SHARPENING AND GENERALIZATIONS OF CARLSON’S INEQUALITY FOR THE ARC COSINE FUNCTION

Bai-Ni Guo∗ and Feng Qi†

Received 03 : 12 : 2009 : Accepted 24 : 02 : 2010

Abstract
In this paper, we sharpen and generalize Carlson’s double inequality for the arc cosine function.

Keywords: Sharpening, Generalization, Carlson’s double inequality, Arc cosine function, Monotonicity

2000 AMS Classification: Primary 33 B 10. Secondary 26 D 05.

1. Introduction and main results

In [1, p. 700, (1.14)] and [3, p. 246, 3.4.30], it was listed that

\[ \frac{6(1-x)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}} < \arccos x < \frac{\sqrt{3}(1-x)^{1/2}}{(1+x)^{1/6}}, \quad 0 \leq x < 1. \]  

In [2], the right-hand side inequality in (1.1) was sharpened and generalized.

On the other hand, the left-hand side inequality in (1.1) was also generalized slightly in [2] as follows: For \( x \in (0, 1) \), the function

\[ F_{1/2,1/2,2\sqrt{2}}(x) = \frac{2\sqrt{2} + (1+x)^{1/2}}{(1-x)^{1/2}} \arccos x \]

is strictly decreasing. Consequently, the double inequality

\[ \frac{6(1-x)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}} < \arccos x < \frac{(1/2 + \sqrt{2})\pi(1-x)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}} \]

holds on \( (0, 1) \) and the constants 6 and \( \frac{1}{2} + \sqrt{2} \) are the best possible.

‡The authors were partially supported by the China Scholarship Council and the Science Foundation of Tianjin Polytechnic University
∗School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China. E-mail: (B.-N. Guo) bai.ni.guo@gmail.com bai.ni.guo@hotmail.com (F. Qi) qifeng618@gmail.com qifeng618@hotmail.com qifeng618@qq.com
†Corresponding Author.
The aim of this paper is to further generalize the left-hand side inequality in (1.1).

Our main results may be stated as follows.

1.1. Theorem. Let $a$ be a real number and

\[ F_a(x) = \frac{a + (1 + x)^{1/2} \arccos x}{(1 - x)^{1/2}}, \quad x \in (0, 1). \]  

(1) If $a \leq \frac{2(\pi - 2) - \pi}{4 - \pi}$, the function $F_a(x)$ is strictly increasing;

(2) If $a \geq 2\sqrt{2}$, then the function $F_a(x)$ is strictly decreasing;

(3) If $\frac{2(\pi - 2) - \pi}{4 - \pi} < a < 2\sqrt{2}$, the function $F_a(x)$ has a unique minimum.

1.2. Theorem. For $a \leq \frac{2(\pi - 2) - \pi}{4 - \pi}$,

\[ \frac{\pi(1 + a)/2(1 - x)^{1/2}}{a + (1 + x)^{1/2}} < \arccos x < \frac{(2 + \sqrt{2}a)(1 - x)^{1/2}}{a + (1 + x)^{1/2}}, \quad x \in (0, 1). \]

For $\frac{2(\pi - 2) - \pi}{4 - \pi} < a < 2\sqrt{2}$,

\[ \frac{8(1 - 2/a^2)(1 - x)^{1/2}}{a + (1 + x)^{1/2}} < \arccos x < \frac{\max\{2 + \sqrt{2}a, \pi(1 + a)/2\}(1 - x)^{1/2}}{a + (1 + x)^{1/2}}, \quad x \in (0, 1). \]

For $a \geq 2\sqrt{2}$, the inequality (1.5) reverses on $(0, 1)$.

Moreover, the constants $2 + \sqrt{2}a$ and $\pi(1 + a)/2$ in (1.5) and (1.6) are the best possible.

2. Remarks

Before proving our theorems, we give several remarks on them as follows.

2.1. Remark. The left-hand side inequality in (1.1) and the double inequality (1.3) are the special case $a = 2\sqrt{2}$ of the double inequality (1.6). This shows that Theorem 1.1 and Theorem 1.2 sharpen and generalize the left-hand side inequality in (1.1).

