

A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

Bilal Seker * and Nak Eun Cho †

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Abstract

In the present paper, we obtain coefficient estimates and distortion and growth theorems for certain subclass of close-to-convex functions. The results presented here contain those given in earlier works as in some special cases.

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1. Introduction

Let \mathcal{A} denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let \mathcal{S} , \mathcal{K} and \mathcal{S}^* denote the usual subclasses of \mathcal{A} whose members are univalent, close-to-convex and starlike in \mathbb{U} , respectively. By $\mathcal{S}^*(\alpha)$, we also denote the class of starlike functions of order α ($0 \leq \alpha < 1$).

For two functions f and g analytic in \mathbb{U} , we say that the function $f(z)$ is subordinate to $g(z)$ in \mathbb{U} , and write as:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in \mathbb{U} with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

*Department of Mathematics, Faculty of Science and Letters, Batman University 72060 - Batman, Turkey. E-Mail: (B. Seker) bilalseker1980@gmail.com

†Department of Applied Mathematics, Pukyong National University Busan 608-737, Korea. E-Mail: (N. E. Cho) necho@pknu.ac.kr

In particular, if the function g is univalent in \mathbb{U} , then $f(z)$ is subordinate to $g(z)$ in \mathbb{U} (cf. [1]) if and only if

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Recently, Kowalczyk et al. [4] discussed a class $K_s(\gamma)$ of analytic functions related to the starlike functions: A function $f(z) \in \mathcal{A}$ is said to be in the class $K_s(\gamma)$ if it satisfies the inequality:

$$\operatorname{Re} \left(\frac{-z^2 f'(z)}{g(z)g(-z)} \right) > \gamma \quad (0 \leq \gamma < 1; z \in \mathbb{U}),$$

where $g(z) \in \mathcal{S}^*(1/2)$.

By simple calculations, we see that the above inequality is equivalent to

$$\left| \frac{z^2 f'(z)}{g(z)g(-z)} + 1 \right| < \left| \frac{z^2 f'(z)}{g(z)g(-z)} - 1 + 2\gamma \right| \quad (0 \leq \gamma < 1; z \in \mathbb{U}).$$

Motivated by the class $K_s(\gamma)$, we introduce a new class $K_s^{(k)}(\gamma, \alpha, \beta)$ of analytic functions related to starlike functions as follows:

1.1. Definition. Let $K_s^{(k)}(\gamma, \alpha, \beta)$ denote the class of functions in \mathcal{A} satisfying the inequality:

$$(1.2) \quad \left| \frac{z^k f'(z)}{g_k(z)} - 1 \right| < \beta \left| \frac{\alpha z^k f'(z)}{g_k(z)} + 1 - (1 + \alpha)\gamma \right|$$

$$(0 \leq \alpha \leq 1; 0 < \beta \leq 1; 0 \leq \gamma < 1; z \in \mathbb{U}),$$

where $g_k(z)$ is defined by

$$(1.3) \quad g_k(z) = \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^{\nu z}) \quad \left(\varepsilon^k = 1; g(z) \in \mathcal{S}^* \left(\frac{k-1}{k} \right); k \geq 1 \right).$$

We note that $K_s^{(2)}(0, 1, 1) = K_s$, where K_s is the class of functions which was defined by Gao and Zhou [2]. Moreover, $K_s^{(2)}(\gamma, 1, 1) = K_s(\gamma)$ and $K_s^{(k)}(\gamma, 1, 1) = K_s^{(k)}(\gamma)$ which were studied by Kowalczyk *et al.* [4] and Seker [6], respectively so the class $K_s^{(k)}(\gamma, \alpha, \beta)$ are generalizations of $K_s(\gamma)$ and $K_s^{(k)}(\gamma)$.

In the present paper, we investigate characterization theorems, coefficient inequalities, growth and distortion theorems for functions belonging to the class $K_s^{(k)}(\gamma, \alpha, \beta)$.

2. Coefficient Estimates

First of all, we show in which way our class is associated with the appropriate subordination.

