NEW RESULTS RELATED TO THE CONVEXITY AND STARLIKENESS OF THE BERNArdI INTEGRAL OPERATOR

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Abstract


\[ L_{\gamma}(f)(z) = F(z) = \gamma + \frac{1}{\gamma} \int_{0}^{z} f(t)t^{\gamma-1}dt, \quad z \in U \]

preserves certain classes of univalent functions, such as the class of starlike functions, the class of convex functions and the class of close-to-convex functions.

In this paper we determine conditions that a function \( f \in A \) needs to satisfy in order that the function \( F \) given by (1) be convex. We also prove two duality theorems between the classes \( K\left(-\frac{1}{2\gamma}\right) \) and \( S^* \), and between \( K\left(-\frac{1}{2\gamma}\right) \) and \( S^*\left(-\frac{1}{2\gamma}\right) \), respectively.

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1. Introduction and preliminaries

Let $U$ be the unit disc of the complex plane:

$$U = \{ z \in \mathbb{C} : |z| < 1 \}.$$ 

Let $\mathcal{H}(U)$ be the space of holomorphic functions in $U$. Also, let

$$A_n = \{ f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U \}$$

with $A_1 = A$ and

$$S = \{ f \in A : f \text{ is univalent in } U \}.$$

Let

$$K = \left\{ f \in A, \ \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) + 1 > 0, \ z \in U \right\},$$

denote the class of normalized convex functions in $U$, 

$$S^* = \left\{ f \in A, \ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \ z \in U \right\}$$

denote the class of starlike functions in $U$, 

$$K(\alpha) = \left\{ f \in A : \ \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) + 1 > \alpha, \ z \in U \right\}$$

denote the class of normalized convex functions of order $\alpha$, where $\alpha < 1$, 

$$S^*(\alpha) = \left\{ f \in A : \ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in U \right\}$$

denote the class of starlike functions of order $\alpha$, with $\alpha < 1$ and 

$$C = \left\{ f \in A : \exists \varphi \in K, \ \text{Re} \left( \frac{f'(z)}{\varphi'(z)} \right) > 0, \ z \in U \right\}$$

denote the class of close-to-convex functions.

In order to prove our original results, we use the following lemmas:

**1.1. Lemma.** [3, 4, 6, Theorem 2.3.i, p. 35] Let $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ satisfy the condition

$$\text{Re} \left( \psi(is, t; z) \right) \leq 0, \ z \in U,$$

for $s, t \in \mathbb{R}$, $t \leq -\frac{1}{2} \left( 1 + s^2 \right)$.

If $p(z) = 1 + p_1z + p_2z^2 + \ldots$ satisfies

$$\text{Re} \left[ p(z), zp'(z); z \right] > 0$$

then

$$\text{Re} \ p(z) > 0, \ z \in U.$$

More general forms of this lemma can be found in [6].

**1.2. Lemma.** [7, Theorem 4.6.3, p. 84] The function $f \in A$, with $f'(z) \neq 0, \ z \in U$ is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] d\theta > -\pi,$$

for all $\theta_1, \theta_2$ with $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and all $r \in (0, 1)$. 
If \( L_\gamma : A \to A \) is the integral operator defined by \( L_\gamma [f] = F \), where \( F \) is given by
\[
L_\gamma [f](z) = F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t)t^{\gamma - 1}dt
\]
and \( \text{Re} \gamma \geq 0, z \in U \), then it is well known that
(i) \( L_\gamma [S^*] \subset S^* \),
(ii) \( L_\gamma [K] \subset K \), and
(iii) \( L_\gamma [C] \subset C \).
These results are obtained in [2] and [8].

2. Main results

We determine conditions such that, for a function \( f \in A \), the image under the Bernardi integral operator is convex or starlike.

2.1. Theorem. Let \( f \in A, \gamma \geq 1 \) and
\[
L_\gamma (f)(z) = F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t)t^{\gamma - 1}dt, \quad z \in U.
\]
If
\[
\text{Re} \left[ \frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2\gamma}, \quad z \in U,
\]
then the function \( F \) given by (1) is convex.

Proof. Let \( f \in A, f(z) = z + a_2z^2 + \cdots, z \in U \). Then, from (1), we have:
\[
F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z (t + a_2t^2 + \cdots)t^{\gamma - 1}dt
= \frac{\gamma + 1}{z^\gamma} \left( \frac{z^{\gamma+1}}{\gamma+1} + \frac{a_2z^{\gamma+2}}{\gamma+2} + \cdots \right)
= z + A_2z^2 + \cdots,
\]
hence \( F \in A \).

