Generalized statistical convergence and some sequence spaces in 2-normed spaces

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Abstract
In this work, we first define the concepts of $A$-statistical convergence and $A^I$-statistical convergence in a 2-normed space and present an example to show the importance of generalized form of convergence through an ideal. We then introduce some new sequence spaces in a 2-Banach space and examine some inclusion relations between these spaces.

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1. Introduction
The idea of statistical convergence was first introduced by Fast [6] and also independently by Buck [2] and Schoenberg [22] for real and complex sequences, but the rapid developments started after the papers of Šalát [18], Fridy [8] and Connor [3].

Let $K \subseteq \mathbb{N}$ and $K_n = \{ k \leq n : k \in K \}$. Then the natural density of $K$ is defined by $\delta(K) = \lim_n n^{-1} |K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of $K_n$.

The number sequence $x = (x_k)$ is said to be statistically convergent to the number $L$ provided that for every $\varepsilon > 0$ the set $K(\varepsilon) := \{ k \in \mathbb{N} : |x_k - L| \geq \varepsilon \}$ has natural density zero. In this case we write $st - \lim x = L$.

Let $X, Y$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix. If for each $x \in X$ the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ converges for all $n$ and the sequence $Ax = (A_n(x)) \in Y$, then we say that $A$ maps $X$ into $Y$. By $(X,Y)$ we denote the set of all matrices which maps $X$ into $Y$, and in addition if the limit is preserved then we denote the class of such matrices by $(X,Y)_{reg}$. A matrix $A$ is called regular if $A \in (c,c)_{reg}$, where $c$ denotes the space of all convergent sequences.

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The well-known Silverman-Toeplitz theorem asserts that $A$ is regular if and only if

$\left( R_1 \right) \quad \| A \| = \sup_n \sum_k |a_{nk}| < \infty$;

$\left( R_2 \right) \quad \lim_n a_{nk} = 0$, for each $k$;

$\left( R_3 \right) \quad \lim_n \sum_k |a_{nk}| = 1$.

Following Freedman and Sember [7], we say that a set $K \subseteq \mathbb{N}$ has $A$-density if

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{nk}$$

exists, where $A = (a_{nk})$ is nonnegative regular matrix.

The idea of statistical convergence was extended to $A$-statistical convergence by Connor [3] and also independently by Kolk [12]. A sequence $x$ is said to be $A$-statistically convergent to $L$ if $\delta_A(K(\varepsilon)) = 0$ for every $\varepsilon > 0$. In this case we write $\text{st}_A\lim x = L$.

Let $X \neq \emptyset$. A class $\mathcal{I} \subseteq 2^X$ of subsets of $X$ is said to be an ideal in $X$ provided: (i) $\emptyset \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (iii) $A \in \mathcal{I}$, $B \subseteq A$ implies $B \in \mathcal{I}$. $\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$, and a nontrivial ideal $\mathcal{I}$ in $X$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Let $\mathcal{I} \subseteq 2^X$ be a nontrivial ideal. Then the sequence $x = (x_k)$ of real numbers is said to be ideal convergent or $\mathcal{I}$-convergent to a number $L$ if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \subseteq \mathcal{I}$ (see [15]).

Note that if $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$, then usual convergence implies $\mathcal{I}$-convergence.

If we take $\mathcal{I} = \mathcal{I}_f$, the ideal of all finite subsets of $\mathbb{N}$, then $\mathcal{I}_f$-convergence coincides with usual convergence. We also note that the ideals $\mathcal{I}_0 = \{B \subseteq \mathbb{N} : \delta(E) = 0\}$ and $\mathcal{I}_{0A} = \{B \subseteq \mathbb{N} : \delta_A(B) = 0\}$ are admissible ideals in $\mathbb{N}$, also $\mathcal{I}_0$-convergence and $\mathcal{I}_{0A}$-convergence coincide with statistical convergence and $A$-statistical convergence respectively.

Savaş et al. (see [21]) have generalized $A$-statistical convergence by using ideals.

Let $A = (a_{nk})$ be a nonnegative regular matrix. A sequence $x = (x_k)$ is said to be $A^1$-statistically convergent (or $S_A(\mathcal{I})$-convergent) to $L$ if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I}.$$  

In this case we shall write $S_A(\mathcal{I})\lim x = L$.

Note that if we take $\mathcal{I} = \mathcal{I}_f$, then $A^1$-statistical convergence coincides with $A$-statistical convergence. Furthermore, the choice of $\mathcal{I} = \mathcal{I}_f$ and $A = C_1$, the Cesàro matrix of order one, gives us $\mathcal{I}$-statistical convergence introduced in [5] and [20].

