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SOME NOTES ON DEDEKIND MODULES

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Abstract

In this paper, we give the relation between a finitely generated torsion free Dedekind module and the endomorphism ring of ${}_{O(M)}M$. In addition it is proved that the endomorphism ring of a finitely generated torsion free Dedekind module M is a Dedekind domain. Also, we give equivalent condition for Dedekind modules, duo modules and uniform modules. Various properties and characterizations of Dedekind modules over integral domains are considered and consequently, necessary and sufficient conditions for an R-module M to be a Dedekind module are given.

Keywords: Dedekind modules and Dedekind domains, Invertible submodules, Duo modules.

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1. Introduction

Throughout this paper all rings are commutative domains with identity and all modules are unitary.

A nonzero ideal I of R is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K : xI \subseteq R\}$. The concept of an invertible submodule was introduced in [7] as a generalization of the concept of an invertible ideal. Let M be an R-module and let $S = R - \{0\}$. Then

$$T = \{t \in S : tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$$

is a multiplicatively closed subset of R. Let N be a submodule of M and $N' = \{x \in R_T : xN \subseteq M\}$. A submodule N is said to be *invertible* in M, if $N'N = M$, [7]. Note that N' is an R-submodule of R_T with $R \subseteq N^{-1}$. A nonzero R-module M is called *Dedekind* provided that each nonzero submodule of M is invertible.

Let $O(M) = \{x \in K : xM \subseteq M\}$, the *order* of an R-module M in K. Then $O(M)$ is a subring of K with $R \subseteq O(M)$ and M is an $O(M)$-module.

Let M be any R-module. We denote the ring of R-endomorphisms of M by $\mathrm{End}(M)$.

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For any prime ideal P of R, $S = R - P$ is a multiplicatively closed set, and we denote $S^{-1}R$ by R_P and $S^{-1}M$ by M_P.

Let M be an R-module, the torsion submodule of M is

$$T(M) = \{m \in M : \exists\, 0 \neq r \in R \text{ such that } rm = 0\}.$$

Then M is called *torsion* if $T(M) = M$, and M is called *torsion free* if $T(M) = 0$.

In this note we prove that if M is a Dedekind R-module, then $T^{-1}R$ is a local ring and $T = R - \mathrm{Ann}(M)$, and M is a Dedekind R-module if and only if M is Dedekind as a $\frac{R}{\mathrm{Ann}(M)}$-module.

We also prove our main theorems, which may be stated as follows:

2.12. Theorem. *Let R be a Dedekind domain. Then the following are equivalent for a finitely generated torsion free R-module M:*

(1) *M is a duo module,*
(2) *M is a multiplication module,*
(3) *M is a Dedekind module,*
(4) *M is a uniform module,*
(5) $\mathrm{rank}_R M = 1$.

2.13. Theorem. *Let M be a finitely generated torsion free Dedekind R-module. Then the following are equivalent:*

(1) *R is a Dedekind domain,*
(2) *R is integrally closed,*
(3) *M is a multiplication module,*
(4) *M is a projective module,*
(5) *M is a flat module,*
(6) *M is a cancellation module,*
(7) *M is an duo module.*

2.21. Theorem. *The following statements are equivalent for a finitely generated torsion free R-module M:*

(1) *M is a Dedekind R-module,*
(2) *$O(M)$ is Noetherian, integrally closed and for each $O(M)$-submodule N of M, $(N :_{O(M)} M) = (O(M) :_{O(M)} N')$,*
(3) *$O(M)$ is a Dedekind domain and* $\mathrm{End}({}_R M) \cong O(M)$,
(4) *$O(M)$ is a Dedekind domain and M is a uniform R-module,*
(5) *$O(M)$ is a Dedekind domain and every nonzero prime submodule of an $O(M)$-module M is maximal.*

2. Dedekind modules

Before starting, we recall some necessary notations and known facts. Let M be an R-module and $S = R - \{0\}$. Then

$$T = \{t \in S : tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$$

is a multiplicatively closed subset of R. It is clear that if M is torsion free, then $T = S$. Now let $T^{-1}R$ be the localization of R at T in the usual sense. Following Naoum and Al-Alwan [7], we say that $xn \in M$, where $x = \frac{r}{t} \in T^{-1}R$ and $n \in M - \{0\}$, as long as there exists an element $m \in M$ such that $tm = rn$ for some $r \in R$. We start with the following proposition:

2.1. Proposition. *Let M be a Dedekind R-module, Then $T^{-1}M$ is a simple $T^{-1}R$-module.*

Proof. Let $\mathcal{N}$ be a nonzero submodule of $T^{-1}M$. Therefore there exists $N \leq M$ such that $\mathcal{N} = T^{-1}N$. Since N is an invertible submodule of M, $\mathcal{N} = T^{-1}N = T^{-1}M$. Thus $T^{-1}M$ is a simple $T^{-1}R$-module. □

2.2. Proposition. *Let M be a Dedekind R-module. Then for each $0 \neq m \in M$, $\mathrm{Ann}_R(m) = \mathrm{Ann}_R(M)$ and $\mathrm{Ann}_R(M)$ is a prime ideal of R.*

Proof. Let $0 \neq m \in M$. Since $T^{-1}M$ is a simple $T^{-1}R$-module, there exists a maximal ideal $\mathcal{P}$ of $T^{-1}R$ such that $\mathrm{Ann}_{T^{-1}R}(m) = \mathcal{P}$. We know that $\mathcal{P} = T^{-1}P$ for some prime ideal P of R. Thus $\mathrm{Ann}_{T^{-1}R}(m) = T^{-1}P$. Consequently, $\mathrm{Ann}_R(m) = P$. Since for each $0 \neq m \in M$, $\mathrm{Ann}_R(m) = P$, thus $\mathrm{Ann}_R(m) = \mathrm{Ann}_R(M) = P$. □

Recall that an R-module M is called *prime* whenever for each $0 \neq N \leq M$, $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(N)$.

2.3. Corollary. *The following hold for a Dedekind R-module M:*

(1) *M is either torsion or torsion free.*
(2) *M is torsion free if and only if it is a faithful R-module.*
(3) *M is a prime module.*

Proof. (1) By Proposition 2.2, if $T(M) \neq 0$ then $T(M) = M$.

(2) It is clear that every torsion free module is faithful.

Conversely, suppose $\mathrm{Ann}(M) = 0$. Then for each $0 \neq m \in M$, $\mathrm{Ann}(m) = 0$. Thus M is torsion free.

(3) Clear by Proposition 2.2. □

The following Theorem gives some important properties of Dedekind modules.

2.4. Theorem. *Let M be a Dedekind R-module.*

(1) *$T^{-1}R$ is a local ring.*
(2) *$\mathrm{Ass}_R(M) = \{\mathrm{Ann}_R(M)\}$*

Proof. (1) Let $\mathrm{Ann}_R(M) = P$. Since $T \cap P = \emptyset$, $T \subseteq R - P$. Let $r \notin P$, then for each $0 \neq m \in M$, $rm \neq 0$. Thus $T = R - P$, hence $T^{-1}R$ is a local ring.

(2) Clear from Proposition 2.2. □

The next proposition gives a condition for a Dedekind R-module to be a simple R-module.

2.5. Proposition. *Let M be a Dedekind R-module. Then M is simple if and only if $\mathrm{Ann}(M)$ is a maximal ideal.*

Proof. The necessity is clear. For the sufficiency, let $\mathrm{Ann}(M)$ be a maximal ideal of R. Then M is a direct sum of some simple modules, hence $M = Rm$. Therefore M is a simple R-module. □

2.6. Corollary. *Let M be a Dedekind R-module and suppose that the Krull dimension of R is 1. If M is a torsion module, then M is a simple R-module.* □

We leave the proof of the following lemma to the reader.

2.7. Lemma. *Let M be a finitely generated Dedekind R-module. Then $(N : M) = \mathrm{Ann}(M)$ if and only if $N = 0$.* □

For a Dedekind R-module M, P denotes $\mathrm{Ann}(M)$. In the next lemma we take M to be an $\frac{R}{P}$-module by natural multiplication.