2.2. Remark. It is easy to verify that the function $a \mapsto \frac{1 + a}{a + (1 + x)^{1/2}}$ is increasing and the function $a \mapsto \frac{2 + \sqrt{2}a}{a + (1 + x)^{1/2}}$ is decreasing. Therefore, the sharp inequalities deduced from (1.5) are

\[ \frac{\pi^2(1 - x)^{1/2}}{2[2(\pi - 2) + (4 - \pi)(1 + x)^{1/2}]} < \arccos x \]

(2.1)

\[ \frac{2[2(2 - \sqrt{2}) + (\sqrt{2} - 1)\pi](1 - x)^{1/2}}{2(\pi - 2) + (4 - \pi)(1 + x)^{1/2}} \]

and

\[ \frac{\pi(1 + 2\sqrt{2})(1 - x)^{1/2}}{2[2\sqrt{2} + (1 + x)^{1/2}]} > \arccos x > \frac{6(1 - x)^{1/2}}{2\sqrt{2} + (1 + x)^{1/2}} \]

(2.2)

on $(0, 1)$.

Furthermore, it is not difficult to see that the double inequalities (2.1) and (2.2) do not include each other.
2.3. Remark. Let

\[ h_x(a) = \frac{1 - 2/a^2}{a + (1 + x)^{1/2}} \]

for \( \frac{2(\pi - 2)}{4 - \pi} < a < 2\sqrt{2} \) and \( x \in (0, 1) \). Direct calculation yields

\[ h'_x(a) = \frac{4\sqrt{1 + x} + 6a - a^3}{a^3(a + \sqrt{1 + x})^2} \]

which satisfies

\[
(2 + a)(\sqrt{3} - 1 + a)(1 + \sqrt{3} - a) = 4 + 6a - a^3
\]

\[
< a^3(a + \sqrt{1 + x})^2 h'_x(a)
\]

\[
= 4\sqrt{1 + x} + 6a - a^3
\]

\[
< 4\sqrt{2} + 6a - a^3
\]

\[
= (a + \sqrt{2})^2 (2\sqrt{2} - a).
\]

Accordingly,

1. When \( \frac{2(\pi - 2)}{4 - \pi} < a \leq 1 + \sqrt{3} \), the function \( a \mapsto h_x(a) \) is increasing;
2. When \( 1 + \sqrt{3} < a < 2\sqrt{2} \), the function \( a \mapsto h_x(a) \) attains its maximum

\[
\frac{4\cos^2\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right) - 1}{4\left[2\sqrt{2} \cos\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right) + \sqrt{1 + x}\right] \cos^2\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right)}
\]

at the point

\[
2\sqrt{2} \cos\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right).
\]

As a result, the sharp inequalities deduced from (1.6) are

\[
\frac{8[1 - 2/(1 + \sqrt{3})^2](1 - x)^{1/2}}{1 + \sqrt{3} + (1 + x)^{1/2}} < \arccos x < \frac{\pi(2 - \sqrt{2})(1 - x)^{1/2}}{4 - \pi + (\pi - 2\sqrt{2})(1 + x)^{1/2}}
\]

and

\[
\frac{2\left[4\cos^2\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right) - 1\right](1 - x)^{1/2}}{2\sqrt{2} \cos\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right) + \sqrt{1 + x}\right] \cos^2\left(\frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right)} < \arccos x
\]

on \((0, 1)\).
In conclusion, we obtain the following best and sharp double inequality

\[
\frac{\pi (2 - \sqrt{2}) (1 - x)^{1/2}}{4 - \pi + (\pi - 2\sqrt{2})(1 + x)^{1/2}} > \arccos x
\]

\[
> \max \left\{ \frac{2[4\lambda^2(x) - 1](1 - x)^{1/2}}{[2\sqrt{2}\lambda(x) + (1 + x)^{1/2}]\lambda^2(x)} \cdot \frac{\pi^2(1 - x)^{1/2}}{2^2[2(\pi - 2) + (4 - \pi)(1 + x)^{1/2}]} \right\}
\]

for \( x \in (0, 1) \), where

\[
\lambda(x) = \cos \left( \frac{1}{3} \arctan \frac{\sqrt{1 - x}}{\sqrt{1 + x}} \right), \quad x \in (0, 1).
\]

**2.5. Remark.** Letting \( \arccos x = t \) in (2.5) leads to

\[
\max \left\{ \frac{2[4\cos^2(t/6) - 1] \sin(t/2)}{2\cos(t/6) + \cos(t/2) \cos^2(t/6)} \cdot \frac{\pi^2 \sin(t/2)}{2\sqrt{2} (\pi - 2) + (4 - \pi) \cos(t/2)} \right\} < t < \frac{2\pi(\sqrt{2} - 1) \sin(t/2)}{4 - \pi + \sqrt{2}(\pi - 2\sqrt{2}) \cos(t/2)}, \quad 0 < t < \frac{\pi}{2}.
\]