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A $2$-norm on $X$ is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies (i) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent; (ii) $\|x, y\| = \|y, x\|$; (iii) $\|ax, y\| = |a| \|x, y\|$, $a \in \mathbb{R}$; (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|.,.\|)$ is then called a $2$-normed space [9]. As an example of a $2$-normed space we may take $X = \mathbb{R}^2$ being equipped with the $2$-norm $\|x, y\| := \text{area of parallelogram spanned by the vectors } x, y$, which may be given explicitly by the formula

$$(1.1) \quad \|x, y\| = |x_1y_2 - x_2y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Recall that $(X, \|.,.\|)$ is a $2$-Banach space if every Cauchy sequence in $X$ is convergent to some $x$ in $X$.

The concept of statistical convergence in $2$-normed spaces has been introduced and examined by Gürdal and Pehlivan [10]. Let $(x_n)$ be a sequence in $2$-normed space $(X, \|.,.\|)$. The sequence $(x_n)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{n} \left| \left\{ n : \| x_n - L, z \| \geq \varepsilon \right\} \right| = 0$$

for all $z \in X$. The sequence $(x_n)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{n} \left| \left\{ n : \| x_n - L, z \| \geq \varepsilon \right\} \right| = 0$$

for all $z \in X$. The sequence $(x_n)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$
for each nonzero $z$ in $X$. In this case we write $st - \lim_{n} \|x_n, z\| = \|L, z\|$. Finally, we recall that a modulus $f$ is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$; (ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$; (iv) $f$ is increasing and (iv) $f$ is continuous from the right at 0.

2. $A^2$-statistical convergence in 2-normed spaces

In this section we introduce the concepts of $A$-statistical convergence and $A^2$-statistical convergence in a 2-normed space when $A = (a_{nk})$ is a nonnegative regular matrix and $\mathcal{I}$ is an admissible ideal of $\mathbb{N}$.

2.1. Definition. Let $(x_k)$ be a sequence in 2-normed space $(X, \|\cdot\|)$. Then $(x_k)$ is said to be $A$-statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_{n} \sum_{k : \|x_k - L, z\| \geq \varepsilon} a_{nk} = 0$$

for each nonzero $z$ in $X$, in other words, $(x_k)$ is said to be $A$-statistically convergent to $L$ provided that $\delta_A (\{k \in \mathbb{N} : \|x_k - L, z\| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$ and each nonzero $z$ in $X$. In this case we write $A - \lim_{k} \|x_k, z\| = \|L, z\|$.

We remark that if we take $A = C_1$ in Definition 2.1, then $A$-statistical convergence coincides with the concept of statistical convergence introduced in [10].

Now we introduce the concept of $A^2$-statistical convergence in a 2-normed space.

2.2. Definition. A sequence $(x_k)$ in 2-normed space $(X, \|\cdot\|)$ is said to be $A^2$-statistically convergent to $L$ provided that for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{ n \in \mathbb{N} : \sum_{k : \|x_k - L, z\| \geq \varepsilon} a_{nk} \geq \delta \right\} \in \mathcal{I}$$

for each nonzero $z$ in $X$. In this case we write $S_A (\mathcal{I}) - \lim_{k} \|x_k, z\| = \|L, z\|$.

We shall denote the space of all $A$-statistically convergent and $A^2$-statistically convergent sequences in a 2-normed space $(X, \|\cdot\|)$ by $S_A (\|\cdot\|)$ and $S_A (\mathcal{I}, \|\cdot\|)$, respectively. It is clear that if $\mathcal{I} = \mathcal{I}_f$, then the space $S_A (\mathcal{I}, \|\cdot\|)$ is reduced to $S_A (\|\cdot\|)$.

Example. Let $X = \mathbb{R}^2$ be equipped with the 2-norm by the formula (1.1). Let $\mathcal{I} \subset 2^\mathbb{N}$ be an admissible ideal, $C = \{p_1 < p_2 < \ldots\} \in \mathcal{I}$ be an infinite set and define the matrix $A = (a_{nk})$ and the sequence $(x_k)$ by

$$a_{nk} = \begin{cases} 1 & \text{if } n = p_i, \ i \in \mathbb{N}, \ k = 2p_i \\ 1 & \text{if } n \neq p_i, \ k = 2n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$x_k = \begin{cases} (0, k) & \text{if } k \text{ is even} \\ (0, 0) & \text{otherwise} \end{cases}$$

respectively. Also let $L = (0, 0)$ and $z = (z_1, z_2)$. If $z_1 = 0$ then

$$\{k : \|x_k - L, z\| \geq \varepsilon\} = \emptyset$$

for each $z$ in $X$. Then $\delta_A (\{k \in \mathbb{N} : \|x_k - L, z\| \geq \varepsilon\}) = 0$. Hence we have $z_1 \neq 0$. For each $\varepsilon > 0$

