RELATED FIXED POINTS FOR TWO PAIRS
OF SET VALUED MAPPINGS ON TWO
METRIC SPACES

Vijendra K. Chourasia* and Brian Fisher†

Received 10. 07. 2003 : Accepted 05. 01. 2004

Abstract
A related fixed point theorem for two pairs of set valued mappings on
two complete metric spaces is proved.

Keywords: Set valued mapping, Complete metric space, Fixed point.

1. Introduction
In the following we let \((X, d)\) be a complete metric space and \(B(X)\) the set of all nonempty subsets of \(X\). As in [1] and [2] we define the function \(\delta(A, B)\) with \(A\) and \(B\) in \(B(X)\) by \(\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}\). If \(A\) consists of a single point \(a\) we write \(\delta(A, B) = \delta(a, B)\). If \(B\) also consists of single point \(b\) we write \(\delta(A, B) = d(a, b)\).

It follows immediately that \(\delta(A, B) = \delta(b, A) \geq 0\), \(\delta(A, B) = 0\) implies \(A = B\) and this set is a singleton, and \(\delta(A, B) \leq \delta(A, C) + \delta(C, B)\) for all \(A, B\) in \(B(X)\).

If now \(\{A_n : n = 1, 2, \ldots\}\) is a sequence of sets in \(B(X)\), we say that it converges to the closed set \(A\) in \(B(X)\) if

(i) each point \(a \in A\) is the limit of some convergent sequence \(\{a_n \in A_n : n = 1, 2, \ldots\}\), and
(ii) for arbitrary \(\epsilon > 0\), there exists an integer \(N\) such that \(A_n \subseteq A\), for \(n > N\), where \(A_\epsilon\) is the union of all open spheres with centres in \(A\) and radius \(\epsilon\).

The set \(A\) is then said to be the limit of the sequence \(\{A_n\}\).

The following lemma was proved in [2].

1.1. Lemma. If \(\{A_n\}\) and \(\{B_n\}\) are sequences of bounded subsets of a complete metric space \((X, d)\) which converge to the bounded subsets \(A\) and \(B\), respectively, then the sequence \(\{\delta(A_n, B_n)\}\) converges to \(\delta(A, B)\).

*Department of Mathematics, R. D. Govt. College, Mandla, M. P. 481661, India.
†Department of Mathematics and Computer Science, University of Leicester, LE1 7RH, U.K.
E-mail: fbr@le.ac.uk
Now let $F$ be a mapping of $X$ into $B(X)$. We say that the mapping $F$ is continuous at a point $x$ if whenever $\{x_n\}$ is a sequence of points in $X$ converging to $x$, the sequence $\{Fx_n\}$ in $B(X)$ converges to $Fx$ in $B(X)$.

We say that $F$ is a continuous mapping of $X$ into $B(X)$ if $F$ is continuous at each point $x$ in $X$. We say that a point $z$ in $X$ is a fixed point of $F$ if $z$ is in $Fz$.

If $A$ is in $B(X)$ we define the set $FA = \bigcup_{a \in A} Fa$.

The following theorem was proved in [4].

1.2. Theorem. Let $(X, d_1)$ and $(Y, d_2)$ be complete metrics spaces, let $F$ be a mapping of $X$ into $B(Y)$ and $G$ a mapping of $Y$ into $B(X)$ satisfying the inequalities

\[
\delta_1(GFx, GFx') \leq c \max\{d_1(x, x'), \delta_1(x, GFx), \delta_1(x', GFx')\},
\]

\[
\delta_2(FGy, FGy') \leq c \max\{d_2(y, y'), \delta_2(y, FGy), \delta_2(y', FGy')\},
\]

for all $x, x'$ in $X$ and $y, y'$ in $Y$, where $0 \leq c < 1$. If $F$ is continuous, then $GF$ has a unique fixed point $z$ in $X$ and $FG$ has a unique fixed point $w$ in $Y$.

2. Results

We now prove the following generalization of Theorem 1.2.

2.1. Theorem. Let $(X, d_1)$ and $(Y, d_2)$ be complete metrics spaces, let $F$ and $G$ be mappings of $X$ into $B(Y)$ and $P$ and $Q$ mappings of $Y$ into $B(X)$ satisfying the inequalities

\[(1)\quad \delta_1(PPx, QGx) \leq \max\{d_1(x, x'), \delta_1(x, PPx), \delta_1(x', QGx)\},\]

\[(2)\quad \delta_1(GPy, FQy) \leq \max\{d_2(y, y'), \delta_2(y, GPy), \delta_2(y', FQy)\},\]

for all $x, x'$ in $X$ and $y, y'$ in $Y$, where $0 \leq c < 1$. If $F$ and $G$ are continuous, then $PF$ and $QG$ have a unique fixed point $z$ in $X$ and $GP$ and $FQ$ have a unique fixed point $w$ in $Y$.

