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ON FUNCTION SPACES WITH WAVELET TRANSFORM IN $\mathbf{L}^p_{\boldsymbol{\omega}}\left(\mathbb{R}^d \times \mathbb{R}_+\right)$

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Abstract

Let ω_1 and ω_2 be weight functions on $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{R}_+$, respectively. Throughout this paper, we define $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ to be the vector space of $f \in L^p_{\omega_1}\left(\mathbb{R}^d\right)$ such that the wavelet transform $W_g f$ belongs to $L^q_{\omega_2}\left(\mathbb{R}^d \times \mathbb{R}_+\right)$ for $1 \leq p, q < \infty$, where $0 \neq g \in S\left(\mathbb{R}^d\right)$. We endow this space with a sum norm and show that $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ becomes a Banach space. We discuss inclusion properties, and compact embeddings between these spaces and the dual of $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$. Later we accept that the variable s in the space $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is fixed. We denote this space by $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$, and show that under suitable conditions $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is an essential Banach Module over $L^1_{\omega_1}\left(\mathbb{R}^d\right)$. We obtain its approximate identities. At the end of this work we discuss the multipliers from $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ into $L^\infty_{\omega_1^{-1}}\left(\mathbb{R}^d\right)$, and from $L^1_{\omega_1}\left(\mathbb{R}^d\right)$ into $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$.

Keywords: Wavelet transform, Essential Banach module, Approximate identity, Compact embedding, Multipliers space.

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1. Introduction

In this paper we work on $\mathbb{R}^d$ with Lebesgue measure dx. $C_c\left(\mathbb{R}^d\right)$ and $S\left(\mathbb{R}^d\right)$ denote the space of complex-valued continuous functions on $\mathbb{R}^d$ with compact support and the space of complex-valued continuous functions on $\mathbb{R}^d$ rapidly decreasing at infinity, respectively. Also $L^p\left(\mathbb{R}^d\right), (1 \leq p < \infty)$ denotes the usual Lebesgue space. For any function $f : \mathbb{R}^d \to \mathbb{C}$, the translation, modulation and dilation operators T_x, M_ω and D_s are given by $T_x f(t) = f(t-x)$, $M_\omega f(t) = e^{2\pi i\omega t} f(t)$ and $D_s f(t) = |s|^{-\frac{d}{2}} f\left(\frac{t}{s}\right)$ for all $x, \omega \in \mathbb{R}^d$, $0 \neq s \in \mathbb{R}$, respectively. The parameters in wavelet theory are "time" x and "scale" s. The dilation operator D_s preserves the shape of f, but it changes the scale. In this paper we also use weight functions, which are positive real valued, measurable and locally bounded functions ω on $\mathbb{R}^d$ which satisfy $\omega(x) \geq 1$, $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}^d$. Let $a \geq 0$. A weight $\omega(x,s) = (1+|x|+|s|)^a$ which is defined on $\mathbb{R}^d \times \mathbb{R}_+$ is called a weight of polynomial type. We have the inequality $\omega(x+z,s) \leq \omega(x,s)\,\omega(z,t)$ for $x, z \in \mathbb{R}^d$ and $s, t \in \mathbb{R}_+$. Indeed

$$\begin{aligned}\omega(x+z,s) &= (1+|x+z|+|s|)^a \leq (1+|x+z|+|s+t|)^a \\ &\leq (1+|x|+|s|)^a (1+|z|+|t|)^a = \omega(x,s)\,\omega(z,t).\end{aligned}$$

We set

$$L^p_\omega\left(\mathbb{R}^d\right) = \left\{f : \; f\omega \in L^p\left(\mathbb{R}^d\right)\right\}$$

for $1 \leq p < \infty$. It is known that $L^p_\omega\left(\mathbb{R}^d\right)$ is a Banach space under the norm $\|f\|_{p,\omega} = \|f\omega\|_p$. Particularly $L^1_\omega\left(\mathbb{R}^d\right)$ is called a Beurling algebra, because it is a Banach convolution algebra. Let ω_1 and ω_2 are two weight functions. We write $\omega_1 \prec \omega_2$ if there exists $C > 0$ such that $\omega_1(x) \leq C\omega_2(x)$ for all $x \in \mathbb{R}^d$. Two weight function ω_1 and ω_2 are called equivalent, written $\omega_1 \approx \omega_2$, if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$.

Let $\langle x,t\rangle = \sum_{i=1}^{d} x_i t_i$ be the usual scalar product on $\mathbb{R}^d$. For $f \in L^1\left(\mathbb{R}^d\right)$, the Fourier transform $\hat{f}$ is given by

$$\hat{f}(t) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i\langle x,t\rangle}\, dx.$$

Given any fixed $0 \neq g \in L^2\left(\mathbb{R}^d\right)$ (called a wavelet function), the wavelet transform of a function $f \in L^2\left(\mathbb{R}^d\right)$ with respect to g is defined by

$$W_g f(x,s) = |s|^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(t)\, \overline{g\left(\frac{t-x}{s}\right)}\, dt = \langle f, T_x D_s g\rangle$$

for $x \in \mathbb{R}^d$ and $0 \neq s \in \mathbb{R}$. We can write the wavelet transform as the convolution $W_g f(x,s) = f * D_s g^*(x)$, where $g^*(t) = \overline{g(-t)}$. Also, the wavelet transform of a function $f \in L^p\left(\mathbb{R}^d\right)$ with respect to $0 \neq g \in L^1\left(\mathbb{R}^d\right)$ is defined similarly. It is easy to see that $W_g(T_z f) = T_{(z,0)} W_g f$.

For $g_1, g_2 \in L^2\left(\mathbb{R}^d\right)$, $d \geq 1$, assume that for almost all $\omega \in \mathbb{R}^d$ with $|\omega| = 1$,

$$\int_0^\infty \left|\hat{g}_1(s\omega)\,\hat{g}_2(s\omega)\right| \frac{ds}{s} < \infty, \tag{1}$$

and

$$\int_0^\infty \overline{\hat{g}_1(s\omega)}\,\hat{g}_2(s\omega)\, \frac{ds}{s} = K. \tag{2}$$

Then for all $f_1, f_2 \in L^2(\mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} W_{g_1} f_1(x,s) \overline{W_{g_2} f_2(x,s)} \frac{dx\, ds}{s^{d+1}} = K \langle f_1, f_2 \rangle .$$

The conditions (1) and (2) are called the wavelet admissibility conditions.

Let $f \in L^2(\mathbb{R}^d)$. If $g_1, g_2 \in L^2(\mathbb{R}^d)$ satisfy the admissibility conditions, then f is reconstructed from its wavelet transform by

$$f = \frac{1}{K} \int_{\mathbb{R}^d} \int_0^\infty W_{g_1} f(x,s)\, T_x D_s g_2 \frac{dx\, ds}{s^{d+1}}.$$

For two Banach modules B_1 and B_2 over a Banach algebra A, we write $M_A(B_1, B_2)$ or $\mathrm{Hom}_A(B_1, B_2)$ for the space of all bounded linear operators from B_1 into B_2 satisfying $T(ab) = aT(b)$ for all $a \in A,\ b \in B_1$. These operators are called (right) multipliers. It is known that

$$Hom_A(B_1, B_2^*) \cong (B_1 \otimes_A B_2)^* ,$$

where B_2^* is the dual of B_2 and $B_1 \otimes_A B_2$ is the A-module tensor product of B_1 and B_2 [18].