2.1. Theorem. A function $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$ if and only if there exists $g_k(z)$ satisfying the condition (1.3) such that

$$(2.1) \quad \frac{z^k f'(z)}{g_k(z)} \prec \frac{1 + \beta[1 - (1 + \alpha)\gamma]z}{1 - \alpha\beta z} \quad (z \in \mathbb{U}).$$

Proof. Let $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$. Then, for $\alpha \neq 1$ and $\beta \neq 1$, squaring and expanding both sides of (1.2), we see that the region of $G(z) = z^k f'(z)/g_k(z)$ for $z \in \mathbb{U}$ is contained in the disk \mathbf{C} whose center is $\{1 + \alpha\beta^2[1 - (1 + \alpha)\gamma]\}/(1 - \alpha^2\beta^2)$ and radius is $\beta(1 + \alpha)(1 - \gamma)/(1 - \alpha^2\beta^2)$. Since $q(z) = \{1 + \beta[1 - (1 + \alpha)\gamma]z\}/(1 - \alpha\beta z)$ maps the unit disk \mathbb{U} to the disk \mathbf{C} and $q(z)$ is univalent in \mathbb{U} , we obtain the relation (2.1). \square

Conversely, assume that the relation (2.1) holds true. Then we have

$$\frac{z^k f'(z)}{g_k(z)} \prec \frac{1 + \beta[1 - (1 + \alpha)\gamma]w(z)}{1 - \alpha\beta w(z)},$$

$$(0 \leq \alpha \leq 1; 0 < \beta \leq 1; 0 \leq \gamma < 1; z \in \mathbb{U}),$$

where $w(z)$ is analytic in \mathbb{U} , $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. Therefore from the above equation, we obtain the inequality (1.2), that is, $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$.

2.2. Remark. From Theorem 2.1, we see that, if $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$, then

$$(2.2) \quad \operatorname{Re} \left(\frac{z f'(z)}{g_k(z)/z^{k-1}} \right) > \gamma \quad (z \in \mathbb{U}),$$

because of

$$\operatorname{Re} \left(\frac{1 + \beta[1 - (1 + \alpha)\gamma]z}{1 - \alpha\beta z} \right) > \gamma \quad (z \in \mathbb{U}).$$

In order to give the coefficient estimate of functions belonging to the class $K_s^k(\gamma, \alpha, \beta)$, we shall require the following lemma.

2.3. Lemma. [7] *Let*

$$(2.3) \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \left(\frac{k-1}{k} \right),$$

then

$$(2.4) \quad G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} B_n z^n \in \mathcal{S}^* \subset \mathcal{S},$$

where $g_k(z)$ is given by (1.3).

2.4. Remark. (i) In particular, for $k = 2$, the coefficients B_n in (2.4) is expressed as follows:

$$B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + \dots + (-1)^n 2b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2.$$

(ii) If $g(z) \in \mathcal{S}^*((k-1)/k)$, then from Lemma 2.3., $G_k(z)$ given by (2.4) belongs to \mathcal{S}^* . Then by (2.2), we see that the class $K_s^k(\gamma, \alpha, \beta)$ is a subclass of the class \mathcal{K} of close-to-convex functions.

Next, we prove the sufficient condition for functions to belong to the class $K_s^k(\gamma, \alpha, \beta)$.

2.5. Theorem. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in \mathbb{U} . If*

$$(2.5) \quad \sum_{n=2}^{\infty} (1 + \alpha\beta)n |a_n| + \sum_{n=2}^{\infty} [1 + \beta|1 - (1 + \alpha)\gamma|] |B_n| \leq \beta(1 + \alpha)(1 - \gamma)$$

$$(0 \leq \alpha \leq 1; 0 < \beta \leq 1; 0 \leq \gamma < 1)$$

where the coefficients B_n ($n = 2, 3, \dots$) are given by (2.4), then $f(z) \in K_s^k(\gamma, \alpha, \beta)$.

Proof. Let the functions $f(z)$ and $g_k(z)$ be given by (1.1) and (1.3), respectively. Now, we obtain

$$\begin{aligned} \Delta &= \left| z f'(z) - \frac{g_k(z)}{z^{k-1}} \right| - \beta \left| \alpha z f'(z) + \frac{[1 - (1 + \alpha)\gamma]g_k(z)}{z^{k-1}} \right| \\ &= \left| \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} B_n z^n \right| - \end{aligned}$$

$$-\beta \left| (1+\alpha)(1-\gamma)z + \alpha \sum_{n=2}^{\infty} n a_n z^n + [1 - (1+\alpha)\gamma] \sum_{n=2}^{\infty} B_n z^n \right|.$$

Thus, for $|z| = r$ ($0 \leq r < 1$), we have, from (2.5),

$$\begin{aligned} \Delta &\leq \sum_{n=2}^{\infty} n |a_n| |z|^n + \sum_{n=2}^{\infty} |B_n| |z|^n \\ &\quad - \beta \left((1+\alpha)(1-\gamma)|z| - \alpha \sum_{n=2}^{\infty} n |a_n| |z|^n - [1 - (1+\alpha)\gamma] \sum_{n=2}^{\infty} |B_n| |z|^n \right) \\ &= -\beta(1+\alpha)(1-\gamma)|z| + \sum_{n=2}^{\infty} (1+\alpha\beta)n |a_n| |z|^n + \\ &\quad \sum_{n=2}^{\infty} [1 + \beta |1 - (1+\alpha)\gamma|] |B_n| |z|^n \\ &< \left(-\beta(1+\alpha)(1-\gamma) + \sum_{n=2}^{\infty} (1+\alpha\beta)n |a_n| + \sum_{n=2}^{\infty} [1 + \beta |1 - (1+\alpha)\gamma|] |B_n| \right) \\ &\leq 0. \end{aligned}$$