Since \( \gamma \geq 1, 0 \leq \frac{1}{2\gamma} \leq \frac{1}{2} \), we have \(-\frac{1}{2} \leq \frac{1}{2\gamma} < 0 \), and
\[
\text{Re} \left[ \frac{zf''(z)}{f'(z)} + 1 \right] \geq -\frac{1}{2\gamma} > -\frac{1}{2}.
\]
Then, according to Lemma 1.2 we obtain \( f \in C \), hence it is univalent. If \( f \in C \) then from (iii), we have \( L_\gamma [f] = F \in C \), hence \( F \) is univalent.

From (1), we have
\[
z^\gamma F(z) = (1 + \gamma) \int_0^z f(t)t^{\gamma - 1}dt, \quad z \in U.
\]
By differentiating (3), we obtain
\[
\gamma F(z) + zF'(z) = (\gamma + 1)f(z), \quad z \in U.
\]
By differentiating (4) and by a simple calculation, we obtain
\[
F'(z) \left[ 1 + \frac{zF''(z)}{F'(z)} \right] + \gamma F'(z) = (\gamma + 1)f'(z), \quad z \in U.
\]
Let
\[
1 + \frac{zF''(z)}{F'(z)} = p(z), \quad z \in U.
\]
Then (5) is equivalent to

\[ F'(z)[p(z) + \gamma] = (\gamma + 1)f'(z), \quad z \in U. \]

Since \( F'(z) \neq 0 \), \( p(z) + \gamma \neq 0 \), \( f \in C \), we have \( f'(z) \neq 0 \), \( z \in U \), and by differentiating (7), we obtain

\[ 1 + \frac{zF''(z)}{F'(z)} + \frac{zp'(z)}{p(z) + \gamma} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U. \]

Using (6), we have

\[ p(z) + \frac{zp'(z)}{p(z) + \gamma} = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in U. \]

Using (2), we obtain

\[ \text{Re} \left[ p(z) + \frac{zp'(z)}{p(z) + \gamma} \right] = \text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1 \]

which is equivalent to

\[ \text{Re} \left[ p(z) + \frac{zp'(z)}{p(z) + \gamma} + \frac{1}{2\gamma} \right] > 0, \quad z \in U, \quad \gamma \geq 1. \]

Let \( \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \),

\[ \psi(p(z), zp'(z); z) = p(z) + \frac{zp'(z)}{p(z) + \gamma} + \frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1. \]

Then (10) is equivalent to

\[ \text{Re} \psi(p(z), zp'(z); z) > 0, \quad z \in U. \]

In order to prove Theorem 2.1, we use Lemma 1.1. For that we calculate

\[ \text{Re} \psi(is, t; z) = \text{Re} \left[ is + \frac{1}{2\gamma} + \frac{t}{is + \gamma} \right] \]
\[ = \text{Re} \left[ is + \frac{1}{2\gamma} + \frac{(\gamma - is)}{\gamma^2 + s^2} \right] \]
\[ = \frac{1}{2\gamma} + \frac{t\gamma}{\gamma^2 + s^2} \]
\[ \leq \frac{1}{2\gamma} - \frac{\gamma(1 + s^2)}{2(\gamma^2 + s^2)} \]
\[ = \frac{\gamma^2 + s^2 - \gamma^2 - \gamma^2 s^2}{2(\gamma^2 + s^2)} \]
\[ = \frac{s^2(1 - \gamma^2)}{2(\gamma^2 + s^2)} \leq 0, \]

since \( \gamma \geq 1 \). Now, using Lemma 1.1 we get that \( \text{Re} p(z) > 0, \quad z \in U \), i.e.

\[ \text{Re} \frac{zF''(z)}{F'(z)} + 1 > 0, \quad z \in U, \quad \text{hence } F \in K. \]

**2.2. Remark.** Since \( \gamma \geq 1, \quad 0 < \frac{1}{2\gamma} \leq \frac{1}{2} \) and \( -\frac{1}{2} \leq \frac{1}{-2\gamma} < 0 \), we let

\[ K \left( -\frac{1}{2\gamma} \right) = \left\{ f \in A; \text{Re} \frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1 \right\}. \]
For the classes $K\left(-\frac{1}{2\gamma}\right)$ and $S^*$ the following duality theorem can be proved:

### 2.3. Theorem

The function $f \in A$ belongs to the class $K\left(-\frac{1}{2\gamma}\right)$ if and only if $F \in S^*$, where $F$ is given by

\[(13) \quad F(z) = z[f'(z)]^{\frac{2\gamma + 1}{2\gamma}}, \quad z \in U\]

and $[f'(z)]^{\frac{2\gamma + 1}{2\gamma}}$ is the holomorphic determination for $[f'(z)]^{\frac{2\gamma + 1}{2\gamma}}|_{z=0} = 1$.

