1-\hat{C}_x^2)}{n}C_x$, $E(e_2^2) = \frac{(1-\hat{C}_x^2)}{n}C_x$, $E(e_0e_1) = \rho_{yx_1}C_yC_{x_1}$, $E(e_0e_2) = \rho_{yx_2}C_yC_{x_2}$ and $E(e_1e_2) = \rho_{x_1x_2}C_{x_1}C_{x_2}$.

The proposed general class of estimators $\bar{Y}_{gc}$ can be written terms of $e_0$, $e_1$ and $e_2$ as

$$\bar{Y}_{gc} = K_1\bar{Y}(1+e_0)\left(\frac{T_1X_1 + P_1}{T_1X_1 (1 + e_1) + P_1}\right) + K_2\bar{Y}(1+e_0)\left(\frac{T_2X_1 + P_2}{T_2X_2 (1 + e_2) + P_2}\right)$$

$$\bar{Y}_{gc} = K_1\bar{Y}(1+e_0)\left(1 + \frac{1}{1 + \frac{T_1X_1 e_1}{T_1X_1 P_1}}\right) + K_2\bar{Y}(1+e_0)\left(1 + \frac{1}{1 + \frac{T_2X_2 e_2}{T_2X_2 P_2}}\right)$$

(3.3) $\bar{Y}_{gc} = K_1\bar{Y}(1+e_0)(1 + \beta_1e_1)^{-1} + K_2\bar{Y}(1+e_0)(1 + \beta_2e_2)^{-1}$

Ignoring the higher order terms and also subtracting $Y$ from both sides of equation (3.4), we get

$$\bar{Y}_{gc} - \bar{Y} \approx \bar{Y}(e_0 - K_1\beta_1 e_1 - K_2\beta_2 e_2)$$

The MSE of the proposed class of estimators are obtained as follows:

$$MSE(\bar{Y}_{gc}) = E(\bar{Y}_{gc} - \bar{Y})^2$$

$$\approx \bar{Y}^2(E(e_0^2) + K_1^2\beta_1^2E(e_1^2) + K_2^2\beta_2^2E(e_2^2) - 2K_1\beta_1E(e_0e_1) - 2K_2\beta_2E(e_0e_2) + 2K_1K_2\beta_1\beta_2E(e_1e_2))$$

So,

$$MSE(\bar{Y}_{gc}) \approx \frac{1 - f}{n}\bar{Y}^2(C_y^2 + K_1^2\beta_1^2C_{x_1}^2 + K_2^2\beta_2^2C_{x_2}^2 - 2K_1\beta_1\rho_{yx_1}C_yC_{x_1} - 2K_2\beta_2\rho_{yx_2}C_yC_{x_2} + 2K_1K_2\beta_1\beta_2\rho_{x_1x_2}C_{x_1}C_{x_2})$$

where, $\beta_1 = \frac{T_1\bar{x}_{1}}{T_1\bar{x}_{1} + P_1}$, $\beta_2 = \frac{T_2\bar{x}_{2}}{T_2\bar{x}_{2} + P_2}$

The optimum values of $K_1$ and $K_2$ to minimize (3.5) for general class of estimators can easily be found as follows:

$$K_1^* = \frac{\beta_2^2C_{x_2}^2 + \beta_1\rho_{yx_1}C_yC_{x_1} - \beta_1\beta_2\rho_{x_1x_2}C_{x_1}C_{x_2} - \beta_2\rho_{yx_2}C_yC_{x_2} - \beta_1^2C_{x_1}^2 - 2\beta_1\beta_2\rho_{x_1x_2}C_{x_1}C_{x_2} + \beta_2^2C_{x_2}^2}{\beta_1^2C_{x_1}^2 - 2\beta_1\beta_2\rho_{x_1x_2}C_{x_1}C_{x_2} + \beta_2^2C_{x_2}^2}$$

$$K_2^* = 1 - K_1^*.$$
So, the minimum MSE of general class of estimators are given by