2. The space $\mathbf{D}^{p,q}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\mathbb{R}^d)$

2.1. Definition. Let $0 \neq g \in S(\mathbb{R}^d)$, and let ω_1, ω_2 be weight functions on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}_+$, respectively. For $1 \leq p, q < \infty$, we set

$$D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) = \left\{ f \in L^p_{\omega_1}\left(\mathbb{R}^d\right) \mid W_g f \in L^q_{\omega_2}\left(\mathbb{R}^d \times \mathbb{R}_+\right) \right\}.$$

It is easy to see that $\|f\|_{D^{p,q}_{\omega_1,\omega_2}} = \|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2}$ is a norm on the vector space $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

2.2. Theorem. $\left(D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d), \|\cdot\|_{D^{p,q}_{\omega_1,\omega_2}}\right)$ *is a Banach space.*

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Clearly $(f_n)_{n\in\mathbb{N}}$ and $(W_g f_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^p_{\omega_1}(\mathbb{R}^d)$ and $L^q_{\omega_2}(\mathbb{R}^d \times \mathbb{R}_+)$, respectively. Since $L^p_{\omega_1}(\mathbb{R}^d)$ and $L^q_{\omega_2}(\mathbb{R}^d \times \mathbb{R}_+)$ are Banach spaces, there exist $f \in L^p_{\omega_1}(\mathbb{R}^d)$ and $h \in L^q_{\omega_2}(\mathbb{R}^d \times \mathbb{R}_+)$ such that $\|f_n - f\|_{p,\omega_1} \to 0$, $\|W_g f_n - h\|_{q,\omega_2} \to 0$. This implies $\|W_g f_n - h\|_q \to 0$.Then $(W_g f_n)_{n\in\mathbb{N}}$ has a subsequence $(W_g f_{n_k})_{n_k\in\mathbb{N}}$ which converges pointwise to h almost everywhere. It is easy to show that $\|f_{n_k} - f\|_p \to 0$. Also by Hölder's inequality, we have

$$\begin{aligned} |W_g f(x,s) - h(x,s)| &= |W_g f(x,s) - h(x,s) + W_g f_{n_k}(x,s) - W_g f_{n_k}(x,s)| \\ &\leq |\langle f_{n_k}, T_x D_s g\rangle - \langle f, T_x D_s g\rangle| + |W_g f_{n_k}(x,s) - h(x,s)| \\ &\leq \int_{\mathbb{R}^d} |(f_{n_k} - f)(t)|\, |T_x D_s g(t)|\, dt + |W_g f_{n_k}(x,s) - h(x,s)| \\ &\leq s^{\frac{d}{r} - \frac{d}{2}} \|f_{n_k} - f\|_p \|g\|_r + |W_g f_{n_k}(x,s) - h(x,s)| . \end{aligned}$$

By using this inequality it is easily seen that $W_g f = h$ almost everywhere. Since the equivalence classes of $W_g f$ and h are equal then $\|f_n - f\|_{D^{p,q}_{\omega_1,\omega_2}} \to 0$ and $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. That means $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is a Banach space. □

2.3. Lemma. *We have the inclusion*

$$C_c\left(\mathbb{R}^d\times\mathbb{R}_+, dxds\right)\subset L^2\left(\mathbb{R}^d\times\mathbb{R}_+, \frac{dxds}{s^{d+1}}\right),$$

where $\frac{dxds}{s^{d+1}}$ *is the weighted Lebesgue measure on* $\mathbb{R}^d\times\mathbb{R}_+$.

Proof. Take any $h\in C_c\left(\mathbb{R}^d\times\mathbb{R}_+, dxds\right)$. Let $\operatorname{supp}h=K$ and $f(x,s)=\frac{|h(x,s)|}{s^{d+1}}$. Since $s>0$ and f is continuous, then $\operatorname{supp}f=K$. If we set $\max f(x,s)=m$, then

$$\begin{aligned}\|h\|_{L^2\left(\mathbb{R}^d\times\mathbb{R}_+,\frac{dxds}{s^{d+1}}\right)} &= \iint\limits_{\mathbb{R}^d\times\mathbb{R}_+}\frac{|h(x,s)|^2}{s^{d+1}}\,dx\,ds\\ &\le m\iint\limits_K dx\,ds=m\mu(K)\end{aligned}$$

is finite. Hence we obtain $h\in L^2\left(\mathbb{R}^d\times\mathbb{R}_+,\ \frac{dxds}{s^{d+1}}\right)$. □

The following example shows that $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)\neq\emptyset$.

2.4. Example. Let ω_2 be any weight function on $\mathbb{R}\times\mathbb{R}_+$. Take the weight function $\omega_1(t)=1+|t|$ on $\mathbb{R}$. Assume that $g\in S(\mathbb{R})$ satisfies the admissibility conditions. Now, we consider the space $D^{2,q}_{\omega_1,\omega_2}(\mathbb{R})$ for $1\le q<\infty$. Take any $F\in C_c(\mathbb{R}\times\mathbb{R}_+,\ dxds)\subset L^2\left(\mathbb{R}\times\mathbb{R}_+,\ \frac{dx\,ds}{s^2}\right)$. Then

$$\frac{1}{K}\iint\limits_{\mathbb{R}\times\mathbb{R}_+}F(x,s)\,T_xD_sg(t)\,\frac{dx\,ds}{s^2}=f(t).$$

Thus we have

$$\begin{aligned}\|f\|_{2,\omega_1} &= \left\|\frac{1}{K}\iint\limits_{\mathbb{R}\times\mathbb{R}_+}F(x,s)\,T_xD_sg(t)\,\frac{dx\,ds}{s^2}\right\|_{2,\omega_1}\\ &\le\frac{1}{K}\iint\limits_{\mathbb{R}\times\mathbb{R}_+}\frac{|F(x,s)|}{s^2}\,\|T_xD_sg\|_{2,\omega_1}\,dx\,ds\\ &\le\frac{1}{K}\iint\limits_{\mathbb{R}\times\mathbb{R}_+}\frac{|F(x,s)|}{s^2}\omega_1(x)\,\|D_sg\|_{2,\omega_1}\,dx\,ds\\ &=\frac{1}{K}\iint\limits_{\mathbb{R}\times\mathbb{R}_+}\frac{|F(x,s)|\,(1+|x|)}{s^2}\,\|D_sg\|_{2,\omega_1}\,dx\,ds.\end{aligned}\tag{3}$$

Also

$$\|D_sg\|^2_{2,\omega_1}\le\left(\frac{1}{\sqrt{s}}\right)^2 s\int\limits_{\mathbb{R}}|g(u)|^2\,\omega_1(u)^2\,\omega_1(s)^2\,du=\omega_1(s)^2\,\|g\|^2_{2,\omega_1}.$$

Hence

$$\|D_sg\|_{2,\omega_1}\le\omega_1(s)\,\|g\|_{2,\omega_1}=(1+s)\,\|g\|_{2,\omega_1}.\tag{4}$$

Combining (3) and (4), we obtain

$$\|f\|_{2,\omega_1}\le\frac{1}{K}\,\|g\|_{2,\omega_1}\iint\limits_{\mathbb{R}\times\mathbb{R}_+}\frac{|F(x,s)|\,(1+|x|)}{s^2}\,(1+s)\,dx\,ds.\tag{5}$$

