A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

Bilal Seker * and Nak Eun Cho †

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Abstract

In the present paper, we obtain coefficient estimates and distortion and growth theorems for certain subclass of close-to-convex functions. The results presented here contain those given in earlier works as in some special cases.

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1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $S$, $K$ and $S^*$ denote the usual subclasses of $A$ whose members are univalent, close-to-convex and starlike in $U$, respectively. By $S^*(\alpha)$, we also denote the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$.

For two functions $f$ and $g$ analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$, and write as:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, analytic in $U$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)) \quad (z \in U).$$

*Department of Mathematics, Faculty of Science and Letters, Batman University 72060 - Batman, Turkey. E-Mail: (B. Seker) bilalseker1980@gmail.com
†Department of Applied Mathematics, Pukyong National University Busan 608-737, Korea. E-Mail: (N. E. Cho) necho@pknu.ac.kr
In particular, if the function \( g \) is univalent in \( U \), then \( f(z) \) is subordinate to \( g(z) \) in \( U, \) cf. [1]) if and only if
\[
f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

Recently, Kowalczyk et al. [4] discussed a class \( K_4(\gamma) \) of analytic functions related to the starlike functions: A function \( f(z) \in A \) is said to be in the class \( K_4(\gamma) \) if it satisfies the inequality:
\[
\Re \left( \frac{-z^2 f'(z)}{g(z)g(-z)} \right) > \gamma \quad (0 \leq \gamma < 1; \ z \in U),
\]
where \( g(z) \in S'((1/2)) \)

Motivated by the class \( K_4(\gamma) \), we introduce a new class \( K_4^k(\gamma, \alpha, \beta) \) of analytic functions related to starlike functions as follows:

1.1. **Definition.** Let \( K_4^k(\gamma, \alpha, \beta) \) denote the class of functions in \( A \) satisfying the inequality:
\[
(1.2) \quad \left| \frac{z^k f'(z)}{g_k(z)} - 1 \right| < \beta \left| \frac{\alpha z^k f'(z)}{g_k(z)} + 1 \right| (1 + \alpha)\gamma
\]
\[(0 \leq \alpha \leq 1; \ 0 < \beta \leq 1; \ 0 \leq \gamma < 1; \ z \in U),\]
where \( g_k(z) \) is defined by
\[
(1.3) \quad g_k(z) = \prod_{i=0}^{k-1} \varepsilon^i \left( z^{\nu} \right) \left( z^{\nu} = 1; \ g(z) \in S^* \left( \frac{k-1}{k} \right); \ k \geq 1 \right).
\]

We note that \( K_4^2(0, 1, 1) = K_4 \), where \( K_4 \) is the class of functions which was defined by Gao and Zhou [2]. Moreover, \( K_4^{(2)}(\gamma, 1, 1) = K_4(\gamma) \) and \( K_4^k(\gamma, 1, 1) = K_4^k(\gamma) \) which were studied by Kowalczyk et al. [4] and Seker [6], respectively so the class \( K_4^k(\gamma, \alpha, \beta) \) are generalizations of \( K_4(\gamma) \) and \( K_4^k(\gamma) \).

In the present paper, we investigate characterization theorems, coefficient inequalities, growth and distortion theorems for functions belonging to the class \( K_4^k(\gamma, \alpha, \beta) \).

2. **Coefficient Estimates**

First of all, we show in which way our class is associated with the appropriate subordination.

2.1. **Theorem.** A function \( f(z) \in K_4^k(\gamma, \alpha, \beta) \) if and only if there exists \( g_k(z) \) satisfying the condition (1.3) such that
\[
(2.1) \quad \frac{z^k f'(z)}{g_k(z)} < 1 + \beta \left[ 1 - (1 + \alpha)\gamma \right] \frac{z}{1 - \alpha \beta z} \quad (z \in U).
\]

Proof. Let \( f(z) \in K_4^k(\gamma, \alpha, \beta) \). Then, for \( \alpha \neq 1 \) and \( \beta \neq 1 \), squaring and expanding both sides of (1.2), we see that the region of \( G(z) = z^k f'(z)/g_k(z) \) for \( z \in U \) is contained in the disk \( C \) whose center is \( \{ 1 + \alpha \beta^2 [1 - (1 + \alpha)\gamma] \} / (1 - \alpha^2 \beta^2) \) and radius is \( \beta(1 + \alpha)(1 - \gamma) / (1 - \alpha^2 \beta^2) \). Since \( q(z) = \{ 1 + \beta[1 - (1 + \alpha)\gamma]z \} / (1 - \alpha \beta z) \) maps the unit disk \( U \) to the disk \( C \) and \( q(z) \) is univalent in \( U \), we obtain the relation (2.1). \( \square \)
Conversely, assume that the relation (2.1) holds true. Then we have
\[
\frac{zf'(z)}{g_k(z)} < \frac{1 + \beta[1 - (1 + \alpha)\gamma]w(z)}{1 - \alpha\beta w(z)},
\]
where \(w(z)\) is analytic in \(U\), \(w(0) = 0\) and \(|w(z)| < 1\) for \(z \in U\). Therefore from the above equation, we obtain the inequality (1.2), that is, \(f(z) \in K_k^A(\gamma, \alpha, \beta)\).