\[
MSE_{\text{min}} \left( \hat{Y}_{ge} \right) \approx \frac{1}{n} \bar{Y}^2 \left( C_y^2 + K_1^2 \beta_1^2 C_{x_1}^2 + K_2^2 \beta_2^2 C_{x_2}^2 \right)
- 2K_1^1 \beta_1 \rho_{yx_1} C_y C_{x_1} - 2K_2^2 \beta_2 \rho_{yx_2} C_y C_{x_2} + 2K_1^1 K_2^2 \beta_1 \beta_2 \rho_{x_1x_2} C_{x_1} C_{x_2}
\]

\[ \Rightarrow (k_1^2 R_{p1j} - \gamma_1^2)C_{x_1}^2 + (k_2^2 R_{p2j} - \gamma_2^2)C_{x_2}^2
- 2(k_1^1 R_{p1j} - \gamma_1^1)\rho_{yx_1} C_y C_{x_1}
+ 2(k_1^1 k_2^2 R_{p1j} R_{p2j} - \gamma_1^1 \gamma_2^2)\rho_{x_1x_2} C_{x_1} C_{x_2} < 0\]

where \( j = 1, 2, \ldots, 16 \).

4. Efficiency comparisons

In this section, the condition for which the proposed ratio estimators will have minimum mean square error compared to usual ratio estimator and existing ratio estimator for estimating the finite population mean have been derived algebraically.

4.1. Comparison with traditional ratio estimator. We compare the MSE of the proposed ratio estimator given in equation (3.2) with the MSE of the classical ratio estimator given in equation (2.1) as follows:

\[
MSE_{\text{min}} \left( \hat{Y}_{pj} \right) < MSE_{\text{min}} \left( \hat{Y}_{MR} \right)
\]

\[
\Rightarrow (k_1^2 R_{p1j} - \gamma_1^2)C_{x_1}^2 + (k_2^2 R_{p2j} - \gamma_2^2)C_{x_2}^2
- 2(k_1^1 R_{p1j} - \gamma_1^1)\rho_{yx_1} C_y C_{x_1}
+ 2(k_1^1 k_2^2 R_{p1j} R_{p2j} - \gamma_1^1 \gamma_2^2)\rho_{x_1x_2} C_{x_1} C_{x_2} < 0
\]

where \( j = 1, 2, \ldots, 16 \).

4.2. Comparison with Singh [23] ratio estimator. The proposed ratio estimator \( \hat{Y}_{pj} \) will be more efficient than that of [23] ratio estimator i.e. \( \hat{Y}_S \) if

\[
MSE_{\text{min}} \left( \hat{Y}_{pj} \right) < MSE \left( \hat{Y}_S \right)
\]

\[
\Rightarrow (k_1^2 R_{p1j} - 1)C_{x_1}^2 + (k_2^2 R_{p2j} - 1)C_{x_2}^2
- 2(k_1^1 R_{p1j} - 1)\rho_{yx_1} C_y C_{x_1}
+ 2(k_1^1 k_2^2 R_{p1j} R_{p2j} - 1)\rho_{x_1x_2} C_{x_1} C_{x_2} - 2k_2^2 R_{p2j} \rho_{yx_2} C_y C_{x_2} < 0
\]

where \( j = 1, 2, \ldots, 16 \).

4.3. Comparison with Singh and Tailor [25] ratio estimator. The proposed ratio estimator \( \hat{Y}_{pj} \) will be more efficient than the Singh and Tailor [25] ratio estimator i.e. \( \hat{Y}_{ST} \) if

\[
MSE_{\text{min}} \left( \hat{Y}_{pj} \right) < MSE \left( \hat{Y}_{ST} \right)
\]

\[
\Rightarrow (k_1^2 R_{p1j} - \delta_1^2 (\delta_1^1 - 2k_{yx1}))C_{x_1}^2
+ (k_2^2 R_{p2j} - \delta_2^2 (\delta_2^1 + 2(k_{yx2} - \delta_1^1 k_{x1x2})) C_{x_2}^2
- 2(k_1^1 R_{p1j} \rho_{yx_1} C_{x_1} + k_2^2 R_{p2j} \rho_{yx_2} C_{x_2}) C_y
+ 2k_1^1 k_2^2 R_{p1j} R_{p2j} \rho_{x_1x_2} C_{x_1} C_{x_2} < 0
\]