Since F is continuous and $s \neq 0$, $\frac{|F(x,s)|(1+|x|)(1+s)}{s^2}$ is continuous. If we set supp$F = A$, then also supp $\left(\frac{|F(x,s)|(1+|x|)(1+s)}{s^2}\right) = A$. Moreover if we set $\max\limits_{(x,s)\in A}\left(\frac{|F(x,s)|(1+|x|)(1+s)}{s^2}\right) = N$, by (5) we have

$$\|f\|_{2,\omega_1} \leq \frac{N}{K}\|g\|_{2,\omega_1}\mu(A) < \infty,$$

where $\mu(A)$ is the area of the set A. Then we obtain $f \in L^2_{\omega_1}(\mathbb{R}) \subset L^2(\mathbb{R})$. Hence by Theorem 10.2 in [9], we have $W_g f \in L^2\left(\mathbb{R}\times\mathbb{R}_+, \frac{dxds}{s^2}\right)$. Since the wavelet transform is one-to-one, this implies $W_g f = F$. It is also known that $C_c(\mathbb{R}\times\mathbb{R}_+) \subset L^q_{\omega_2}(\mathbb{R}\times\mathbb{R}_+)$. Thus we have $W_g f \in L^q_{\omega_2}(\mathbb{R}\times\mathbb{R}_+)$. That means $f \in D^{2,q}_{\omega_1,\omega_2}(\mathbb{R})$.

2.5. Theorem. *Let ω_1 be a weight function and ω_2 a weight function of polynomial type. Then*

(1) *$D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is invariant under translations.*

(2) *The mapping $f \mapsto T_z f$ is continuous from $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ into $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ for every fixed $z \in \mathbb{R}^d$.*

Proof. 1) Let $f \in D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$. Then we have $f \in L^p_{\omega_1}\left(\mathbb{R}^d\right)$ and $W_g f \in L^q_{\omega_2}\left(\mathbb{R}^d\times\mathbb{R}_+\right)$. Since $\|T_z f\|_{p,\omega_1} \leq \omega_1(z)\|f\|_{p,\omega_1}$, we see that $T_z f \in L^p_{\omega_1}\left(\mathbb{R}^d\right)$ for all $z \in \mathbb{R}^d$ [7]. Also, since ω_2 is a weight function of polynomial type then we write $\omega_2(x+z,s) \leq \omega_2(x,s)\,\omega_2(z,t)$ for every *fixed* $t \in \mathbb{R}_+$. By using the equality $W_g(T_z f) = T_{(z,0)}W_g f$, we have

$$\|W_g(T_z f)\|_{q,\omega_2} \leq \omega_2(z,t)\|W_g f\|_{q,\omega_2}$$

for all fixed $z \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$. Thus, we obtain

$$\|T_z f\|_{D^{p,q}_{\omega_1,\omega_2}} \leq \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z,t)\|W_g f\|_{q,\omega_2}.$$

Hence $T_z f \in D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$. This means that $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is invariant under translations.

2) Let $f \in D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$. Since $f \mapsto T_z f$ is linear, it is enough to prove the theorem for $f = 0$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ to be $\delta = \frac{\varepsilon}{\omega_1(z)+\omega_2(z,t)}$. Thus, if $\|f\|_{D^{p,q}_{\omega_1,\omega_2}} < \delta$, then $\|f\|_{p,\omega_1} \leq \|f\|_{D^{p,q}_{\omega_1,\omega_2}} < \delta$ and $\|f\|_{q,\omega_2} \leq \|f\|_{D^{p,q}_{\omega_1,\omega_2}} < \delta$. Also, similarly to the proof of $\|W_g(T_z f)\|_{q,\omega_2} \leq \omega_2(z,t)\|W_g f\|_{q,\omega_2}$ in 1), we obtain

$$\begin{aligned}\|T_z f\|_{D^{p,q}_{\omega_1,\omega_2}} &= \|T_z f\|_{p,\omega_1} + \|W_g(T_z f)\|_{q,\omega_2}\\ &< \delta\{\omega_1(z)+\omega_2(z,t)\} = \varepsilon.\end{aligned}$$

□

3. Inclusion properties of the space $\mathbf{D}^{p,q}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}\left(\mathbb{R}^d\right)$

3.1. Proposition. *For every $0 \neq f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$ there exists $C(f) > 0$ such that*

$$C(f)\,\omega_1(z) \leq \|T_z f\|_{D^{p,q}_{\omega_1,1}} \leq \omega_1(z)\|f\|_{D^{p,q}_{\omega_1,1}}.$$

Proof. Let $0 \neq f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$. By [7, Proposition 1.7], there exists $C(f) > 0$ such that

$$C(f)\,\omega_1(z) \leq \|T_z f\|_{p,\omega_1} \leq \omega_1(z)\|f\|_{p,\omega_1}.$$

By using $W_g(T_z f) = T_{(z,0)}W_g f$, we write

$$\begin{aligned}C(f)\,\omega_1(z) &\leq \|T_z f\|_{p,\omega_1} + \|W_g(T_z f)\|_q \leq \omega_1(z)\|f\|_{p,\omega_1} + \|W_g f\|_q\\ &\leq \omega_1(z)\|f\|_{p,\omega_1} + \omega_1(z)\|W_g f\|_q\\ &= \omega_1(z)\left\{\|f\|_{p,\omega_1} + \|W_g f\|_q\right\} = \omega_1(z)\|f\|_{D^{p,q}_{\omega_1,1}}\end{aligned}$$

for all $f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$. Hence, we obtain

$$C(f)\,\omega_1(z) \le \|T_z f\|_{D^{p,q}_{\omega_1,1}} \le \omega_1(z)\,\|f\|_{D^{p,q}_{\omega_1,1}}. \qquad \square$$

3.2. Lemma. *Let $\omega_1,\ \omega_2,\ \omega_3$ and ω_4 be weight functions. If $D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right) \subset D^{p,q}_{\omega_2,\omega_4}\left(\mathbb{R}^d\right)$, then $D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right)$ is a Banach space under the norm $\|f\|_D = \|f\|_{D^{p,q}_{\omega_1,\omega_3}} + \|f\|_{D^{p,q}_{\omega_2,\omega_4}}$.*

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right), \|\cdot\|_D\right)$. Then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right), \|\cdot\|_{D^{p,q}_{\omega_1,\omega_3}}\right)$ and $\left(D^{p,q}_{\omega_2,\omega_4}\left(\mathbb{R}^d\right), \|\cdot\|_{D^{p,q}_{\omega_2,\omega_4}}\right)$. Since these spaces are Banach spaces, there exist $f \in D^{p,q}_{\omega_2,\omega_4}\left(\mathbb{R}^d\right)$ and $h \in D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right)$ such that $\|f_n - f\|_{D^{p,q}_{\omega_2,\omega_4}} \to 0$, $\|f_n - h\|_{D^{p,q}_{\omega_1,\omega_3}} \to 0$. Using the inequalities $\|\cdot\|_p \le \|\cdot\|_{D^{p,q}_{\omega_2,\omega_4}}$ and $\|\cdot\|_p \le \|\cdot\|_{D^{p,q}_{\omega_1,\omega_3}}$, we obtain $\|f_n - f\|_p \to 0$, and $\|f_n - h\|_p \to 0$. Also by using the inequality $\|f-h\|_p \le \|f_n - f\|_p + \|f_n - h\|_p$, we see that $\|f-h\|_p = 0$, and then $f = h$. Thus $\|f_n - f\|_D \to 0$ and $f \in \left(D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right), \|\cdot\|_D\right)$. That means $\left(D^{p,q}_{\omega_1,\omega_3}\left(\mathbb{R}^d\right), \|\cdot\|_D\right)$ is a Banach space. $\square$

It is easy to prove the following Lemma 3.3.