This may be rearranged as

\[
\max \left\{ \frac{2\cos(t/6) + \cos(t/2) \cos^2(t/6)}{4 \cos^2(t/6) - 1}, \frac{4\sqrt{2} (\pi - 2) + (4 - \pi) \cos(t/2)}{\pi^2} \right\} > \frac{\sin(t/2)}{t/2}, \quad 0 < t < \frac{\pi}{2}.
\]

Therefore, we have

\[
\max \left\{ \frac{2\cos(t/3) + \cos t \cos^2(t/3)}{4 \cos^2(t/3) - 1}, \frac{4\sqrt{2} (\pi - 2) + (4 - \pi) \cos t}{\pi^2} \right\} > \frac{\sin t}{t}, \quad 0 < t < \frac{\pi}{4}.
\]

It is noted that the double inequality (2.9) improves related inequalities surveyed in [4, Section 3] and [8, Section 1.7].

**2.6. Remark.** The approach used in this paper to prove Theorem 1.1 and Theorem 1.2 has been utilized in [2, 5, 6, 7, 9, 10] to establish similar monotonicity and inequalities related to the arc sine, arc cosine and arc tangent functions. For more information on this topic, please see the expository and survey article [8].

### 3. Proofs of Theorem 1.1 and Theorem 1.2

Now we are in a position to verify our theorems.
Proof of Theorem 1.1. Straightforward differentiation yields
\[
F'_a(x) = \frac{\sqrt{1 - x^2}(a\sqrt{x^2 + 1} + 2)}{2(x - 1)^2(x + 1)} \left[ \frac{2(x - 1)(a\sqrt{x^2 + 1} + x + 1)}{\sqrt{1 - x^2}(a\sqrt{x^2 + 1} + 2)} + \arccos x \right]
\]
\[
\triangleq \frac{\sqrt{1 - x^2}(a\sqrt{x^2 + 1} + 2)}{2(x - 1)^2(x + 1)} G_a(x),
\]
and
\[
G'_a(x) = \frac{(a^2\sqrt{x^2 + 1} - ax - a - 4\sqrt{x^2 + 1})\sqrt{1 - x}}{(1 + x)(a\sqrt{x^2 + 1} + 2)^2}
\]
\[
\triangleq \frac{H_a(x)\sqrt{1 - x}}{(1 + x)(a\sqrt{x^2 + 1} + 2)^2}
\]
It is clear that only if \( a \in (2, \sqrt{2}) \) the denominators of \( G'_a(x) \) and \( G_a(x) \) do not equal zero on \((0, 1)\) and that the function \( H_a(x) \) has two zeros
\[
\alpha_1(x) = x + 1 - \frac{\sqrt{x^2 + 18x + 17}}{2\sqrt{x + 1}} \quad \text{and} \quad \alpha_2(x) = x + 1 + \frac{\sqrt{x^2 + 18x + 17}}{2\sqrt{x + 1}}
\]
whose derivatives are
\[
\alpha'_1(x) = \frac{\sqrt{x^2 + 18x + 17} - x - 1}{4\sqrt{(1 + x)(x^2 + 18x + 17)}} > 0
\]
and
\[
\alpha'_2(x) = \frac{1 + x + \sqrt{x^2 + 18x + 17}}{4\sqrt{(1 + x)(x^2 + 18x + 17)}} > 0
\]
with
\[
\lim_{x \to 0^+} \alpha_1(x) = \frac{1 - \sqrt{17}}{2}, \quad \lim_{x \to 1^-} \alpha_1(x) = -\sqrt{2},
\]
\[
\lim_{x \to 0^+} \alpha_2(x) = \frac{1 + \sqrt{17}}{2}, \quad \lim_{x \to 1^-} \alpha_2(x) = 2\sqrt{2}.
\]
Since the functions \( \alpha_1(x) \) and \( \alpha_2(x) \) are strictly increasing on \((0, 1)\), the following conclusions can be derived:

(1) When \( a \leq -2 < \frac{1 + \sqrt{17}}{2} < \sqrt{2} \) or \( a \geq 2\sqrt{2} \), the function \( H_a(x) \) and the derivative \( G'_a(x) \) are always positive on \((0, 1)\), and so the function \( G_a(x) \) is strictly increasing on \((0, 1)\). From
\[
(3.1) \quad \lim_{x \to 0^+} G_a(x) = \frac{(\pi - 4)a + 2(\pi - 2)}{2(a + 2)} \quad \text{and} \quad \lim_{x \to 1^-} G_a(x) = 0,
\]
it follows that the functions \( G_a(x) \) and \( F'_a(x) \) are negative, and so the function \( F_a(x) \) is strictly decreasing on \((0, 1)\).