2.8. Lemma. *Let M be an R-module. Then M is a Dedekind R-module if and only if it is a Dedekind $\frac{R}{P}$-module.*

Proof. Let $N \leq M$, and put

$$N'' = \left\{\gamma = \frac{r}{t} + T^{-1}P \in \frac{T^{-1}R}{T^{-1}P} : \gamma N \subseteq M\right\},$$
$$N' = \left\{\delta = \frac{r'}{t'} \in T^{-1}R : \delta N \subseteq M\right\}.$$

By natural multiplication, $N''N = M$ if and only if $N'N = M$. Therefore M is a Dedekind R-module if and only if it is a Dedekind $\frac{R}{P}$-module. □

2.9. Corollary. *Let M be a finitely generated Dedekind R-module. If M is a divisible $\frac{R}{\mathrm{Ann}(M)}$-module, then M is a simple R-module.*

Proof. By [1, Corollary 3.8], $M = Rm$ for some $m \in M$. Thus $\frac{R}{\mathrm{Ann}(M)}$ is a field. Therefore by Proposition 2.5, M is a simple R-module. □

From the previous properties we can extend [1, Theorem 3.14]. A proper submodule N of M is *P-prime* if for any $r \in R$ and $m \in M$ such that $rm \in N$, either $rM \subseteq N$ or $m \in M$ where $P = (N : M)$.

2.10. Corollary. *The following are equivalent for a finitely generated Dedekind R-module M such that $\frac{R}{P}$ is an integrally closed domain:*

(1) *M is Dedekind,*
(2) *M is Noetherian and every nonzero prime submodule of M is maximal,*
(3) *M is Noetherian and for any maximal ideal $\bar{m}$ of R containing P, there exists $\pi \in R_{\bar{m}}$ such that every submodule of $M_{\bar{m}}$ is of the form $\pi^n M_{\bar{m}}$ for some $n \in \mathbb{N}$.*

Proof. Clear by [1, Theorem 3.14]. □

Recall that an R-module M is *uniform* in case any two nonzero submodules of M have nonzero intersection.

2.11. Lemma. *Let M be a Dedekind R-module. Then M is uniform module.*

Proof. Let $N,\ N'$ be two nonzero submodules of M. We will show that $N \cap N' \neq 0$. Let $N \cap N' = 0$. Let $n \in N$ be such that $n \notin N'$. Since N' is an invertible submodule of M, there exists $t \in T$ such that $tn \in N'$. Thus $tn = 0$, so $n = 0$, a contradiction. Thus for each nonzero submodule $N,\ N'$ of M, $N \cap N' \neq 0$. □

From now on, unless otherwise stated, we take M to be a finitely generated torsion free module over a domain R.

Now we recall that an R-module M is called *multiplication* when for each submodule N of M, there exists an ideal I of R such that $N = IM$. If M is a torsion module then it is easily seen that M is multiplication as an R-module if and only if M is multiplication as an $\frac{R}{\mathrm{Ann}(M)}$-module by natural multiplication.

Recall that if R is an integral domain with the quotient field K, the *rank* of an R-module M is defined to be the greatest number of elements of M linearly independent over R. It is easy to see that $\mathrm{rank}_R M = \dim_K S^{-1}M$.

A submodule N of M is called *fully invariant* if $f(N) \subseteq N$, for each R-endomorphism f of M. An R-module M is called *duo* provided that every submodule of M is fully invariant, [9].

2.12. Theorem. *Let R be a Dedekind domain. Then the following are equivalent for an R-module M:*

(1) *M is a duo module,*
(2) *M is multiplication,*
(3) *M is a Dedekind module,*
(4) *M is a uniform module,*
(5) $\mathrm{rank}_R M = 1$.

Proof. (1) $\Longrightarrow$ (2) It is well-known that every torsion free module over a Dedekind domain is projective. By [11, Theorem A], since M is projective and every submodule of M is fully invariant, M is a multiplication R-module.

(2) $\Longrightarrow$ (3) This is [7, Theorem 3.4].

(3) $\Longrightarrow$ (4) This follows from Lemma 2.11.

(4) $\Longrightarrow$ (1) Since R is integrally closed, therefore M is a duo module [9, Corollary 3.4].

(3) $\Longrightarrow$ (5) Let M be a Dedekind R-module. Then $S^{-1}M$ is a simple $S^{-1}R$-module. Thus $\mathrm{rank}_R M = 1$.