**Proof.** If $f \in K\left(-\frac{1}{2\gamma}\right)$, according to Lemma 1.2 we obtain $f \in C$, hence $f'(z) \neq 0$, $z \in U$. Then function $F$ given by (13) is

\[F(z) = z[f'(z)]^{\frac{2\gamma + 1}{2\gamma}} = z(1 + 2a_2z + 3a_3z^2 + \cdots)^{\frac{2\gamma + 1}{2\gamma}},\]

holomorphic in $U$, with $F(0) = 0$, $F'(0) = 1$, $\frac{F(z)}{z} \neq 0$, $z \in U$.

Relation (13) is equivalent to

\[(14) \quad \left(\frac{F(z)}{z}\right)^{\frac{2\gamma + 1}{2\gamma}} = f'(z), \quad z \in U.\]

By differentiating (14), we obtain

\[\frac{2\gamma + 1}{2\gamma} \left[zF'(z) - 1\right] = zF''(z), \quad z \in U,\]

which is equivalent to

\[\frac{2\gamma + 1}{2\gamma} zF'(z) - 1 = zF''(z) + 1, \quad z \in U.\]

Since

\[\frac{2\gamma + 1}{2\gamma} \text{Re} \frac{zF'(z)}{F(z)} - 1 = \text{Re} \left(\frac{zF''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma}\]

we obtain

\[\text{Re} \frac{zF'(z)}{F(z)} > 0, \quad z \in U, \text{ i.e. } F \in S^*.\]

For the converse, if $F \in S^*$, let $f$ be the function:

\[(15) \quad f'(z) = \left[\frac{F(z)}{z}\right]^{\frac{2\gamma + 1}{2\gamma}}, \quad z \in U, \quad \gamma \geq 1.\]

Differentiating (15), we have

\[(16) \quad \frac{zf''(z)}{f'(z)} = 2\gamma + 1 \left[\frac{zF'(z)}{F(z)} - 1\right], \quad z \in U, \quad \gamma \geq 1,\]

which is equivalent to

\[\frac{zf''(z)}{f'(z)} + 1 = 2\gamma + 1 \frac{zF'(z)}{F(z)} - 1, \quad z \in U, \quad \gamma \geq 1.\]

Since

\[\text{Re} \left[\frac{zf''(z)}{f'(z)} + 1\right] = 2\gamma + 1 \text{Re} \frac{zF'(z)}{F(z)} - 1 > -\frac{1}{2\gamma}\]

we deduce

\[\text{Re} \left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma}, \text{ i.e. } f \in K\left(-\frac{1}{2\gamma}\right).\]
2.4. Theorem. The function \( f \in A \), belongs to the class \( K\left(-\frac{1}{2\gamma}\right) \) if and only if
\[
F \in S^*\left(-\frac{1}{2\gamma}\right), \quad \text{where } F \text{ is given by}
\]
(17) \[ F(z) = zf'(z), \quad z \in U. \]

Proof. If \( f \in K\left(-\frac{1}{2\gamma}\right) \), then according to Lemma 1.2 we obtain \( f \in C \). Hence \( f'(z) \neq 0, \quad z \in U \). Then \( F \) given by (17) is
\[
F(z) = z(1 + 2a_2z + \ldots) = z + A_2z^2 + \cdots \in A,
\]
holomorphic in \( U \), with \( F(0) = 0, \quad F'(0) = 1, \quad \frac{F(z)}{z} \neq 0, \quad z \in U \). Relation (17) is equivalent to
(18) \[ \frac{F(z)}{z} = f'(z), \quad z \in U. \]

By differentiating (18), we obtain
\[
\frac{zF''(z)}{F(z)} - 1 = \frac{zf''(z)}{f'(z)}, \quad z \in U,
\]
which is equivalent to
(19) \[ \frac{zF''(z)}{F(z)} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U. \]

Since
\[
\text{Re} \left( \frac{zF''(z)}{F(z)} - 1 \right) = \text{Re} \left( \frac{zf''(z)}{f'(z)} + 1 \right) > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1
\]
we deduce \( F \in S^*\left(-\frac{1}{2\gamma}\right) \).

Conversely, if \( F \in S^*\left(-\frac{1}{2\gamma}\right) \), we have
(20) \[ f'(z) = \frac{F(z)}{z}, \quad z \in U. \]

Differentiating (20), we obtain
\[
\frac{zf''(z)}{f'(z)} = \frac{zF''(z)}{F(z)} - 1, \quad z \in U,
\]
which is equivalent to
\[
\frac{zf''(z)}{f'(z)} + 1 = \frac{zF''(z)}{F(z)}, \quad z \in U.
\]

Since
\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} + 1 \right) = \text{Re} \left( \frac{zF''(z)}{F(z)} \right) > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1
\]
we have \( f \in K\left(-\frac{1}{2\gamma}\right) \). □
References