3.3. Lemma. *Let k be a constant number and $1 \le p < \infty$. If $\omega \approx k$, then*

$$L^p_\omega\left(\mathbb{R}^d \times \mathbb{R}_+\right) = L^p\left(\mathbb{R}^d \times \mathbb{R}_+\right). \qquad \square$$

3.4. Theorem. *Suppose that ω_1 and ω_2 are weight functions. Then $D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right) \subset D^{p,q}_{\omega_2,1}\left(\mathbb{R}^d\right)$ if and only if $\omega_2 \prec \omega_1$.*

Proof. Let $\omega_2 \prec \omega_1$. Then there exists $C > 0$ such that $\omega_2(z) \le C\omega_1(z)$ for all $z \in \mathbb{R}^d$. We can choose $C > 1$. Take any $f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$. Thus we write $\|f\|_{p,\omega_2} \le C\,\|f\|_{p,\omega_1}$. Furthermore, since $\|W_g f\|_q < \infty$, we have

$$\begin{aligned}\|f\|_{D^{p,q}_{\omega_2,1}} &= \|f\|_{p,\omega_2} + \|W_g f\|_q \\ &\le C\,\|f\|_{p,\omega_1} + C\,\|W_g f\|_q = C\,\|f\|_{D^{p,q}_{\omega_1,1}} < \infty.\end{aligned}$$

Therefore, $D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right) \subset D^{p,q}_{\omega_2,1}\left(\mathbb{R}^d\right)$.

Conversely, suppose that $D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right) \subset D^{p,q}_{\omega_2,1}\left(\mathbb{R}^d\right)$. For every $f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$, we have $f \in D^{p,q}_{\omega_2,1}\left(\mathbb{R}^d\right)$. By Proposition 3.1, there are constants $C_1,\ C_2,\ C_3,\ C_4 > 0$ such that

$$C_1\omega_1(z) \le \|T_z f\|_{D^{p,q}_{\omega_1,1}} \le C_2\omega_1(z) \tag{6}$$

and

$$C_3\omega_2(z) \le \|T_z f\|_{D^{p,q}_{\omega_2,1}} \le C_4\omega_2(z) \tag{7}$$

for all $z \in \mathbb{R}^d$. Also, from Lemma 3.2 the space $D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$ is a Banach space under the norm $\|f\|_D = \|f\|_{D^{p,q}_{\omega_1,1}} + \|f\|_{D^{p,q}_{\omega_2,1}}$. Then by the closed graph theorem, there exists $C > 0$ such that

$$\|f\|_{D^{p,q}_{\omega_2,1}} \le C\,\|f\|_{D^{p,q}_{\omega_1,1}} \tag{8}$$

for all $f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$. Furthermore, by Proposition 3.1 $T_z f \in D^{p,q}_{\omega_1,1}\left(\mathbb{R}^d\right)$, and by (8) we write

$$\|T_z f\|_{D^{p,q}_{\omega_2,1}} \le C\,\|T_z f\|_{D^{p,q}_{\omega_1,1}}. \tag{9}$$

Hence, combining (6), (7) and (9), we obtain

$$C_3\omega_2(z) \le \|T_z f\|_{D^{p,q}_{\omega_2,1}} \le C\|T_z f\|_{D^{p,q}_{\omega_1,1}} \le CC_2\omega_1(z).$$

Thus, $\omega_2(z) \le \frac{CC_2}{C_3}\omega_1(z)$. If we take $k = \frac{CC_2}{C_3}$, then we find $\omega_2(z) \le k\omega_1(z)$. □

3.5. Proposition. *Let $\omega_1,\ \omega_2$ be weight functions and $\omega_3 \approx k_1,\ \omega_4 \approx k_2$, where $k_1,\ k_2$ are constant numbers. Then $D^{p,q}_{\omega_1,\omega_3}(\mathbb{R}^d) \subset D^{p,q}_{\omega_2,\omega_4}(\mathbb{R}^d)$ if and only if $\omega_2 \prec \omega_1$.*

Proof. Since $\omega_3 \approx k_1$ and $\omega_4 \approx k_2$, by Lemma 3.3 we can write $L^p_{\omega_3}(\mathbb{R}^d \times \mathbb{R}_+) = L^p(\mathbb{R}^d \times \mathbb{R}_+)$ and $L^p_{\omega_4}(\mathbb{R}^d \times \mathbb{R}_+) = L^p(\mathbb{R}^d \times \mathbb{R}_+)$. By using Theorem 3.4, we obtain $D^{p,q}_{\omega_1,\omega_3}(\mathbb{R}^d) \subset D^{p,q}_{\omega_2,\omega_4}(\mathbb{R}^d)$ if and only if $\omega_2 \prec \omega_1$. □

3.6. Corollary. *Let $\omega_3 \approx k_1$ and $\omega_4 \approx k_2$. Then $D^{p,q}_{\omega_1,\omega_3}(\mathbb{R}^d) = D^{p,q}_{\omega_2,\omega_4}(\mathbb{R}^d)$ if and only if $\omega_1 \approx \omega_2$.*

Proof. Follows easily from Proposition 3.5. □

3.7. Proposition. *Assume that $\omega_1,\ \omega_2,\ \omega_3$, and ω_4 are weight functions. If $\omega = \max\{\omega_1,\ \omega_3\}$ and $m = \max\{\omega_2,\ \omega_4\}$, then we have*

$$D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) \cap D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right) = D^{p,q}_{\omega,m}\left(\mathbb{R}^d\right).$$

Proof. For every $f \in D^{p,q}_{\omega,m}(\mathbb{R}^d)$, we have

$$\|f\|_{D^{p,q}_{\omega_1,\omega_2}} = \|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2} \le \|f\|_{p,\omega} + \|W_g f\|_{q,m} < \infty.$$

Hence, $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Similarly we have $f \in D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d)$. Then we obtain $D^{p,q}_{\omega,m}(\mathbb{R}^d) \subset D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d)$.