Thus we have

$$\left| \frac{z^k f'(z)}{g_k(z)} - 1 \right| < \beta \left| \frac{\alpha z^k f'(z)}{g_k(z)} + 1 - (1+\alpha)\gamma \right|,$$

that is, $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$. This completes the proof of Theorem 2.5. \square

In the following theorem, we give the coefficient estimates of functions belonging to the class $K_s^{(k)}(\gamma, \alpha, \beta)$.

2.6. Theorem. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, and satisfy the inequality (2.1). Then, for, $n \geq 2$, we have*

$$(2.6) \quad \begin{aligned} &|n a_n - B_n|^2 - [\beta(1+\alpha)(1-\gamma)]^2 \\ &\leq (1+\beta)|(1+\alpha)\gamma - 1| \sum_{k=2}^{n-1} \{2k |a_k B_k| + [1 + \beta |(1+\alpha)\gamma - 1|] |B_k|^2\}, \end{aligned}$$

where B_n is given by (2.4).

Proof. Suppose that the condition (1.2) is satisfied. Then, by using the a similar method as in the proof of (p. 30, [5]), we have

$$(2.7) \quad \frac{z f'(z)}{G_k(z)} = \frac{1 + [(1+\alpha)\gamma - 1]z\phi(z)}{1 + \alpha z\phi(z)} \quad (z \in \mathbb{U}),$$

where ϕ is analytic in \mathbb{U} , $|\phi(z)| \leq \beta$ for $z \in \mathbb{U}$ and $G_k(z)$ is given by (2.4). Then from (2.7), we have

$$(\alpha z f'(z) - [(1+\alpha)\gamma - 1]G_k(z)) z\phi(z) = G_k(z) - z f'(z)$$

Thus, putting

$$z\phi(z) = \sum_{n=1}^{\infty} t_n z^n,$$

we obtain

$$(2.8) \quad \begin{aligned} & \left((1 + \alpha)(1 - \gamma)z + \alpha \sum_{n=2}^{\infty} na_n z^n - [(1 + \alpha)\gamma - 1] \sum_{n=2}^{\infty} B_n z^n \right) \sum_{n=1}^{\infty} t_n z^n \\ & = \sum_{n=2}^{\infty} B_n z^n - \sum_{n=2}^{\infty} na_n z^n. \end{aligned}$$

Equating the coefficient of z^n in (2.8), we have

$$\begin{aligned} B_n - na_n & = (1 + \alpha)(1 - \gamma)t_{n-1} + \{2\alpha a_2 - [(1 + \alpha)\gamma - 1] B_2\}t_{n-2} \\ & \quad + \dots + \{(n - 1)\alpha a_{n-1} - [(1 + \alpha)\gamma - 1] B_{n-1}\}t_1. \end{aligned}$$

Thus the coefficient combination on the right side of (2.8) depends only upon the coefficient combinations:

$$\{2\alpha a_2 - [(1 + \alpha)\gamma - 1] B_2\}, \dots, \{(n - 1)\alpha a_{n-1} - [(1 + \alpha)\gamma - 1] B_{n-1}\}.$$

Hence for $n \geq 2$, the equation (2.8) can be written as

$$(2.9) \quad \begin{aligned} & \left[(1 + \alpha)(1 - \gamma)z + \sum_{k=2}^{n-1} (k\alpha a_k - [(1 + \alpha)\gamma - 1] B_k) z^k \right] z\phi(z) \\ & = \sum_{k=2}^n (B_k - ka_k) z^k + \sum_{k=n+1}^{\infty} c_k z^k. \end{aligned}$$

Then, squaring the modulus of the both sides of (2.9) and integrating along $|z| = r < 1$, so that by Parseval's identity (p. 192, [1]), we obtain

$$(2.10) \quad \begin{aligned} & \sum_{k=2}^n |ka_k - B_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} \\ & \leq \beta^2 \left([(1 + \alpha)(1 - \gamma)]^2 r^2 + \sum_{k=2}^{n-1} |k\alpha a_k - [(1 + \alpha)\gamma - 1] B_k|^2 r^{2k} \right). \end{aligned}$$