2.2. Remark. From Theorem 2.1, we see that, if \(f(z) \in K_k^A(\gamma, \alpha, \beta)\), then
\[
\text{Re} \left( \frac{zf'(z)}{g_k(z)/z^{k-1}} \right) > \gamma \quad (z \in U),
\]
because of
\[
\text{Re} \left( \frac{1 + \beta[1 - (1 + \alpha)\gamma]z}{1 - \alpha\beta z} \right) > \gamma \quad (z \in U).
\]

In order to give the coefficient estimate of functions belonging to the class \(K_k^A(\gamma, \alpha, \beta)\), we shall require the following lemma.

2.3. Lemma. [7] Let
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^* \left( k - \frac{1}{k} \right),
\]
then
\[
G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} B_n z^n \in S^* \subset S,
\]
where \(g_k(z)\) is given by (1.3).

2.4. Remark. (i) In particular, for \(k = 2\), the coefficients \(B_n\) in (2.4) is expressed as follows:
\[
B_{2n-1} = 2b_{2n-1} - 2b_2b_{2n-2} + \cdots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1} b_2^2.
\]
(ii) If \(g(z) \in S^*((k-1)/k)\), then from Lemma 2.3., \(G_k(z)\) given by (2.4) belongs to \(S^*\). Then by (2.2), we see that the class \(K_k^A(\gamma, \alpha, \beta)\) is a subclass of the class \(K\) of close-to-convex functions.

Next, we prove the sufficient condition for functions to belong to the class \(K_k^A(\gamma, \alpha, \beta)\).

2.5. Theorem. Let \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) and \(g(z) = z + \sum_{n=2}^{\infty} b_n z^n\) be analytic in \(U\). If
\[
\sum_{n=2}^{\infty} (1 + \alpha\beta)n |a_n| + \sum_{n=2}^{\infty} [1 + \beta(1 - (1 + \alpha)\gamma)] |B_n| \leq \beta(1 + \alpha)(1 - \gamma) \quad (0 \leq \alpha \leq 1; \ 0 < \beta \leq 1; \ 0 \leq \gamma < 1)
\]
where the coefficients \(B_n\) \((n = 2, 3, \cdots)\) are given by (2.4), then \(f(z) \in K_k^A(\gamma, \alpha, \beta)\).

Proof. Let the functions \(f(z)\) and \(g_k(z)\) be given by (1.1) and (1.3), respectively. Now, we obtain
\[
\Delta = \left| zf'(z) - \frac{g_k(z)}{z^{k-1}} \right| - \beta \left| \alpha zf'(z) + \frac{[1 - (1 + \alpha)\gamma]g_k(z)}{z^{k-1}} \right|
\]
\[
= \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} B_n z^n.
\]
Thus, for \( |z| = r (0 \leq r < 1) \), we have, from (2.5),

\[
\Delta \leq \sum_{n=2}^{\infty} n |a_n| |z|^n + \sum_{n=2}^{\infty} |B_n| |z|^n
- \beta \left( (1 + \alpha)(1 - \gamma) |z| - \alpha \sum_{n=2}^{\infty} n |a_n| |z|^n - |1 - (1 + \alpha)\gamma| \sum_{n=2}^{\infty} |B_n| |z|^n \right)
\]

\[
= -\beta(1 + \alpha)(1 - \gamma) |z| + \sum_{n=2}^{\infty} (1 + \alpha \beta) n |a_n| |z|^n + \\
\sum_{n=2}^{\infty} [1 + \beta |1 - (1 + \alpha)\gamma|] |B_n| |z|^n
< \left( -\beta(1 + \alpha)(1 - \gamma) + \sum_{n=2}^{\infty} (1 + \alpha \beta) n |a_n| + \sum_{n=2}^{\infty} [1 + \beta |1 - (1 + \alpha)\gamma|] |B_n| \right)
\]

\[
\leq 0.
\]

Thus we have

\[
\left| \frac{z^k f'(z)}{g_k(z)} - 1 \right| < \beta \left| \frac{az^k f'(z)}{g_k(z)} + 1 - (1 + \alpha)\gamma \right|
\]

that is, \( f(z) \in K_{\gamma}^{(k)}(\alpha, \beta) \). This completes the proof of Theorem 2.5. \( \square \)

In the following theorem, we give the coefficient estimates of functions belonging to the class \( K_{\gamma}^{(k)}(\alpha, \beta) \).