where \( j = 1, 2, \ldots, 16 \).
4.4. Comparison with Lu and Yan [16] ratio estimators. We compare the MSE of the proposed ratio estimator given in equation (3.4) with the MSE of the ratio estimator proposed by [16] given in equation (2.4) as follows:

\[
MSE_{\min}\left(\hat{Y}_{pj}\right) < MSE_{\min}\left(\hat{Y}_{i}\right)
\]

\[
\Leftrightarrow (k_1^2R_{p1j}^2 - a_1^2R_{1i}^2)C_{x1}^2 + (k_2^2R_{p2j}^2 - a_2^2R_{2i}^2)C_{x2}^2
\]

\[
-2(k_1^2R_{p1j} - a_1^2R_{1i})\rho_{yx1}C_yC_{x1} - 2(k_2^2R_{p2j} - a_2^2R_{2i})\rho_{yx2}C_yC_{x2}
\]

\[
+2(k_1^2k_2^2R_{p1j}R_{p2j} - a_1^2a_2^2R_{1i}R_{2i})\rho_{x1x2}C_{x1}C_{x2} < 0
\]

where \(j = 1, 2, \ldots, 16\) and \(i = 1, 2, \ldots, 9\). If the above condition is satisfied, then the proposed estimator \(\hat{Y}_{pj}\) will be more efficient than the \(\hat{Y}_{i}\) ratio estimator.

**Table 1.** The suitable choices of constant \(T_1\), \(T_2\), \(P_1\), and \(P_2\) for existing and proposed estimators.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>(T_1)</th>
<th>(P_1)</th>
<th>(T_2)</th>
<th>(P_2)</th>
</tr>
</thead>
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<td>(C_{x1})</td>
<td>1</td>
<td>(C_{x2})</td>
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Table 2. The values of constants and MSEs of the existing ratio estimators.

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<td>$\hat{Y}_{ST}$</td>
<td>–</td>
<td>–</td>
<td>10076.26</td>
<td>–</td>
</tr>
<tr>
<td>$\hat{Y}_{1}$</td>
<td>0.9966</td>
<td>0.9963</td>
<td>10555.66</td>
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<tr>
<td>$\hat{Y}_{2}$</td>
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<td>0.9816</td>
<td>10480.16</td>
<td>0.9976</td>
</tr>
<tr>
<td>$\hat{Y}_{3}$</td>
<td>0.9988</td>
<td>0.9990</td>
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<tr>
<td>$\hat{Y}_{4}$</td>
<td>0.9810</td>
<td>0.9757</td>
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<tr>
<td>$\hat{Y}_{5}$</td>
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<td>0.9977</td>
<td>10564.99</td>
<td>0.9968</td>
</tr>
<tr>
<td>$\hat{Y}_{6}$</td>
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<td>0.9970</td>
<td>10559.00</td>
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<tr>
<td>$\hat{Y}_{7}$</td>
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<td>0.9921</td>
<td>10525.67</td>
<td>0.9964</td>
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<tr>
<td>$\hat{Y}_{8}$</td>
<td>0.9993</td>
<td>0.9994</td>
<td>10575.24</td>
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<tr>
<td>$\hat{Y}_{9}$</td>
<td>0.9700</td>
<td>0.9601</td>
<td>10362.63</td>
<td>0.9973</td>
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</table>

Table 3. The values of constants and MSEs of the existing ratio estimators.