(5) $\Longrightarrow$ (3) Let $\mathrm{rank}_R M = 1$. By [1, Corollary 3.7], $M \cong I$ for some ideal I of R. Thus M is a Dedekind R-module. □

An R - module M is called *cancellation*, if for all ideals I and J of R, $IM \subseteq JM$ implies $I \subseteq J$, [2].

An R-module M is said to be *integrally closed* whenever $y^n m_n + \cdots + y m_1 + m_0 = 0$ for some $n \in \mathbb{N}$, $y \in T^{-1}R$ and $m_i \in M$, then $y m_n \in M$, [1].

In the following theorem, we extend [1, Theorem 3.12].

2.13. Theorem. *Let M be a Dedekind R-module. Then the following are equivalent:*

(1) *R is a Dedekind domain,*
(2) *R is integrally closed,*
(3) *M is multiplication,*
(4) *M is a projective module,*
(5) *M is a flat module,*
(6) *M is a cancellation module,*
(7) *M is a duo module.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) are by [1, Theorem 3.12].

(4) $\Longleftrightarrow$ (5) By [10, Corollary 6], M is a finitely presented R-module. Therefore M is a projective R-module if and only if it is a flat R-module.

(1) $\Longrightarrow$ (6) Since every Dedekind module over a Dedekind domain is multiplication, M is cancellation, [4, Theorem 3.1].

(6) $\Longrightarrow$ (1) Let M be a cancellation module and I any nonzero ideal of R. Then

$$(IM)' = \{q \in K : \ qIM \subseteq M\} = \{q \in K : \ qI \subseteq R\} = I^{-1}.$$

Therefore $I^{-1}IM = M$. Since M is cancellation, $I^{-1}I = R$. Thus R is a Dedekind domain.

(2) $\Longrightarrow$ (7) Since M is uniform and R is integrally closed, M is a duo module, [9, Corollary 3.4].

(7) $\Longrightarrow$ (2) By using [9, Theorem 3.7] and [10, Lemma 2], since M is a duo module, R is an integrally closed domain. □

Now let $O(M) = \{x \in K : xM \subseteq M\}$, the *order* of M in K. Then $O(M)$ is a subring of K with $R \subseteq O(M)$, and M is an $O(M)$-module. We will use the notation ${}_{O(M)}M$ to indicate that M is regarded as an $O(M)$-module.

In the following, $\bar{R}$ denote the integral closure of R.

2.14. Lemma. *Let M be an integrally closed R-module. Then $O(M) = \bar{R}$.*

Proof. Let $x \in O(M)$. Then $xM \subseteq M$. By a determinant argument, $x \in \bar{R}$.

Let $x \in \bar{R}$, therefore there exist $b_0, b_1, \ldots, b_{n-1} \in R$ and $n \in \mathbb{N}$ such that $b_0 + b_1 x + \cdots + b_{n-1}x^{n-1} + x^n = 0$. Thus for each $m \in M$, $b_0 m + b_1 mx + \cdots + b_{n-1}mx^{n-1} + mx^n = 0$. Since M is integrally closed, $xm \in M$. This completes the proof. □

2.15. Theorem. *The R-module M is Dedekind if and only if*

(1) *$O(M)$ is integrally closed.*
(2) *$O(M)$ is Noetherian.*
(3) *For each $O(M)$ - submodule N of M, $(N :_{O(M)} M) = (O(M) :_{O(M)} N')$.*

Proof. $\Longrightarrow$ By [10, Lemma 2] and [1, Proposition 3.10)].

$\Longleftarrow$ We will show that every maximal submodule of ${}_{O(M)}M$ is invertible. Let L be a maximal submodule of ${}_{O(M)}M$. Since $L'L \subseteq M$ and $L \subseteq L'L$, thus $L'L = L$ or $L'L = M$. If $L'L = L$, then $L' = O(M)$ because L is finitely generated and $O(M)$ is integrally closed. Thus by (3), $(N :_{O(M)} M) = O(M)$, therefore $N = M$, a contradiction. Therefore $LL' = M$, thus every maximal submodule of ${}_{O(M)}M$ is invertible and by [1, Proposition 3.4], M is a Dedekind $O(M)$-module. Therefore, M is a Dedekind R-module. □

If M is an integrally closed R-module, then by using Lemma 2.14, $O(M) = \bar{R}$, thus $O(M)$ is integrally closed. Therefore in Theorem 2.15 we can replace (1) by 'M is an integrally closed R-module'. The converse is true because if M is a Dedekind R-module, then by [10, Lemma 2] and [1, Theorem 3.12], M is an integrally closed $O(M)$-module, therefore M is an integrally closed R-module.