$$\{k : \|x_k - L, z\| \geq \varepsilon\} \text{ if } k \text{ is even} \left\{ k : k \geq \frac{\varepsilon}{|z_1|} \right\}.$$
hence for each \( \delta > 0 \) we obtain
\[
\left\{ n \in \mathbb{N} : \sum_{k : \|x_k - L, z\| \geq \varepsilon} a_{nk} \geq \delta \right\} = \{ n \in \mathbb{N} : n = p_i \} = C \cap \mathbb{J}.
\]
This means that \( S_A (J) - \lim_k \|x_k, z\| = \|(0, 0), z\| \), but \( s_{tA} - \lim_k \|x_k, z\| \neq \|(0, 0), z\| \) since
\[
\lim_n \sum_{k : \|x_k - L, z\| \geq \varepsilon} a_{nk} = 1 \neq 0.
\]
This example also shows that \( A^3 \)-statistical convergence is more general than \( A \)-statistical convergence in a 2-normed space.

3. Some New Sequence Spaces

Following the study of Maddox [16], who introduced the notion of strongly Cesáro summability with respect to a modulus, several authors used modulus function to construct some new sequence spaces by using different methods of summability. For instance, summability with respect to a modulus, several authors used modulus function to construct some new sequence spaces by using different methods of summability. For instance, summability with respect to a modulus, several authors used modulus function to construct some new sequence spaces by using different methods of summability.

Let \( A = (a_{nk}) \) be a nonnegative regular matrix, \( J \) be an admissible ideal of \( \mathbb{N} \) and let \( p = (p_k) \) be a bounded sequence of positive real numbers. By \( s(2 - X) \) we denote the space of all sequences defined over \( (X, \|\cdot\|_2) \). Throughout the paper \( \mathcal{F} = (F_k) \) is assumed to be a sequence of modulus functions such that \( \lim_{t \to 0^+} \sup_k F_k (t) = 0 \) and further let \( (X, \|\cdot\|_2) \) be a \( 2 \)-Banach space. Now we define the following sequence space:
\[
w^p (A, \mathcal{F}, p, \|\cdot\|_2) = \left\{ x \in s(2 - X) : \{ n \in \mathbb{N} : \sum_k a_{nk} \left[ F_k \left( \|x_k - L, z\| \right) \right]^{p_k} \geq \delta \} \in \mathbb{J} \right\}
\]
for each \( \delta > 0 \) and \( z \in X \), for some \( L \in X \).

If \( x \in w^p (A, \mathcal{F}, p, \|\cdot\|_2) \) then \( x \) is said to be strongly \( (A, \mathcal{F}, p, \|\cdot\|_2) \)-summable to \( L \in X \).

Note that if \( 0 < p_k \leq \sup_k p_k = H, D := \max(1, 2^{H-1}) \), then
\[
|a_k + b_k|^{p_k} \leq D \left( |a_k|^{p_k} + |b_k|^{p_k} \right)
\]
for all \( k \) and \( a_k, b_k \in \mathbb{C} \).

3.1. Theorem. \( w^p (A, \mathcal{F}, p, \|\cdot\|_2) \) is a linear space.

Proof. Assume that the sequences \( x \) and \( y \) are strongly \( (A, \mathcal{F}, p, \|\cdot\|_2) \)-summable to \( L \) and \( L' \), respectively and let \( \alpha, \beta \in \mathbb{C} \). By using the definitions of modulus function and 2-norm and also from (3.1), we have
\[
\sum_{k=1}^{\infty} a_{nk} \left[ F_k \left( \| (\alpha x_k + \beta y_k) - (\alpha L + \beta L') , z \| \right) \right]^{p_k} \leq DM_{\alpha}^H \sum_{k=1}^{\infty} a_{nk} \left[ F_k \left( \| x_k - L, z \| \right) \right]^{p_k}
\]
\[
+ DM_{\beta}^H \sum_{k=1}^{\infty} a_{nk} \left[ F_k \left( \| y_k - L, z \| \right) \right]^{p_k}
\]
where \( M_\alpha \) and \( M_\beta \) are positive numbers such that \( |\alpha| \leq M_\alpha \) and \( |\beta| \leq M_\beta \). From the last inequality, we conclude that \( \alpha x + \beta y \in w^p (A, \mathcal{F}, p, \|\cdot\|_2) \).
If we take $f_k(t) = t$ for all $k$ and $t$, then the space $w^j(A,F,p,\|\cdot\|)$ is reduced to
$$w^j(A,p,\|\cdot\|) = \{x \in s(2 - X) : \{n \in N : \sum_k a_{nk} (\|x_k - L, z\|)^p_k \geq \delta \} \in \mathcal{I} \}
$$
for each $\delta > 0$ and $z \in X$, for some $L \in X$.

If $x \in w^j(A,p,\|\cdot\|)$ then we say that $x$ is strongly $(A,\|\cdot\|)$-summable to $L \in X$.