Proof. Let $x_1$ be an arbitrary point in $X$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $X$ and $Y$ respectively as follows. Choose a point $y_1$ in $Fx_1$, a point $x_2$ in $Py_1$, a point $y_2$ in $Gx_2$ and then a point $x_3$ in $Qy_2$. In general, having chosen $x_n$ in $X$ and $y_n$ in $Y$, choose a point $y_{n+1}$ in $Fx_{n+1}$, a point $x_{n+2}$ in $Py_{n+1}$, a point $y_{n+2}$ in $Gx_{n+2}$ and then a point $x_{n+3}$ in $Qy_{n+2}$ for $n = 1, 2, \ldots$. Then, using inequality $(1)$, we have

\[
d_1(x_{2n+2}, x_{2n+1}) \leq \delta_1(PPx_{2n+1}, QGx_{2n}) \leq c \max\{d_1(x_{2n+1}, y_{2n+1}), \delta_1(x_{2n+1}, PPx_{2n+1})\} \leq c \max\{d_1(x_{2n+1}, QGx_{2n}), \delta_1(x_{2n+1}, GFx_{2n})\} \leq c \max\{d_1(QGx_{2n}, PPx_{2n-1}), \delta_1(QGx_{2n}, PFx_{2n+1})\} \leq \delta_2(PFx_{2n-1}, QGx_{2n}), \delta_2(Fx_{2n+1}, Gx_{2n})\} \leq \delta_1(PFx_{2n-1}, QGx_{2n}), \delta_2(GPy_{2n-1}, FQy_{2n})\},\]

since

\[
\delta_2(GFx_{2n+1}, Gx_{2n}) \leq \delta_2(GPy_{2n-1}, FQy_{2n}).
\]

Similarly, using inequality $(1)$ again, we have

\[
d_1(x_{2n+2}, x_{2n+3}) \leq \delta_2(GPy_{2n+2}, FQy_{2n+3}) \leq \delta_2(GPy_{2n+1}, FQy_{2n+2}), \delta_2(GPy_{2n+1}, FQy_{2n+2})\}.
\]
Using inequality (2), we have
\[ d_2(y_{2n+1}, y_{2n+2}) \leq \delta_2(FQy_{2n}, GPy_{2n+1}) \]
\[ \leq c \max\{d_2(y_{2n}, y_{2n+1}), \delta_2(y_{2n+1}, GPy_{2n+1}), \]
\[ \delta_2(y_{2n}, FQy_{2n}), \delta_1(Py_{2n+1}, Qy_{2n})\} \]
\[ \leq c \max\{\delta_2(GPy_{2n-1}, FQy_{2n}), \delta_2(FQy_{2n}, GPy_{2n+1}), \]
\[ \delta_2(GPy_{2n-1}, FQy_{2n}), \delta_1(Py_{2n+1}, Qy_{2n})\} \]
\[ \leq c \max\{\delta_2(GPy_{2n-1}, FQy_{2n}), \delta_1(PFx_{2n+1}, QGx_{2n})\}, \]

(5)

since
\[ \delta_1(Py_{2n+1}, Qy_{2n}) \leq \delta_1(PFx_{2n+1}, QGx_{2n}). \]

Similarly, using inequality (2) again, we have
\[ d_2(y_{2n+2}, y_{2n+3}) \leq \delta_2(GPy_{2n+1}, FQy_{2n+2}) \]
\[ \leq c \max\{\delta_2(GPy_{2n+1}, FQy_{2n}), \delta_1(PFx_{2n+1}, QGx_{2n+2})\}. \]

(6)

We will now prove that
\[ \delta_1(PFx_{2n+1}, QGx_{2n}) \leq c^n K, \]
\[ \delta_1(PFx_{2n+1}, QGx_{2n+2}) \leq c^n K, \]
\[ \delta_2(FQy_{2n}, GPy_{2n+1}) \leq c^n K, \]
\[ \delta_2(GPy_{2n+1}, FQy_{2n+2}) \leq c^n K, \]

(7), (8), (9), (10)

where
\[ K = \max\{\delta_1(PFx_1, QGx_2), \delta_1(PFx_3, QGx_2), \delta_1(PFx_3, QGx_4), \]
\[ \delta_2(GPy_1, FQy_2), \delta_2(GPy_3, FQy_2)\}, \]

for \( n = 1, 2, \ldots \).