Conversely take any $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d)$. Since $\omega = \max\{\omega_1,\ \omega_3\}$ and $m = \max\{\omega_2,\ \omega_4\}$, it easily shown that

$$\|f\|_{D^{p,q}_{\omega,m}} = \|f\|_{p,\omega} + \|W_g f\|_{q,m} < \infty.$$

Thus, we may write $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d) \subset D^{p,q}_{\omega,m}(\mathbb{R}^d)$. Finally, we obtain

$$D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) \cap D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right) = D^{p,q}_{\omega,m}\left(\mathbb{R}^d\right).$$ □

3.8. Proposition. *Let $\omega_1,\ \omega_2,\ \omega_3$, and ω_4 be weight functions. If $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$, then*

$$D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) \subset D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right)$$

for all $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

Proof. Let $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$. Then there exist $C_1, C_2 > 0$ such that $\omega_3(t) \le C_1\omega_1(t)$ and $\omega_4(z,u) \le C_2\omega_2(z,u)$ for all $t \in \mathbb{R}^d$, $(z,u) \in \mathbb{R}^d \times \mathbb{R}_+$. Take any $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Since $f \in L^p_{\omega_1}(\mathbb{R}^d)$ and $W_g f \in L^q_{\omega_2}(\mathbb{R}^d \times \mathbb{R}_+)$, we have $\|f\|_{p,\omega_3} \le C_1\|f\|_{p,\omega_1}$ and $\|W_g f\|_{q,\omega_4} \le C_2\|W_g f\|_{q,\omega_2}$. Therefore, we find $f \in D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d)$, and hence $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d)$. □

3.9. Proposition. *Let $\omega_1,\ \omega_2,\ \omega_3$, and ω_4 be weight functions. If $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset D^{p,q}_{\omega_3,\omega_4}(\mathbb{R}^d)$, then there exists a $C > 0$ such that*

$$\|f\|_{D^{p,q}_{\omega_3,\omega_4}} \le C\|f\|_{D^{p,q}_{\omega_1,\omega_2}}$$

for every $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

Proof. We endow the space $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$ with the norm $\|\cdot\|_D = \|\cdot\|_{D^{p,q}_{\omega_1,\omega_2}} + \|\cdot\|_{D^{p,q}_{\omega_3,\omega_4}}$. By Lemma 3.2, the space $\left(D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d), \|\cdot\|_D\right)$ is Banach space. If we use the closed graph theorem, then there exists $C > 0$ such that $\|f\|_{D^{p,q}_{\omega_3,\omega_4}} \leq C\|f\|_{D^{p,q}_{\omega_1,\omega_2}}$ for every $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. □

3.10. Lemma. *Let ω_1 be any weight function and ω_2 a weight function of polynamial type. Then, there exists $C(f) > 0$ such that*

$$C(f)\,\omega_1(z) \leq \|T_z f\|_{D^{p,q}_{\omega_1,\omega_2}} \leq (\omega_1(z) + \omega_2(z,t))\,\|f\|_{D^{p,q}_{\omega_1,\omega_2}}$$

for every $0 \neq f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$ and $t \in \mathbb{R}_+$.

Proof. Let $0 \neq f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$ be given. Since $f \in L^p_{\omega_1}(\mathbb{R}^d)$, then by [7, Proposition 1.7] there exists $C(f) > 0$ such that

$$C(f)\,\omega_1(z) \leq \|T_z f\|_{p,\omega_1} \leq \omega_1(z)\,\|f\|_{p,\omega_1}.$$

Furthermore, using the inequality $\|W_g(T_z f)\|_{q,\omega_2} \leq \omega_2(z,t)\,\|W_g f\|_{q,\omega_2}$ in the proof of Theorem 2.5, we have

$$\begin{aligned} C(f)\,\omega_1(z) &\leq \|T_z f\|_{p,\omega_1} + \|W_g(T_z f)\|_{q,\omega_2} \\ &\leq \omega_1(z)\,\|f\|_{p,\omega_1} + \omega_2(z,t)\,\|W_g f\|_{q,\omega_2} \\ &\leq \omega_1(z)\,\|f\|_{D^{p,q}_{\omega_1,\omega_2}} + \omega_2(z,t)\,\|f\|_{D^{p,q}_{\omega_1,\omega_2}} \\ &= \{\omega_1(z) + \omega_2(z,t)\}\,\|f\|_{D^{p,q}_{\omega_1,\omega_2}} \end{aligned}$$

for all $t \in \mathbb{R}_+$. □

4. Compact embeddings of the space $\mathbf{D}^{p,q}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\mathbb{R}^d)$

4.1. Lemma. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. If $(f_n)_{n\in\mathbb{N}}$ converges to zero in $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} f_n(x)\,k(x)\,dx \to 0$$

as $n \to \infty$ for all $k \in C_c(\mathbb{R}^d)$.

Proof. Let $k \in C_c(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{s} = 1$. Then we may write

$$\left|\int_{\mathbb{R}^d} f_n(x)\,k(x)\,dx\right| \leq \|k\|_s\,\|f_n\|_p \leq \|k\|_s\,\|f_n\|_{D^{p,q}_{\omega_1,\omega_2}}. \tag{10}$$

Therefore, by the assumption and (10), we obtain $\int_{\mathbb{R}^d} f_n(x)\,k(x)\,dx \to 0$ as $n \to \infty$ for all $k \in C_c(\mathbb{R}^d)$. □

4.2. Theorem. *Let ω_1, ω_2 be weight functions of polynomial type on $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{R}_+$ respectively, and let ν be a weight function on $\mathbb{R}^d$. If $\nu \prec \omega_1$ and $\frac{\nu(x)}{\omega_1(x)+\omega_2(x,s)} \nrightarrow 0$ for every fixed s and for $x \to \infty$, then the embedding of the space $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$ into $L^p_\nu(\mathbb{R}^d)$ is never compact.*

Proof. Since $\nu \prec \omega_1$, there exists $C_1 > 0$ such that $\nu(x) \le C_1\omega_1(x)$ for all $x \in \mathbb{R}^d$. This implies $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset L^p_\nu(\mathbb{R}^d)$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence with $t_n \to \infty$ as $n \to \infty$ in $\mathbb{R}^d$. Since $\frac{\nu(x)}{\omega_1(x)+\omega_2(x,s)}$ does not tend to zero as $x \to \infty$, then there exists $\delta > 0$ such that $\frac{\nu(x)}{\omega_1(x)+\omega_2(x,s)} \ge \delta > 0$ for $x \to \infty$. For any fixed $f \in D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$ and fixed $t_0 \in \mathbb{R}_+$, define a sequence $(f_n)_{n\in\mathbb{N}}$ by

$$f_n = (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} T_{t_n} f.$$

This sequence is bounded in $D^{p,q}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Indeed, since the wavelet transform is linear, we can write

$$\begin{aligned}\|f_n\|_{D^{p,q}_{\omega_1,\omega_2}} &= \left\|(\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} T_{t_n} f\right\|_{D^{p,q}_{\omega_1,\omega_2}} \\ &= (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} \|T_{t_n} f\|_{D^{p,q}_{\omega_1,\omega_2}}.\end{aligned} \tag{11}$$

By using (11) and Lemma 3.10, we obtain

$$\begin{aligned}\|f_n\|_{D^{p,q}_{\omega_1,\omega_2}} &\le (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} \|T_{t_n} f\|_{D^{p,q}_{\omega_1,\omega_2}} \\ &\le (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} (\omega_1(t_n) + \omega_2(t_n, t_0)) \|f\|_{D^{p,q}_{\omega_1,\omega_2}} \\ &= \|f\|_{D^{p,q}_{\omega_1,\omega_2}}.\end{aligned}$$