<table>
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<tr>
<th>Estimator</th>
<th>Population 1</th>
<th>Population 1</th>
<th>Population 1</th>
<th>Population 1</th>
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<tr>
<td></td>
<td>$R_{pij}$</td>
<td>$R_{p2j}$</td>
<td>MSE</td>
<td>$R_{pij}$</td>
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<td>0.5464</td>
<td>8887.83</td>
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<td>$\hat{Y}_{p3}$</td>
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<td>0.5201</td>
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</tr>
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<td>$\hat{Y}_{p4}$</td>
<td>0.4831</td>
<td>0.4908</td>
<td>8899.02</td>
<td>0.5080</td>
</tr>
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<td>$\hat{Y}_{p5}$</td>
<td>0.6812</td>
<td>0.6992</td>
<td>9064.65</td>
<td>0.2578</td>
</tr>
<tr>
<td>$\hat{Y}_{p6}$</td>
<td>0.7893</td>
<td>0.8180</td>
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<td>0.4902</td>
</tr>
<tr>
<td>$\hat{Y}_{p7}$</td>
<td>0.7619</td>
<td>0.8017</td>
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<td>0.4442</td>
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<td>$\hat{Y}_{p8}$</td>
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<td>$\hat{Y}_{p9}$</td>
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<tr>
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<tr>
<td>$\hat{Y}_{p11}$</td>
<td>0.4418</td>
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<tr>
<td>$\hat{Y}_{p12}$</td>
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</tr>
<tr>
<td>$\hat{Y}_{p13}$</td>
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<td>9496.34</td>
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<tr>
<td>$\hat{Y}_{p14}$</td>
<td>0.3668</td>
<td>0.3515</td>
<td>9124.94</td>
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<tr>
<td>$\hat{Y}_{p15}$</td>
<td>0.3310</td>
<td>0.3278</td>
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</tr>
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<td>$\hat{Y}_{p16}$</td>
<td>0.2961</td>
<td>0.3025</td>
<td>9283.07</td>
<td>0.4858</td>
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</tbody>
</table>
5. Empirical Study

The performance of the proposed ratio estimators and the existing ratio estimators is evaluated by using two natural populations. The population 1 is taken from Singh and Chaudhary [22] page 177 and population 2 is taken from Murthy [18] page 228. The characteristics of the two populations are given below:

Population 1 (Singh and Chaudhary [22])

\[ Y = \text{Area under Wheat in 1974} \]
\[ X_1 = \text{Area under Wheat in 1971} \]
\[ X_2 = \text{Area under Wheat in 1973} \]

\( N = 34, \ n = 20, \ Y = 856.412, \ X_1 = 208.882, \ X_2 = 199.441, \ C_y = 0.8561, \)
\( C_{x_1} = 0.721, \ C_{x_2} = 0.753, \ \rho_{yx_1} = 0.449, \ \rho_{yx_2} = 0.445, \ \rho_{x_1x_2} = 0.980, \)
\( \beta_{2(x_1)} = 2.910, \ \beta_{2(x_2)} = 3.732, \ MR_1 = 284.500, \ MR_2 = 320.000, \)
\( TM_1 = 162.250, \ TM_2 = 165.562, \ HL_1 = 190.000, \ HL_2 = 184.000, \)
\( DM_1 = 223.467, \ DM_2 = 206.944. \)

Population 2 (Murthy [18])

\[ Y = \text{Output} \]
\[ X_1 = \text{Number of workers} \]
\[ X_2 = \text{Fixed capital} \]

\( N = 80, \ n = 20, \ Y = 5182.637, \ X_1 = 285.125, \ X_2 = 1126.463, \ C_y = 0.354, \)
\( C_{x_1} = 0.948, \ C_{x_2} = 0.751, \ \rho_{yx_1} = 0.915, \ \rho_{yx_2} = 0.941, \ \rho_{x_1x_2} = 0.988, \)
\( \beta_{2(x_1)} = 0.698, \ \beta_{2(x_2)} = 1.050, \ MR_1 = 573.000, \ MR_2 = 1795.500, \)
\( TM_1 = 206.937, \ TM_2 = 931.562, \ HL_1 = 249.000, \ HL_2 = 1040.500, \)
\( DM_1 = 276.189, \ DM_2 = 1150.700. \)

The values of constants and the MSE of the existing and proposed ratio estimators using the information of two auxiliary variables are given in Tables 2 and 3, respectively. It can be observed that the constants and the MSE of the suggested ratio estimators are smaller than the usual ratio estimator and the existing ratio estimators consider in this study (cf. Tables 2-3). From Table 3, it is evident that the proposed estimators perform better than the usual ratio estimator and the existing ratio estimators in terms of MSE, which shows that the proposed estimators are more efficient.