2.6. Theorem. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \), \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S \), and satisfy the inequality (2.1). Then, for, \( n \geq 2 \), we have

\[
|na_n - B_n|^2 - [\beta(1 + \alpha)(1 - \gamma)]^2
\]

\[
\leq (1 + \beta [(1 + \alpha)\gamma - 1]) \sum_{k=2}^{n-1} \left\{ 2k |a_k B_k| + [1 + \beta (1 + \alpha)\gamma - 1] |B_k|^2 \right\}
\]

where \( B_n \) is given by (2.4).

Proof. Suppose that the condition (1.2) is satisfied. Then, by using the a similar method as in the proof of (p. 30, [5]), we have

\[
\frac{zf'(z)}{G_k(z)} = 1 + [1 + \alpha \beta - 1] z\phi(z) \quad (z \in \mathbb{U}),
\]

where \( \phi \) is analytic in \( \mathbb{U} \), \( |\phi(z)| \leq \beta \) for \( z \in \mathbb{U} \) and \( G_k(z) \) is given by (2.4). Then from (2.7), we have

\[
(\alpha z f'(z) - [(1 + \alpha)\gamma - 1] G_k(z)) \phi(z) = G_k(z) - zf'(z)
\]

Thus, putting

\[
z\phi(z) = \sum_{n=1}^{\infty} t_n z^n,
\]
A subclass of close-to-convex functions

we obtain

\[
(1 + \alpha)(1 - \gamma)z + \alpha \sum_{n=2}^{\infty} na_n z^n - [(1 + \alpha)\gamma - 1] \sum_{n=2}^{\infty} B_n z^n \sum_{n=1}^{\infty} t_n z^n
\]

(2.8)

\[
= \sum_{n=2}^{\infty} B_n z^n - \sum_{n=2}^{\infty} na_n z^n.
\]

Equating the coefficient of \(z^n\) in (2.8), we have

\[
B_n - na_n = (1 + \alpha)(1 - \gamma) t_{n-1} + (2\alpha a_2 - [(1 + \alpha)\gamma - 1] B_2) t_{n-2} + \ldots + ((n-1)\alpha a_{n-1} - [(1 + \alpha)\gamma - 1] B_{n-1}) t_1.
\]

Thus the coefficient combination on the right side of (2.8) depends only upon the coefficient combinations:

\[
\{2\alpha a_2 - [(1 + \alpha)\gamma - 1] B_2\}, \ldots, \{(n-1)\alpha a_{n-1} - [(1 + \alpha)\gamma - 1] B_{n-1}\}.
\]

Hence for \(n \geq 2\), the equation (2.8) can be written as

\[
\left[(1 + \alpha)(1 - \gamma)z + \sum_{k=2}^{n-1} (ka_k - [(1 + \alpha)\gamma - 1] B_k) z^k\right] z\phi(z)
\]

(2.9)

\[
= \sum_{k=2}^{n} (B_k - ka_k) z^k + \sum_{k=n+1}^{\infty} c_k z^k.
\]

Then, squaring the modulus of the both sides of (2.9) and integrating along \(|z| = r < 1\), so that by Parseval's identity (p. 192, [1]), we obtain

\[
\sum_{k=2}^{n} |ka_k - B_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k}
\]

(2.10)

\[
\leq \beta^2 \left([(1 + \alpha)(1 - \gamma)]^2 r^2 + \sum_{k=2}^{n-1} |ka_k - [(1 + \alpha)\gamma - 1] B_k|^2 r^{2k}\right).
\]

Letting \(r \to 1\) on the left side of (2.10), we obtain

\[
\sum_{k=2}^{n} |ka_k - B_k|^2 \leq \beta^2 \left([(1 + \alpha)(1 - \gamma)]^2 + \sum_{k=2}^{n-1} |ka_k - [(1 + \alpha)\gamma - 1] B_k|^2\right).
\]