Now we show that there cannot exist a norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\nu(\mathbb{R}^d)$. For all $k \in C_c(\mathbb{R}^d)$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} f_n(x) k(x)\, dx\right| &\le \frac{1}{\omega_1(t_n) + \omega_2(t_n, t_0)} \int_{\mathbb{R}^d} |(T_{t_n} f)(x)| \, |k(x)|\, dx \\ &\le \frac{1}{\omega_1(t_n) + \omega_2(t_n, t_0)} \|k\|_s \|T_{t_n} f\|_p \\ &= \frac{1}{\omega_1(t_n) + \omega_2(t_n, t_0)} \|k\|_s \|f\|_p,\end{aligned} \tag{12}$$

where $\frac{1}{p} + \frac{1}{s} = 1$. Since the right hand side of (12) tends to zero as $n \to \infty$, then we have

$$\int_{\mathbb{R}^d} f_n(x) k(x)\, dx \to 0.$$

Therefore, by Lemma 4.1 the only possible limit of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\nu(\mathbb{R}^d)$ is zero. On the other hand it is known by [6] that $\|T_{t_n} f\|_{p,\nu} \approx \nu(t_n)$. Thus there exist $C_1, C_2 > 0$ such that

$$C_1 \nu(t_n) \le \|T_{t_n} f\|_{p,\nu} \le C_2 \nu(t_n). \tag{13}$$

By using the inequality (13), we obtain

$$\begin{aligned}\|f_n\|_{p,\nu} &= \left\|(\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} T_{t_n} f\right\|_{p,\nu} = (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} \|T_{t_n} f\|_{p,\nu} \\ &\ge C_1 (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} \nu(t_n).\end{aligned} \tag{14}$$

Also, since $\frac{\nu(t_n)}{\omega_1(t_n)+\omega_2(t_n,t_0)} \ge \delta > 0$ for all t_n, by using (14), we can write

$$\|f_n\|_{p,\nu} \ge C_1 (\omega_1(t_n) + \omega_2(t_n, t_0))^{-1} \nu(t_n) \ge \delta C_1 > 0.$$

This means that it is not possible to find a norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\nu(\mathbb{R}^d)$, and the proof is complete. □

4.3. Corollary. *Let ω_1, ω_2 be weight functions of polynamial type on $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{R}_+$, respectively. Also, let ω_3, ω_4 be any weight functions on $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{R}_+$ respectively. If $\omega_3 \prec \omega_1$, $\omega_4 \prec \omega_2$ and $\frac{\omega_3(x)}{\omega_1(x)+\omega_2(x,s)} \nrightarrow 0$ for every fixed s as $x \to \infty$, then the embedding of the space $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ into $D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right)$ is never compact.*

Proof. Since $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$, by Proposition 3.8, we have $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) \subset D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right)$. Also, the unit map is continuous from $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ into $D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right)$. Now, assume that the unit map is compact. Take any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$. If there exists a convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $D^{p,q}_{\omega_3,\omega_4}\left(\mathbb{R}^d\right)$, this sequence also converges in $L^p_{\omega_3}\left(\mathbb{R}^d\right)$. But this is not possible by Theorem 4.2. This completes the proof. □

5. Dual space of $\mathbf{D}^{p,q}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}\left(\mathbb{R}^d\right)$

Consider for each p, q, $(1 \le p, q < \infty)$, the mapping $\Phi : D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) \to L^p_{\omega_1}\left(\mathbb{R}^d\right) \times L^q_{\omega_2}\left(\mathbb{R}^d \times \mathbb{R}_+\right)$ defined by $\Phi(f) = (f, W_g f)$. Let $H = \Phi\left(D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)\right)$. Then

$$|||\Phi(f)||| = |||(f, W_g f)||| = \|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2}$$

is a norm on $L^p_{\omega_1}\left(\mathbb{R}^d\right) \times L^q_{\omega_2}\left(\mathbb{R}^d \times \mathbb{R}_+\right)$. Also, Φ is an linear isometry from $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ into $L^p_{\omega_1}\left(\mathbb{R}^d\right) \times L^q_{\omega_2}\left(\mathbb{R}^d \times \mathbb{R}_+\right)$. Now, we define a set K by

$$K = \left\{ (\varphi, \psi) \in L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d \times \mathbb{R}_+\right) \;\middle|\; \int_{\mathbb{R}^d} f(y)\,\varphi(y)\,dy + \iint_{\mathbb{R}^d\times\mathbb{R}_+} W_g f(x,s)\,\psi(x,s)\,dx\,ds = 0,\ \forall\,(f, W_g f) \in H \right\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

5.1. Proposition. *The dual space of $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is $L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d \times \mathbb{R}_+\right)/K$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. Since $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is a Banach space, then $H = \Phi\left(D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)\right)$ is closed. If we use the duality theorem in [15], we obtain

$$H^* \cong L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d \times \mathbb{R}_+\right)/K, \tag{15}$$

where H^* is the dual of H. Moreover, since Φ is an isometry, then $\left(D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)\right)^* \cong H^*$. Finally by using (15) we obtain

$$\left(D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)\right)^* \cong L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d \times \mathbb{R}_+\right)/K.$$ □

6. The space $\left(\mathbf{D}^{p,q}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}\right)_{\mathbf{s}}\left(\mathbb{R}^{\mathbf{d}}\right)$

Throughout this section we accept that the scale s in $D^{p,q}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is fixed. We denote this new space by $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$. That means $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is the vector space of functions $f \in L^p_{\omega_1}\left(\mathbb{R}^d\right)$ such that their wavelet transforms $W_g f$ are in $L^q_{\omega_2}\left(\mathbb{R}^d\right)$, where s is fixed. We endow this space with the sum norm $\|f\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} = \|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2}$. By using the method in Theorem 2.2, it is easy to see that this space is a Banach space with this sum norm.

6.1. Proposition. *Let ω_2 be a weight function of polynomial type. Then $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is dense in $L^p_{\omega_1}\left(\mathbb{R}^d\right)$.*

Proof. Since ω_2 is a weight of polynomial type, then $D_sg^* \in L^1_{\omega_2}\left(\mathbb{R}^d\right)$. Take any $f \in C_c\left(\mathbb{R}^d\right)$. Then $f \in L^p_{\omega_1}\left(\mathbb{R}^d\right)$. Also, by [7, Theorem 1.11], $L^q_{\omega_2}\left(\mathbb{R}^d\right)$ is a Banach convolution module over $L^1_{\omega_2}\left(\mathbb{R}^d\right)$. Thus if we use the equality $W_gf = f * D_sg^*$, we obtain

$$\|W_gf\|_{q,\omega_2} = \|f * D_sg^*\|_{q,\omega_2} \le \|f\|_{q,\omega_2}\|D_sg^*\|_{1,\omega_2} < \infty.$$

Hence $C_c\left(\mathbb{R}^d\right) \subset \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right) \subset L^p_{\omega_1}\left(\mathbb{R}^d\right)$. Since $C_c\left(\mathbb{R}^d\right)$ is dense in $L^p_{\omega_1}\left(\mathbb{R}^d\right)$, the proof is complete. □

6.2. Proposition. *Let k be a constant number and $\omega_2 \approx k$. Then the spaces $\left(D^{q,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ and $L^q_{\omega_1}\left(\mathbb{R}^d\right)$ are algebraically isomorphic and homeomorphic.*

Proof. By the definition of the space $\left(D^{q,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$, we have $\left(D^{q,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right) \subset L^q_{\omega_1}\left(\mathbb{R}^d\right)$. Since $\omega_2 \approx k$, there exists $C > 0$ such that $\|\cdot\|_{q,\omega_2} \le C\|\cdot\|_q$. Now, take any $f \in L^q_{\omega_1}\left(\mathbb{R}^d\right)$. By using $W_gf = f * D_sg^*$, we have

$$\|f\|_{q,\omega_1} + \|W_gf\|_{q,\omega_2} \le \|f\|_{q,\omega_1} + C\|f * D_sg^*\|_q . \tag{16}$$