The comparison of the proposed ratio estimators with the traditional ratio and the existing ratio estimators are also shown by graphically for all the populations considered in this study. From Figures 1-2, it can be seen that the proposed estimators have smaller values of MSE as compared to the usual ratio estimator and the existing ratio estimators, which indicates that the performance of the proposed estimators are better as compared to the traditional ratio estimator, Singh [23] estimator, Singh and Tailor [25] estimator and Lu and Yan [16] estimators.
Figure 1. Mean squared error of the proposed and existing estimators of population 1.
Figure 2. Mean squared error of the proposed and existing estimators of population 2.
Figure 3. Scatter graph of first auxiliary and study variables.
Figure 4. Scatter graph of second auxiliary and study variables.
Table 4. The values MSEs of the existing and proposed estimators for outliers data.

<table>
<thead>
<tr>
<th>Existing</th>
<th>Proposed</th>
<th>MSE</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{Y}_{MR}$</td>
<td>$\hat{Y}_{p1}$</td>
<td>151170.40</td>
<td>23377.04</td>
</tr>
<tr>
<td>$\hat{Y}_{S}$</td>
<td>$\hat{Y}_{p2}$</td>
<td>2928277.00</td>
<td>18051.03</td>
</tr>
<tr>
<td>$\hat{Y}_{ST}$</td>
<td>$\hat{Y}_{p3}$</td>
<td>1121996.00</td>
<td>19849.40</td>
</tr>
<tr>
<td>$\hat{Y}_1$</td>
<td>$\hat{Y}_{p4}$</td>
<td>152013.10</td>
<td>18406.64</td>
</tr>
<tr>
<td>$\hat{Y}_2$</td>
<td>$\hat{Y}_{p5}$</td>
<td>151555.80</td>
<td>25556.96</td>
</tr>
<tr>
<td>$\hat{Y}_3$</td>
<td>$\hat{Y}_{p6}$</td>
<td>156380.50</td>
<td>24281.94</td>
</tr>
<tr>
<td>$\hat{Y}_4$</td>
<td>$\hat{Y}_{p7}$</td>
<td>151367.60</td>
<td>21726.77</td>
</tr>
<tr>
<td>$\hat{Y}_5$</td>
<td>$\hat{Y}_{p8}$</td>
<td>151824.10</td>
<td>19016.20</td>
</tr>
<tr>
<td>$\hat{Y}_6$</td>
<td>$\hat{Y}_{p9}$</td>
<td>151648.40</td>
<td>25035.24</td>
</tr>
<tr>
<td>$\hat{Y}_7$</td>
<td>$\hat{Y}_{p10}$</td>
<td>152047.30</td>
<td>19106.17</td>
</tr>
<tr>
<td>$\hat{Y}_8$</td>
<td>$\hat{Y}_{p11}$</td>
<td>152160.60</td>
<td>18854.91</td>
</tr>
<tr>
<td>$\hat{Y}_9$</td>
<td>$\hat{Y}_{p12}$</td>
<td>151562.80</td>
<td>18698.78</td>
</tr>
<tr>
<td>$\hat{Y}_{p13}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{Y}_{p14}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{Y}_{p15}$</td>
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<td></td>
</tr>
<tr>
<td>$\hat{Y}_{p16}$</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

5.1. Robustness of the proposed estimators.
As in the earlier sections, it is mentioned that measures used in this study such as tri-mean, mid-range, Hodges-Lehmann and decile mean are robust against outliers. Thus when there is an outlier in the data these measures are perform efficiently as compared to other measures of locations. So, in this section, we check the efficiency of our proposed estimators in case of outliers. For this purpose, we considered the data of Population 2 and introduced some outliers in this data. From Figures 3 and 4, we clearly see that there are outliers in the data, so we can expect the proposed estimators to perform better than the usual and existing estimators considered in this study.

We obtain the MSE values of the existing and proposed estimators as defined in Sections 2 and 3, respectively by using outliers data. The MSE values of the existing and proposed estimators are given in Table 4. From Table 4, it is observed that the proposed estimators have smaller values of MSE as compared to the usual ratio estimator and existing estimators, which indicates that the proposed estimators are more efficient in the presence of outliers.