Inequalities (7) to (10) clearly hold when \( n = 1 \). Suppose inequalities (7) to (10) hold for some \( n \). Then it follows from inequality (3) that
\[ \delta_1(PFx_{2n+3}, QGx_{2n+2}) \leq c \max\{\delta_1(PFx_{2n+1}, QGx_{2n+2}), \delta_2(GPy_{2n+1}, FQy_{2n+2})\} \]
\[ \leq c^{n+1} K \]
on using our assumptions on inequalities (8) and (10). Inequality (7) now follows by induction.

Using inequality (5), we have
\[ \delta_2(FQy_{2n+2}, GPy_{2n+3}) \leq c \max\{\delta_2(GPy_{2n+1}, FQy_{2n+2}), \delta_1(PFx_{2n+3}, QGx_{2n+2})\} \]
\[ \leq c^{n+1} K, \]
on using inequality (7) and our assumption on inequality (10). Inequality (9) now follows by induction.

Using inequality (4), we have
\[ \delta_1(PFx_{2n+3}, QGx_{2n+4}) \leq c \max\{\delta_1(PFx_{2n+3}, QGx_{2n+2}), \delta_2(GPy_{2n+3}, FQy_{2n+2})\} \]
\[ \leq c^{n+1} K, \]
on using inequalities (7) and (9). Inequality (8) now follows by induction.

Finally, using inequality (6), we have
\[ \delta_2(GPy_{2n+3}, FQy_{2n+4}) \leq c \max\{\delta_2(GPy_{2n+3}, FQy_{2n+2}), \delta_1(PFx_{2n+3}, QGx_{2n+4})\} \]
\[ \leq c^{n+1} K, \]
on using inequalities (8) and (9). Inequality (10) now follows by induction.

It follows that, for \( r = 1, 2, \ldots \),

\[
d_1(x_{2n+1}, x_{2n+r+1}) \leq d_1(x_{2n+1}, x_{2n+2}) + d_1(x_{2n+2}, x_{2n+3}) + \ldots
\]

\[
+ d_1(x_{2n+r}, x_{2n+r+1})
\]

\[
\leq \delta_1(QGx_{2n}, PFx_{2n+1}) + \delta_1(PFx_{2n+1}, QGx_{2n+2}) + \ldots
\]

\[
\leq (c^n + c^{n+1} + c^{n+1} + \ldots)K < \epsilon,
\]

for \( n \) greater than some \( N \), since \( c < 1 \). The sequence \( \{x_n\} \) is therefore a Cauchy sequence in the complete metric space \( X \), and so has a limit \( z \) in \( X \). Similarly the sequence \( \{y_n\} \) is a Cauchy sequence in the complete metric space \( Y \) and so has a limit \( w \) in \( Y \).

Further, with \( m > n \), we have

\[
\delta_1(QGx_{2n}, PFx_{2m+1}) \leq \delta_1(QGx_{2n}, PFx_{2n+1}) + \delta_1(PFx_{2n+1}, QGx_{2n+2}) + \ldots + \delta_1(QGx_{2m}, PFx_{2m+1})
\]

\[
\leq (c^n + c^{n+1} + c^{n+1} + \ldots)K < \epsilon
\]

for \( n > N \). Next, we have

\[
\delta_1(z, QGx_{2n}) \leq d_1(z, x_{2m+2}) + \delta_1(x_{2m+2}, QGx_{2n})
\]

\[
\leq d_1(z, x_{2m+2}) + \delta_1(PFx_{2m+1}, QGx_{2n}),
\]

since \( x_{2m+2} \in PFx_{2m+1} \). Thus, on using inequality (11), we have

\[
\delta_1(z, QGx_{2n}) \leq d_1(z, x_{2m+2}) + \epsilon
\]

for \( m > n > N \). Letting \( m \) tend to infinity it follows that

\[
\delta_1(z, QGx_{2n}) \leq \epsilon
\]

for \( n > N \), and so

\[
\lim_{n \to \infty} QGx_{2n} = \{z\},
\]

since \( \epsilon \) is arbitrary.

Similarly,

\[
\lim_{n \to \infty} PFx_{2n+1} = \{z\},
\]

\[
\lim_{n \to \infty} GFy_{2n+1} = \{w\} = \lim_{n \to \infty} FQy_{2n}.
\]

From the continuity of \( F \) of \( G \), we have

\[
\lim_{n \to \infty} FX_{2n+1} = Fz = \{w\},
\]

\[
\lim_{n \to \infty} GX_{2n} = Gz = \{w\}.
\]

Using inequality (1), we now have

\[
\delta_1(PFx, QGx_{2n}) \leq c \max\{d_1(z, x_{2n}), \delta_1(z, PFx), \delta_1(x_{2n}, QGx_{2n}), \delta_2(Fz, Gx_{2n})\}.
\]