Letting $r \rightarrow 1$ on the left side of (2.10), we obtain

$$\sum_{k=2}^n |ka_k - B_k|^2 \leq \beta^2 \left([(1 + \alpha)(1 - \gamma)]^2 + \sum_{k=2}^{n-1} |k\alpha a_k - [(1 + \alpha)\gamma - 1] B_k|^2 \right).$$

Hence we have

$$\begin{aligned} |na_n - B_n|^2 & < [\beta(1 + \alpha)(1 - \gamma)]^2 + \beta^2 \sum_{k=2}^{n-1} |k\alpha a_k - [(1 + \alpha)\gamma - 1] B_k|^2 - \\ & \quad - \sum_{k=2}^{n-1} |ka_k - B_k|^2 = \\ & = [\beta(1 + \alpha)(1 - \gamma)]^2 + (\beta^2 \alpha^2 - 1) \sum_{k=2}^{n-1} k^2 |a_k|^2 + \\ & \quad + \{(\beta [(1 + \alpha)\gamma - 1])^2 - 1\} \sum_{k=2}^{n-1} |B_k|^2 + (\alpha \beta^2 |(1 + \alpha)\gamma - 1| + 1) \sum_{k=2}^{n-1} 2k |a_k| |B_k| \leq \end{aligned}$$

$$\begin{aligned} &\leq [\beta(1+\alpha)(1-\gamma)]^2 + (\beta|(1+\alpha)\gamma-1|+1)^2 \sum_{k=2}^{n-1} |B_k|^2 + \\ &+ (\beta|(1+\alpha)\gamma-1|+1) \sum_{k=2}^{n-1} 2k |a_k| |B_k|, \end{aligned}$$

which implies the inequality (2.6). Therefore, we complete the proof of Theorem 2.6. \square

Finally, we provide the growth and the distortion theorems for functions belonging to the class $K_s^{(k)}(\gamma, \alpha, \beta)$.

2.7. Theorem. *If $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$, then*

$$(2.11) \quad \frac{1-\beta[1-(1+\alpha)\gamma]r}{(1+\alpha\beta r)(1+r^2)} \leq |f'(z)| \leq \frac{1+\beta[1-(1+\alpha)\gamma]r}{(1-\alpha\beta r)(1-r^2)} \quad (|z|=r < 1)$$

and

$$(2.12) \quad \begin{aligned} &\frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2} \ln \frac{1+\alpha\beta r}{1+r} + \frac{(1+\beta[1-(1+\alpha)\gamma])r}{(1-\alpha\beta)(1+r)} \leq |f(z)| \\ &\leq \frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2} \ln(1-\alpha\beta r)(1-r) - \frac{(1+\beta[1-(1+\alpha)\gamma])r}{(1-\alpha\beta)(1-r)} \quad (|z|=r < 1), \end{aligned}$$

The results are sharp.

Proof. If $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$, then there exists function $g_k(z)$ satisfying (1.2). Then it follows from the Lemma 2.3. that the function $G_k(z)$ given by (2.4) is a starlike function. Hence from (p. 70, [1]), we have

$$(2.13) \quad \frac{r}{1+r^2} \leq |G_k(z)| \leq \frac{r}{1-r^2} \quad (|z|=r < 1).$$

Let us define $p(z)$ by

$$p(z) = \frac{zf'(z)}{G_k(z)} \quad (z \in \mathbb{U}).$$

Then by using a similar method as in (p. 105, [3]), we have

$$(2.14) \quad \frac{1-\beta[1-(1+\alpha)\gamma]r}{1+\alpha\beta r} \leq |p(z)| \leq \frac{1+\beta[1-(1+\alpha)\gamma]r}{1-\alpha\beta r} \quad (|z|=r < 1).$$

Thus from (2.13) and (2.14), we have

$$\frac{1-\beta[1-(1+\alpha)\gamma]r}{(1+\alpha\beta r)(1+r^2)} \leq |f'(z)| \leq \frac{1+\beta[1-(1+\alpha)\gamma]r}{(1-\alpha\beta r)(1-r^2)} \quad (|z|=r < 1),$$

which gives us (2.11). Upon integrating (2.11) from 0 to r , we have the inequality (2.12). Moreover, the results are sharp for the functions given, respectively, by

$$f_1(z) = \frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2} \ln \frac{1+\alpha\beta z}{1+z} + \frac{(1+\beta[1-(1+\alpha)\gamma])z}{(1-\alpha\beta)(1+z)} \quad (z \in \mathbb{U})$$

and

$$f_2(z) = \frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2} \ln(1-\alpha\beta z)(1-z) - \frac{(1+\beta[1-(1+\alpha)\gamma])z}{(1-\alpha\beta)(1-z)} \quad (z \in \mathbb{U}).$$

\square

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