To show the dominance of the proposed ratio estimators over the existing estimators, we have also found the percent relative efficiencies (PREs) for population 2 in case of excluded and included outliers in the data. The PREs of the proposed
estimators (p) with respect to the existing estimators (e) is computed as

\[(5.1) \quad PRE(e, p) = \frac{MSE(e)}{MSE(p)} * 100\]

and are given in Tables 5 and 6, respectively (see Appendix).

If the percentage relative efficiency value found from (5.1) is greater than 100, then it is seeming that the proposed estimators are more efficient as compared to the usual ratio estimator and existing estimators. Therefore, from Tables 5 and 6, we see that all the proposed estimators are more efficient than the traditional ratio estimator and existing estimators consider in this study. It is to be also noted that the values of relative efficiencies of the proposed estimators with respect to the existing estimators in Table 6 would increase dramatically, which shows that the efficiencies of the proposed estimators would increase significantly, if there were more outliers in the data.

6. Summary and Conclusions

The study has proposed a variety of two auxiliary based ratio estimators using tri-mean, mid-range, Hodges-Lehmann, decile mean, coefficient of variation, coefficient of kurtosis and correlation coefficient. It is observed that the proposed estimators outperform the usual ratio estimator and the existing ratio estimators in terms of mean squared error under all the populations considered for the numerical study. Moreover, robustness to extreme observations is an added feature of the proposed estimators. Hence, we recommended the use of the proposed ratio estimators over the usual and other existing ratio estimators, especially in the presence of unusual observations in the data.

Acknowledgments

The authors are grateful to the Editor-in-chief Prof. Dr. Cem Kadilar and referees for their constructive comments that led to substantial improvements in the article. The third author is indebted to the King Fahd University of Petroleum and Minerals (KFUPM), Dhahran, Saudi Arabia, for providing excellent research facilities.

References


[16] Lu, J., and Yan, Z. A class of ratio estimators of a finite population mean using two auxiliary variables, PLOS ONE 2(9), 1-6, 2014.


### Table 5. Percentage Relative Efficiency of existing estimators with respect to proposed estimators of Population 2 without outlier data.

<table>
<thead>
<tr>
<th>Proposed Estimators</th>
<th>$\hat{Y}_{MR}$</th>
<th>$\hat{Y}_S$</th>
<th>$\hat{Y}_{ST}$</th>
<th>$\hat{Y}_1$</th>
<th>$\hat{Y}_2$</th>
<th>$\hat{Y}_3$</th>
<th>$\hat{Y}_4$</th>
<th>$\hat{Y}_5$</th>
<th>$\hat{Y}_6$</th>
<th>$\hat{Y}_7$</th>
<th>$\hat{Y}_8$</th>
<th>$\hat{Y}_9$</th>
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<tbody>
<tr>
<td>$\hat{Y}_{p1}$</td>
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<td>14804.2</td>
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<td>425.3</td>
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<td>421.5</td>
<td>420.9</td>
<td>422.7</td>
<td>424.4</td>
<td>419.9</td>
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<td>$\hat{Y}_{p2}$</td>
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<td>504.8</td>
<td>507.1</td>
<td>509.0</td>
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<td>6941.9</td>
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<td>509.3</td>
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<td>507.7</td>
<td>510.0</td>
<td>512.0</td>
<td>506.5</td>
</tr>
</tbody>
</table>
Table 6. Percentage Relative Efficiency of existing estimators with respect to proposed estimators of Population 2 with outlier data.