Letting \( n \) tend to infinity, and using equations (12) and (16), we have

\[
\delta_1(PFx, z) \leq c \delta_1(PFx, z).
\]

Since \( c < 1 \), we must have

\[
PFz = \{z\} = Pw,
\]

on using equation (15), proving that \( z \) is a fixed point of \( PF \).
Using inequality (1) again, we now have
\[ \delta_1(x_{2n+2}, QGz) \leq \delta_1(PFz_{2n+1}, QGz) \]
\[ \leq c \max\{d_1(x_{2n+1}, z), \delta_1(x_{2n+1}, PFz_{2n+1}), \delta_1(z, QGz), \delta_1(Fz_{2n+1}, Gz)\}. \]

Letting \( n \) tend to infinity, and using equations (13), (15) and (16), we have
\[ \delta_1(z, QGz) \leq c \delta_1(z, QGz). \]

Since \( c < 1 \), we must have
\[ (18) \quad QGz = \{z\} = Qw, \]
on using equation (16), proving that \( z \) is also a fixed point of \( QG \).

It now follows from equations (15) and (18) that
\[ FQw = Fz = \{w\}, \]
and it follows from equations (16) and (17) that
\[ GPw = Gz = \{w\}. \]

Therefore, \( w \) is a fixed point of \( FQ \) and \( GP \).

To prove uniqueness, suppose that \( PF \) and \( QG \) have a second common fixed point \( z' \).

Then using inequalities (1) and (2), we have
\[ \max\{\delta_1(z', QGz'), \delta_1(z', PFz')\} \leq \delta_1(PFz', QGz') \]
\[ \leq c \max\{d_1(z', z), \delta_1(z', PFz'), \delta_1(z', QGz'), \delta_2(PFz', Gz')\} \]
\[ = c \delta_2(Fz', Gz') \]
\[ \leq c \max\{\delta_2(Fz', Gz'), \delta_2(Fz', GPFz'), \delta_2(Gz', FQGz'), \delta_2(PFz', QGz')\} \]
\[ \leq c^2 \max\{\delta_2(GPFz', FQGz'), \delta_1(PFz', QGz')\} \]
\[ = c^2 \delta_2(PFz', QGz') \]

and it follows that
\[ \max\{\delta_1(z', QGz'), \delta_1(z', PFz')\} = \delta_1(PFz', QGz') = \delta_2(Fz', Gz') = 0, \]
since \( c < 1 \). Thus \( Fz' \) and \( Gz' \) are singletons and
\[ PFz' = QGz' = \{z'\}. \]

Using inequalities (1) and (2) again, we have
\[ d_1(z, z') = d_1(PFz, QGz') \]
\[ \leq c \max\{d_1(z, z'), d_1(z, PFz), \delta_1(z', QGz'), \delta_2(Fz, Gz')\} \]
\[ = cd_2(Fz, Gz') \]
\[ = cd_2(Gz, Fz') \]
\[ \leq c \delta_2(GPFz, FQGz') \]
\[ \leq c^2 \max\{d_2(Fz, Gz'), \delta_2(Fz, GPFz), \delta_2(Gz', FQGz'), \delta_1(PFz, QGz')\} \]
\[ = c^2 \max\{d_2(Fz, Fz'), d_1(z, z')\} \]
\[ = c^2 d_1(z, z'). \]

Since \( c < 1 \), the uniqueness of \( z \) follows.
Similarly, \( w \) is the unique fixed point of \( GP \) and \( FQ \). This completes the proof of the theorem.

If we let \( F \) and \( G \) be single valued mappings of \( X \) into \( Y \) and let \( P \) and \( Q \) be single valued mappings of \( Y \) into \( X \), we obtain the following corollary, which generalizes a result given in [3].

**2.2. Corollary.** Let \((X, d_1)\) and \((Y, d_2)\) be complete metric spaces. If \( F \) and \( G \) are continuous mappings of \( X \) into \( Y \) and \( P \) and \( Q \) are mappings of \( Y \) into \( X \) satisfying the inequalities

\[
\begin{align*}
    d_1(PFx, QGx') &\leq c \max\{d_1(x, x'), d_1(x, PFx), d_1(x', QGx'), d_2(Fx, Gx')\}, \\
    d_2(GPy, FQy') &\leq c \max\{d_2(y, y'), d_2(y, GPy), d_2(y', FQy'), d_1(Py, Qy')\}
\end{align*}
\]

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \), where \( 0 \leq c < 1 \), then \( PF \) and \( QG \) have a unique fixed point \( z \) in \( X \) and \( GP \) and \( FQ \) have a unique fixed point \( w \) in \( Y \).

**References**