Hence we have

\[
|na_n - B_n|^2 < \beta^2 [(1 + \alpha)(1 - \gamma)]^2 + \beta^2 \sum_{k=2}^{n-1} |ka_k - [(1 + \alpha)\gamma - 1] B_k|^2 - \left[\sum_{k=2}^{n-1} |ka_k - B_k|^2 = \right.
\]

\[
= \beta [(1 + \alpha)(1 - \gamma)]^2 + \beta^2 \sum_{k=2}^{n-1} k^2 |a_k|^2 + \left.\{(\beta [(1 + \alpha)\gamma - 1])^2 - 1\} \sum_{k=2}^{n-1} |B_k|^2 + (\alpha \beta^2 |(1 + \alpha)\gamma - 1| + 1) \sum_{k=2}^{n-1} 2k |a_k||B_k| \leq \right.
\]

\[
\sum_{k=2}^{n-1} 2k |a_k||B_k|.
\]
\[
\leq [\beta(1 + \alpha)(1 - \gamma)]^2 + (\beta |(1 + \alpha)\gamma - 1| + 1)^2 \sum_{k=2}^{n-1} |B_k|^2 + \\
+ (\beta |(1 + \alpha)\gamma - 1| + 1) \sum_{k=2}^{n-1} 2k|a_k||B_k|,
\]
which implies the inequality (2.6). Therefore, we complete the proof of Theorem 2.6. \(\Box\)

Finally, we provide the growth and the distortion theorems for functions belonging to the class \(K_{\alpha}^{(k)}(\gamma, \alpha, \beta)\).

2.7. Theorem. If \(f(z) \in K_{\alpha}^{(k)}(\gamma, \alpha, \beta)\), then

\[
(2.11) \quad \frac{1 - \beta |1 - (1 + \alpha)\gamma|}{(1 + \alpha\beta r)(1 + r^2)} \leq |f'(z)| \leq \frac{1 + \beta |1 - (1 + \alpha)\gamma|}{(1 - \alpha\beta r)(1 + r^2)} \quad (|z| = r < 1)
\]

and

\[
(2.12) \quad \frac{\beta(1 + \alpha)(1 - \gamma)}{(1 - \alpha\beta)^2} \ln \frac{1 + \alpha\beta r}{1 + r} + \frac{1 + \beta |1 - (1 + \alpha)\gamma|r}{(1 - \alpha\beta)(1 + r)} \leq |f(z)| \\
\leq \frac{\beta(1 + \alpha)(1 - \gamma)}{(1 - \alpha\beta)^2} \ln(1 - \alpha\beta r)(1 - r) - \frac{1 + \beta |1 - (1 + \alpha)\gamma|r}{(1 - \alpha\beta)(1 - r)} \quad (|z| = r < 1),
\]

The results are sharp.

Proof. If \(f(z) \in K_{\alpha}^{(k)}(\gamma, \alpha, \beta)\), then there exists function \(g_k(z)\) satisfying (1.2). Then it follows from the Lemma 2.3. that the function \(G_k(z)\) given by (2.4) is a starlike function. Hence from (p. 70, [1]), we have

\[
(2.13) \quad \frac{r}{1 + r^2} \leq |G_k(z)| \leq \frac{r}{1 - r^2} \quad (|z| = r < 1).
\]

Let us define \(p(z)\) by

\[
p(z) = \frac{zf'(z)}{G_k(z)} \quad (z \in U).
\]

Then by using a similar method as in (p. 105, [3]), we have

\[
(2.14) \quad \frac{1 - \beta |1 - (1 + \alpha)\gamma|}{1 + \alpha\beta r} \leq |p(z)| \leq \frac{1 + \beta |1 - (1 + \alpha)\gamma|}{1 - \alpha\beta r} \quad (|z| = r < 1).
\]

Thus from (2.13) and (2.14), we have

\[
\frac{1 - \beta |1 - (1 + \alpha)\gamma|}{(1 + \alpha\beta r)(1 + r^2)} \leq |f'(z)| \leq \frac{1 + \beta |1 - (1 + \alpha)\gamma|}{(1 - \alpha\beta r)(1 - r^2)} \quad (|z| = r < 1),
\]

which gives us (2.11). Upon integrating (2.11) from 0 to \(r\), we have the inequality (2.12). Moreover, the results are sharp for the functions given, respectively, by

\[
f_1(z) = \frac{\beta(1 + \alpha)(1 - \gamma)}{(1 - \alpha\beta)^2} \ln \frac{1 + \alpha\beta z}{1 + z} + \frac{1 + \beta |1 - (1 + \alpha)\gamma|z}{(1 - \alpha\beta)(1 + z)} \quad (z \in U)
\]

and

\[
f_2(z) = \frac{\beta(1 + \alpha)(1 - \gamma)}{(1 - \alpha\beta)^2} \ln(1 - \alpha\beta z)(1 - z) - \frac{1 + \beta |1 - (1 + \alpha)\gamma|z}{(1 - \alpha\beta)(1 - z)} \quad (z \in U).
\]

\(\Box\)
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