| Proposed Estimators | \( \hat{\bar{Y}}_{MR} \) | \( \hat{\bar{Y}}_{S} \) | \( \hat{\bar{Y}}_{ST} \) | \( \hat{\bar{Y}}_{1} \) | \( \hat{\bar{Y}}_{2} \) | \( \hat{\bar{Y}}_{3} \) | \( \hat{\bar{Y}}_{4} \) | \( \hat{\bar{Y}}_{5} \) | \( \hat{\bar{Y}}_{6} \) | \( \hat{\bar{Y}}_{7} \) | \( \hat{\bar{Y}}_{8} \) | \( \hat{\bar{Y}}_{9} \) |
|---------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \hat{\bar{Y}}_{p1} \) | 646.7            | 12526.3         | 4799.6          | 650.3           | 648.3           | 668.9           | 647.5           | 649.5           | 648.7           | 650.4           | 650.9           | 648.3           |
| \( \hat{\bar{Y}}_{p2} \) | 837.5            | 16222.2         | 6215.7          | 842.1           | 839.6           | 866.3           | 838.6           | 841.1           | 840.1           | 842.3           | 842.9           | 839.6           |
| \( \hat{\bar{Y}}_{p3} \) | 761.6            | 14752.5         | 5652.5          | 765.8           | 763.5           | 787.8           | 762.6           | 764.9           | 764.0           | 766.0           | 766.6           | 763.6           |
| \( \hat{\bar{Y}}_{p4} \) | 821.3            | 15908.8         | 6095.6          | 825.9           | 823.4           | 849.6           | 822.4           | 824.8           | 823.9           | 826.0           | 826.7           | 823.4           |
| \( \hat{\bar{Y}}_{p5} \) | 591.5            | 11457.8         | 4390.2          | 594.8           | 593.0           | 611.9           | 592.3           | 594.1           | 593.4           | 594.9           | 595.4           | 593.0           |
| \( \hat{\bar{Y}}_{p6} \) | 622.6            | 12059.5         | 4620.7          | 626.0           | 624.2           | 644.0           | 623.4           | 625.3           | 624.5           | 626.2           | 626.6           | 624.2           |
| \( \hat{\bar{Y}}_{p7} \) | 695.8            | 13477.7         | 5164.1          | 699.7           | 697.6           | 719.8           | 696.7           | 698.8           | 698.0           | 699.8           | 700.3           | 697.6           |
| \( \hat{\bar{Y}}_{p8} \) | 795.0            | 15398.9         | 5900.2          | 799.4           | 797.0           | 822.4           | 796.0           | 798.4           | 797.5           | 799.6           | 800.2           | 797.0           |
| \( \hat{\bar{Y}}_{p9} \) | 603.8            | 11696.6         | 4481.7          | 607.2           | 605.4           | 624.6           | 604.6           | 606.4           | 605.7           | 607.3           | 607.8           | 605.4           |
| \( \hat{\bar{Y}}_{p10} \) | 791.2            | 15326.3         | 5872.4          | 795.6           | 793.2           | 818.5           | 792.2           | 794.6           | 793.7           | 795.8           | 796.4           | 793.3           |
| \( \hat{\bar{Y}}_{p11} \) | 813.4            | 15756.2         | 6037.1          | 817.9           | 815.5           | 841.4           | 814.5           | 816.9           | 816.0           | 818.1           | 818.7           | 815.5           |
| \( \hat{\bar{Y}}_{p12} \) | 808.5            | 15660.3         | 6000.4          | 813.0           | 810.5           | 836.3           | 809.5           | 811.9           | 811.0           | 813.1           | 813.7           | 810.5           |
| \( \hat{\bar{Y}}_{p13} \) | 619.8            | 12006.3         | 4600.3          | 623.3           | 621.4           | 641.2           | 620.6           | 622.5           | 621.8           | 623.4           | 623.9           | 621.4           |
| \( \hat{\bar{Y}}_{p14} \) | 740.5            | 14344.9         | 5496.4          | 744.7           | 742.4           | 766.1           | 741.5           | 743.7           | 742.9           | 744.8           | 745.4           | 742.5           |
| \( \hat{\bar{Y}}_{p15} \) | 781.1            | 15130.2         | 5797.3          | 785.4           | 783.1           | 808.0           | 782.1           | 784.5           | 783.6           | 785.6           | 786.2           | 783.1           |
| \( \hat{\bar{Y}}_{p16} \) | 808.7            | 15665.9         | 6002.5          | 813.3           | 810.8           | 836.6           | 809.8           | 812.2           | 811.3           | 813.4           | 814.0           | 810.8           |