It is also known that $L^q\left(\mathbb{R}^d\right)$ is a Banach convolution module over $L^1\left(\mathbb{R}^d\right)$. Thus from (16), we have

$$\begin{aligned}\|f\|_{q,\omega_1} + C\|f * D_sg^*\|_q &\le \|f\|_{q,\omega_1} + C\|f\|_q\|D_sg^*\|_1 \\ &\le \|f\|_{q,\omega_1}\left\{1 + C\|D_sg^*\|_1\right\} < \infty.\end{aligned} \tag{17}$$

Combining (16) and (17), we find $f \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$, and $L^q_{\omega_1}\left(\mathbb{R}^d\right) \subset \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$. Finally we have $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right) = L^q_{\omega_1}\left(\mathbb{R}^d\right)$. Moreover, if we take $M = \left\{1 + C\|D_sg^*\|_1\right\}$, by (16) and (17) we have

$$\|f\|_{q,\omega_1} \le \|f\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} \le M\|f\|_{q,\omega_1}$$

for all $f \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$. That means $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ and $L^q_{\omega_1}\left(\mathbb{R}^d\right)$ are algebraically isomorphic and homeomorphic. □

6.3. Theorem. *$\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is invariant under translations and the translation mapping $z \mapsto T_zf$ is continuous from $\mathbb{R}^d$ into $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$.*

Proof. Let $f \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ be given. Then we write

$$\begin{aligned}\|T_zf\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} &= \|T_zf\|_{p,\omega_1} + \|W_g(T_zf)\|_{q,\omega_2} \\ &\le \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z)\|W_gf\|_{q,\omega_2} < \infty.\end{aligned}$$

Hence $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is translation invariant. Moreover, it is known that the translation mapping is continuous from $\mathbb{R}^d$ into $L^p_{\omega_1}\left(\mathbb{R}^d\right)$ [7]. Thus, for any given $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that if $\|z - u\| < \delta_1$ for $z, u \in \mathbb{R}^d$, then

$$\|T_zf - T_uf\|_{p,\omega_1} < \frac{\varepsilon}{2}.$$

Also, since the translation mapping is continuous from $\mathbb{R}^d$ into $L^q_{\omega_2}\left(\mathbb{R}^d\right)$, then for the same $\varepsilon > 0$, there exists $\delta_2(\varepsilon) > 0$ such that if $\|z - u\| < \delta_2$ for all $z, u \in \mathbb{R}^d$, then

$$\|W_g(T_zf - T_uf)\|_{q,\omega_2} < \frac{\varepsilon}{2}.$$

If we set $\delta = \min\{\delta_1, \delta_2\}$, and if $\|z-u\| < \delta$ for $z, u \in \mathbb{R}^d$, then

$$\|T_z f - T_u f\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} = \|T_z f - T_u f\|_{p,\omega_1} + \|W_g (T_z f - T_u f)\|_{q,\omega_2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

6.4. Proposition. *$\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is Banach function space.*

Proof. Take any function $f \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$, and a compact subset $K \subset \mathbb{R}^d$. Since $K \subset \mathbb{R}^d$ is compact and $p \geq 1$, then there exists $C > 0$ such that

$$\int_K |f(x)|\, dx \leq C\|f\|_p .$$

Then

$$\int_K |f(x)|\, dx \leq C\left\{\|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2}\right\} = C\|f\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} .$$

Since $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is Banach space, the proof is complete. □

6.5. Theorem. *Suppose that $\omega_2 = k$, where k is a constant number. Then $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is an essential Banach module over $L^1_{\omega_1}(\mathbb{R}^d)$.*

Proof. It is known that $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is a Banach space. Now we take any $f \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ and $h \in L^1_{\omega_1}(\mathbb{R}^d)$. Since $L^p_{\omega_1}(\mathbb{R}^d)$ is a Banach convolution module over $L^1_{\omega_1}(\mathbb{R}^d)$, we can write

$$\|f * h\|_{p,\omega_1} \leq \|f\|_{p,\omega_1}\|h\|_{1,\omega_1} . \tag{18}$$

Thus by using $W_g f = f * D_s g^*$, we have

$$\begin{aligned} \|W_g(f*h)\|_{q,\omega_2} &= \|(f*h) * D_s g^*\|_{q,\omega_2} \\ &\leq \|h\|_1 \|f * D_s g^*\|_{q,\omega_2} = \|h\|_1 \|W_g f\|_{q,\omega_2} . \end{aligned} \tag{19}$$

Thus $W_g(f*h) \in L^q_{\omega_2}(\mathbb{R}^d)$. Combining (18) and (19), we obtain

$$\begin{aligned} \|f*h\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} &= \|f*h\|_{p,\omega_1} + \|W_g(f*h)\|_{q,\omega_2} \\ &\leq \|f\|_{p,\omega_1}\|h\|_{1,\omega_1} + \|h\|_{1,\omega_1}\|W_g f\|_{q,\omega_2} = \|h\|_{1,\omega_1}\|f\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} . \end{aligned}$$

Hence $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is a Banach module over $L^1_{\omega_1}(\mathbb{R}^d)$.

In order to show that $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is an essential Banach module over$L^1_{\omega_1}(\mathbb{R}^d)$, we will use the Module Factorization Theorem [20]. For this, it suffices to prove that $L^1_{\omega_1}(\mathbb{R}^d) * \left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$ is dense in $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$. It is known that $L^1_{\omega_1}(\mathbb{R}^d)$ has a bounded approximate identity [8]. Let U be a neighbourhood of the unit element of $\mathbb{R}^d$. We can choose an approximate identity $(e_\alpha)_{\alpha\in I}$ which is positive bounded and satisfies supp$e_\alpha \subset U$, $\|e_\alpha\|_1 = 1$ for all $\alpha \in I$. Take any $h \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^d)$. For fixed $\alpha_0 \in I$, we have

$$\begin{aligned}\|e_{\alpha_0} * h - h\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} &= \left\| \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, T_z h(y)\ dz - \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, h(y)\ dz \right\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} \\ &= \left\| \int_{\mathbb{R}^d} e_{\alpha_0}(z) \left(T_z h(y) - h(y)\right) dz \right\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} \\ &\le \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, \|T_z h - h\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s}\ dz.\end{aligned}$$

We know by Theorem 6.3 that the translation mapping $z \mapsto T_z f$ is continious from $\mathbb{R}^d$ into $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$. Hence for given any $\varepsilon > 0$, we can make $\|T_z h - h\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} < \varepsilon$. Then, we obtain

$$\|e_\alpha * h - h\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} \le \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, \varepsilon dz = \varepsilon.$$

Therefore $L^1_{\omega_1}\left(\mathbb{R}^d\right) * \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is dense in $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$. Finally, from the Module Factorization Theorem, the proof is complete. □

By using [4, Theorem 6.5 and Corollary 15.3], it easy to prove following Corollary 6.6.