3.2. Lemma. Let $f$ be any modulus function and $0 < \delta < 1$. Then for each $t \geq \delta$ we have $f(t) \leq 2f(1)\delta^{-1}t$ [16].

3.3. Theorem. If $x$ is strongly $(A,\|\cdot\|)$-summable to $L$ then $x$ is strongly $(A,F,p,\|\cdot\|)$-summable to $L$, i.e. the inclusion
$$w^j(A,p,\|\cdot\|) \subset w^j(A,F,p,\|\cdot\|)
$$
holds.

Proof. Let $x = (x_k) \in w^j(A,p,\|\cdot\|)$. Since a modulus function is continuous at $t = 0$ from the right and $\lim_{t \to 0^+} \sup_k f_k(t) = 0$, then for any $\varepsilon > 0$ we can choose $0 < \delta < 1$ such that for every $t$ with $0 \leq t \leq \delta$, we have $f_k(t) < \varepsilon$ ($k \in N$). Then, from Lemma 3.2, we have
$$\sum_{k=1}^{\infty} a_{nk} [f_k(\|x_k - L, z\|)]^{p_k} = \sum_{k: \|x_k - L, z\| \leq \delta} a_{nk} [f_k(\|x_k - L, z\|)]^{p_k} + \sum_{k: \|x_k - L, z\| > \delta} a_{nk} [f_k(\|x_k - L, z\|)]^{p_k} \leq \max (\varepsilon^{\inf p_k}, \varepsilon^{\sup p_k}) \sum_{k=1}^{\infty} a_{nk} + \max (M_1, M_2) \sum_{k=1}^{\infty} a_{nk} (\|x_k - L, z\|)^{p_k}
$$
where $M_1 = (2 \sup f_k(1)\delta^{-1})^{\inf p_k}$ and $M_2 = (2 \sup f_k(1)\delta^{-1})^{\sup p_k}$. Let $M := \max (M_1, M_2)$ and $N := \max (\varepsilon^{\inf p_k}, \varepsilon^{\sup p_k})$. Now by considering the inequality $\sum_k a_{nk} \leq \|A\|$ for each $n \in N$, choose a $\sigma > 0$ such that $\sigma - N \|A\| > 0$. Then we obtain
$$\begin{align*}
\{n \in N : \sum_k a_{nk} [f_k(\|x_k - L, z\|)]^{p_k} \geq \sigma \} \\
\subset \{n \in N : \sum_k a_{nk} [f_k(\|x_k - L, z\|)]^{p_k} \geq \frac{\sigma - N \|A\|}{M} \}
\end{align*}
$$
From the assumption we conclude that $x \in w^j(A,F,p,\|\cdot\|)$.

3.4. Theorem. Let $F = (f_k)$ be the sequence of modulus functions such that $\lim_{t \to \infty} \inf_k \frac{f_k(t)}{t} > 0$. Then $w^j(A,F,p,\|\cdot\|) \subset w^j(A,p,\|\cdot\|)$.

Proof. Let $x \in w^j(A,F,p,\|\cdot\|)$. If $\lim_{t \to \infty} \inf_k \frac{f_k(t)}{t} > 0$ then there exists a $c > 0$ such that $f_k(t) > ct$ for every $t > 0$ and for all $k \in N$. Thus, for each $\delta > 0$ we have
$$\begin{align*}
\{n \in N : \sum_{k=1}^{\infty} a_{nk} [f_k(\|x_k - L, z\|)]^{p_k} \geq \delta \} \\
\supset \{n \in N : \min (\varepsilon^{\inf p_k}, \varepsilon^{\sup p_k}) \sum_{k=1}^{\infty} a_{nk} (\|x_k - L, z\|)^{p_k} \geq \delta \}.
\end{align*}$$
conclude that for all $\delta > 3.6$.

**Theorem.** Let $x \in \j (A,p,\|\cdot\|)$ and $w^3 (A,\j,\|\cdot\|)$.

**Proof.** Let $x \in \j (A,\j,\|\cdot\|)$ and $\epsilon > 0$. Then for every $\epsilon > 0$ we have

$$
\sum_{k=1}^{\infty} a_{nk} [f_k (\|x_k - L,z\|)]^p \geq \min \left( c^{\inf p_k}, c^{\sup p_k} \right) \sum_{k \in K(\epsilon)} a_{nk}.
$$

Let $C := \min (c^{\inf p_k}, c^{\sup p_k})$. Thus we have

$$
\left\{ n \in \mathbb{N} : \sum_{k \in K(\epsilon)} a_{nk} \geq \delta \right\} \subset \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} [f_k (\|x_k - L,z\|)]^p \geq \frac{\delta}{C} \right\}
$$

for all $\delta > 0$. Since the set on the right-hand of the above inclusion belongs to $l$, we conclude that $x \in \j (A,\|\cdot\|)$. This completes the proof.

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