ON A SUM OF THE PSI FUNCTION WITH A LOGARITHM

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Abstract
Let $\psi$ be the psi function, that is the logarithmic derivative of the Euler gamma function. The aim of this paper is to establish an asymptotic formula for the function $\psi(x) + \log \left( e^{1/x} - 1 \right)$ and to improve some results of Batir (Some new inequalities for gamma and polygamma functions, J. Ineq. Pure Appl. Math. 6(4), Art 103, 2005) and Alzer (Sharp inequalities for the harmonic numbers, Expo. Math. 24, 385-388, 2006). Finally we give a short proof of, respectively, the monotonicity and concavity of the function $\psi(x) + \log \left( e^{1/x} - 1 \right)$, previously stated by Alzer above, and by Guo and Qi (Some properties of the psi and polygamma functions, Hacet. J. Math. Stat. 39(2), 219–231, 2010).

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1. Introduction
Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the $n$th harmonic number. Alzer [3] proved

$$a - \log \left( e^{1/(n+1)} - 1 \right) \leq H_n < b - \log \left( e^{1/(n+1)} - 1 \right), \quad (n \geq 1),$$

with the best possible constants $a = 1 + \log \left( \sqrt{e} - 1 \right)$ and $b = \gamma$, where $\gamma = 0.577215\ldots$ is the Euler-Mascheroni constant. One year earlier, Batir [4, Cor. 2.2] established an inequality of type (1.1) with $a = \log \left( \pi^2/6 \right)$ and $b = \gamma$.

The harmonic numbers are related to the psi function $\psi$, which is the logarithmic derivative of the Euler gamma function:

$$\psi(x) = \frac{d}{dx} \log \Gamma(x).$$

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