6.6. Corollary. *Let $(e_\alpha)_{\alpha\in I}$ be an approximate identity in $L^1_{\omega_1}\left(\mathbb{R}^d\right)$, and let $\omega_2 = k$ where k is constant number. Then $(e_\alpha)_{\alpha\in I}$ is an approximate identity of the space $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$.* □

7. Space of multipliers of $\left(\mathbf{D}^{p,q}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}\right)_s\left(\mathbb{R}^d\right)$

Consider the mapping Φ from $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ into $L^p_{\omega_1}\left(\mathbb{R}^d\right) \times L^q_{\omega_2}\left(\mathbb{R}^d\right)$ defined by $\Phi(f) = (f, W_g f)$. This mapping is a linear isometry of $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ into $L^p_{\omega_1}\left(\mathbb{R}^d\right) \times L^q_{\omega_2}\left(\mathbb{R}^d\right)$ with the norm

$$|||\Phi(f)||| = |||(f, W_g f)||| = \|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2}$$

for all $f \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$. Let $H = \Phi\left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)\right)$. Define the set K to be

$$K = \left\{ (\varphi,\psi) \in L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d\right) \;\middle|\; \int_{\mathbb{R}^d} f(y)\,\varphi(y)\ dy + \right. \\ \left. + \int_{\mathbb{R}^d} W_g f(x,s)\,\psi(x,s)\ dx = 0,\ \forall (f, W_g f) \in H \right\}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

7.1. Proposition. *The dual space $\left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)\right)^*$ is isomorphic to $L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d\right)/K$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. This result follows easily from the Duality Theorem in [15]. □

7.2. Theorem. *Let $\omega_2 = k$, where k is a constant number. Then the spaces* $L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d\right)/K$ *and* $\mathrm{Hom}_{L^1_{\omega_1}}\left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right), L^{\infty}_{\omega_1^{-1}}\left(\mathbb{R}^d\right)\right)$ *are algebrically isomorphic and topologically homeomorphic.*

Proof. By Theorem 6.5, $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is an essential Banach module over $L^1_{\omega_1}\left(\mathbb{R}^d\right)$. If we use [17, Theorem 1.4] and Proposition 7.1, we obtain

$$\begin{aligned}\mathrm{Hom}_{L^1_{\omega_1}}&\left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right), L^{\infty}_{\omega_1^{-1}}\left(\mathbb{R}^d\right)\right)\\ &= \mathrm{Hom}_{L^1_{\omega_1}}\left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right), \left(L^1_{\omega_1}\left(\mathbb{R}^d\right)\right)^*\right)\\ &\cong \left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right) * L^1_{\omega_1}\left(\mathbb{R}^d\right)\right)^*\\ &= \left(\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)\right)^* \cong L^{p'}_{\omega_1^{-1}}\left(\mathbb{R}^d\right) \times L^{q'}_{\omega_2^{-1}}\left(\mathbb{R}^d\right)/K,\end{aligned}$$

and the proof is complete. □

Let $\omega_2 = k$. Suppose that $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity in $L^1_{\omega_1}\left(\mathbb{R}^d\right)$. The relative completion $\left(\widetilde{D}^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ of $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is defined by

$$\left(\widetilde{D}^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right) = \Big\{ f \in L^p_{\omega_1}\left(\mathbb{R}^d\right) \;\Big|\; f * e_\alpha \in \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right) \text{ for all } \alpha \in I \text{ and } \sup_{\alpha\in I}\|f * e_\alpha\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s} < \infty \Big\}.$$

$\left(\widetilde{D}^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is a Banach space with the norm

$$\|f\|_{\left(\widetilde{D}^{p,q}_{\omega_1,\omega_2}\right)_s} = \sup_{\alpha\in I}\|f * e_\alpha\|_{\left(D^{p,q}_{\omega_1,\omega_2}\right)_s},$$

and this space does not depend on the approximate identity [5].

7.3. Theorem. *Let $\omega_2 = k$ for a constant number k.Then the spaces* $\left(\widetilde{D}^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ *and* $M\left(L^1_{\omega_1}\left(\mathbb{R}^d\right), \left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)\right)$ *are algebrically isomorphic and topologically homeomorphic*

Proof. Since $\left(\widetilde{D}^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$ is the relative completion of $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s\left(\mathbb{R}^d\right)$, it is easy to prove this theorem using [5, Theorem 2.6]. □

References

[1] Daubechies, I. *Ten Lectures on Wavelets* (CBMS-NSF, SIAM, Philadelphia, 1992).

[2] Doğan, M. and Gürkanlı, A. T. *On functions with Fourier transforms in S_w*, Bull. Cal. Math. Soc. **92** (2), 111-120, 2000.

[3] Doğan, M. and Gürkanlı, A. T. *Multipliers of the space $S_w(G)$*, Mathematica Balkanica, New Series **15** (3-4), 199–212, 2001.

[4] Doran, R. S. and Wichmann, J. *Approximate Identity and Factorization in Banach Modules* (Lecture Notes in Math. **768**, Springer-Verlag, Berlin, Heidelberg, New York, 1979).

[5] Duyar, C. and Gürkanlı, A. T. *Multipliers and relative completion in weighted Lorentz space*, Acta Mathematica Scientia **23B** (4), 467–476, 2003.

[6] Feichtinger, H. G. and Gürkanlı, A. T. *On a family of weighted convolution algebras*, Internat. J. Math. Sci. **13** (3), 517–526, 1990.

[7] Fischer, R. H., Gürkanlı, A. T. and Liu, T. S. *On a family of weighted spaces*, Math. Slovaca **46** (1), 71–82, 1996.

[8] Gaudry, G. I. *Multipliers of weighted Lebesque and measure spaces*, Proc. London Math. Soc. **19** (3), 327–340, 1969.

[9] Gröchenig, K. *Foundations of Time-Frequency Analysis* (Birkhauser, Boston, 2001).
[10] Gürkanlı, A. T. *Multipliers of some Banach ideals and Wiener-Ditkin Sets*, Math. Slovaca **55** (2), 237–248, 2005.
[11] Gürkanlı, A. T. *Time frequency analysis and multipliers of the space* $M(p,q)(\mathbb{R}^d)$, $S(p,q)\left(\mathbb{R}^d\right)$, J. Math. Kyoto Univ. **46** (3), 595–616, 2006.
[12] Gürkanlı, A. T. *Tensor product factorization and multipliers of some Banach modules*, International Journal of Applied Mathematics **20** (5), 661–670, 2007.
[13] Gürkanlı, A. T. *Compact embeddings of the spaces* $A^p_{w,\omega}\left(\mathbb{R}^d\right)$, Taiwanese Journal of Mathematics **12** (7), 1757–1767, 2008.
[14] Larsen, R. *Banach Algebras an Introduction* (Marcel Dekker Inc., New York, 1973).
[15] Liu, T. S. and Rooij, A. Van. *Sums and intersections of normed linear spaces*, Mathematische Nachrichten **42**, 29–42, 1969.
[16] Reiter, H. *Classical Harmonic Analysis and Locally Compact Groups* (Oxford Universty Pres, Oxford, 1968).
[17] Rieffel, M. A. *Induced Banach representation of Banach algebras and locally compact groups*, J. Funct. Anal. **1**, 443–491, 1967.
[18] Rieffel, M. A. *Multipliers and tensor products of* L^p*-spaces of locally compact groups*, Studia Math. **33**, 71–82, 1969.
[19] Sandıkçı, A. and Gürkanlı, A. T. *The space* $\Omega^p_m\left(\mathbb{R}^d\right)$ *and some properties*, Ukranian Mathematical Journal **58** (1), 139–145, 2006.
[20] Wang, H. C. *Homogeneous Banach Algebras* (Marcel Dekker INC., New York, 1977).