Recall that an R-module M is called a *weak multiplication* module if the set of all prime submodules is the empty set, or for every prime submodule N of M, we have $N = IM$, where I is an ideal of R, [3].

We need to prove the following lemmas for our last theorem.

2.16. Lemma. *Let M be an R-module. Then every nonzero prime submodule of M is maximal if and only if M is a multiplication module and* $\dim R = 1$.

Proof. Let every nonzero prime submodule of M be maximal, thus by [1, Lemma 3.13], every prime submodule of M has the form PM where P is prime ideal of R and $\dim R = 1$. Thus M is a weak multiplication R-module. By [3, Theorem 2.7] M is a multiplication module.

Conversely, it is clear that every nonzero prime submodule of M is maximal. □

2.17. Lemma. *Let M be an R-module. Then* $\mathrm{End}({}_R M) = \mathrm{End}({}_{O(M)}M)$.

Proof. Let f be an R-endomorphism of M. It is easy to see that $f(\gamma m) = \gamma f(m)$ for each $m \in M$, $\gamma \in O(M)$. Therefore $\mathrm{End}({}_R M) = \mathrm{End}({}_{O(M)}M)$. □

The following proposition characterizes the endomorphism ring of a Dedekind R-module M.

2.18. Proposition. *Let M be a Dedekind R-module. Then* $\mathrm{End}({}_R M) \cong O(M)$.

Proof. By [10, Lemma 2], M is a Dedekind $O(M)$-module and $O(M)$ is a Dedekind domain. Therefore M is a multiplication module. By [8, Corollary 3.3], $\mathrm{End}(_{O(M)}M) \cong O(M)$. Thus Lemma 2.17 completes the proof. □

The following theorem explains the interrelation between Dedekind modules and the endomorphism ring of Dedekind modules.

2.19. Theorem. *The R-module M is Dedekind if and only if $O(M) \cong \mathrm{End}(_RM)$ and $O(M)$ is a Dedekind domain.*

Proof. The necessity is clear. For the sufficiency, since M is a finitely generated torsion free $O(M)$-module, we can regard M as a subspace of $\mathcal{S}^{-1}M = \sum_{i=1}^{n} \mathcal{S}^{-1}O(M)x_i$ over K, where $\mathcal{S} = O(M) - \{0\},\ x_i \in M$. Therefore M is a submodule of the free $O(M)$-module $M' = \sum_{i=1}^{n} O(M)x_i$. Since $O(M)$ is a Dedekind domain, M is a projective $O(M)$-module. It is proved that if M is projective and $\mathrm{End}(M)$ is a commutative ring, then M is a multiplication module, [11, Theorem A]. Thus M is a multiplication $O(M)$-module. By Lemma 2.16 every nonzero prime submodule of the $O(M)$-module M is maximal, therefore M is a Dedekind R-module, [10, Theorem 9]. □

2.20. Proposition. *An R-module M is Dedekind if and only if M is uniform and $O(M)$ is a Dedekind domain.*

Proof. Let M be a Dedekind R-module. Then by Lemma 2.11 and [10, Lemma 2], M is a uniform R-module and $O(M)$ is a Dedekind domain.

Conversely, since M is uniform, $\mathrm{End}(_RM) \cong O(M)$, [9, Lemma 3.2]. Thus Theorem 2.19 completes the proof. □

We can summarize the above theorems in the following theorem.

2.21. Theorem. *The following statements are equivalent for an R-module M,*

(1) *M is a Dedekind R-module,*
(2) *$O(M)$ is Noetherian, integrally closed and for each $O(M)$-submodule N of M, $(N :_{O(M)} M) = (O(M) :_{O(M)} N')$,*
(3) *$O(M)$ is a Dedekind domain and $\mathrm{End}(_RM) \cong O(M)$,*
(4) *$O(M)$ is a Dedekind domain and M is a uniform R-module,*
(5) *$O(M)$ is a Dedekind domain and every nonzero prime submodule of the $O(M)$-module M is maximal.*

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