(2) When \(-\sqrt{2} \leq a \leq \frac{1 + \sqrt{17}}{2} \), the function \( H_a(x) \) and the derivative \( G'_a(x) \) are negative on \((0, 1)\), and so the function \( G_a(x) \) is strictly decreasing on \((0, 1)\). From \( (3.1) \), it is obtained that the function \( G_a(x) \) and the derivative \( F'_a(x) \) are positive. So the function \( F_a(x) \) is strictly increasing on \((0, 1)\).

(3) When \( \frac{1 + \sqrt{17}}{2} < a < 2\sqrt{2} \), the functions \( H_a(x) \) and \( G'_a(x) \) have a unique zero which is the unique maximum point of \( G_a(x) \). From \( (3.1) \), it is deduced that
(a) If \( \frac{1 + \sqrt{17}}{2} < a < \frac{2(\pi - 2)}{4\pi} \), the functions \( G_a(x) \) and \( F'_a(x) \) are positive, and so the function \( F_a(x) \) is strictly increasing on \((0, 1)\).

(b) If \( \frac{2(\pi - 2)}{4\pi} < a < 2\sqrt{2} \), the functions \( G_a(x) \) and \( F'_a(x) \) have a unique zero which is the unique minimum point of the function \( F_a(x) \) on \((0, 1)\).
On the other hand, the derivative $F_a'(x)$ can be rearranged as

$$F_a'(x) = \frac{\sqrt{1-x^2}}{2(x-1)^2(x+1)} \left[ 2(x-1)(a\sqrt{x+1} + x + 1) \frac{1}{\sqrt{1-x^2}} + (a\sqrt{x+1} + 2) \arccos x \right]$$

with

$$Q_a'(x) = \frac{\arccos x}{2\sqrt{x+1}} \left( a - \frac{4\sqrt{1-x}}{\arccos x} \right)$$

and

$$P'(x) = \frac{2(x+1)}{\sqrt{x+1}\sqrt{1-x^2}(\arccos x)^2} \left[ \frac{2\sqrt{1-x^2}}{x+1} - \arccos x \right]$$

and

$$R'(x) = \frac{x-1}{(x+1)\sqrt{1-x^2}} < 0.$$ 

From $\lim_{x \to 0^+} R(x) = 0$ and the decreasingly monotonic property of $R(x)$, we obtain that $R(x) > 0$, and so the function $P(x)$ is strictly increasing. Since

$$\lim_{x \to 0^+} P(x) = \frac{8}{\pi} \quad \text{and} \quad \lim_{x \to 1^-} P(x) = 2\sqrt{2},$$

the function $Q_a(x)$ is strictly decreasing (or increasing, respectively) with respect to $x \in (0,1)$ for $a \leq \frac{8}{\pi}$ (or $a \geq 2\sqrt{2}$, respectively). By virtue of $\lim_{x \to 1^-} Q_a(x) = 0$, it follows that

1. If $a \leq \frac{8}{\pi}$, the function $Q_a(x)$ is positive on $(0,1)$;
2. If $a \geq 2\sqrt{2}$, the function $Q_a(x)$ is negative on $(0,1)$.

These imply that the function $F_a(x)$ is strictly increasing for $a \leq \frac{8}{\pi} < \frac{2(\pi - 2)}{4-\pi}$ and strictly decreasing for $a \geq 2\sqrt{2}$. The proof of Theorem 1.1 is complete. \(\square\)

**Proof of Theorem 1.2.** Easy calculation gives

$$\lim_{x \to 0^+} F_a(x) = \frac{\pi}{2}(1+a) \quad \text{and} \quad \lim_{x \to 1^-} F_a(x) = 2 + \sqrt{2}a.$$ 

By the monotonicity of $F_a(x)$ procured in Theorem 1.1, it follows that

1. If $a \leq \frac{2(\pi - 2)}{4-\pi}$, then

$$\frac{\pi}{2}(1+a) < F_a(x) < 2 + \sqrt{2}a$$

on $(0,1)$, which can be rearranged as the inequality (1.5);
2. If $a \geq 2\sqrt{2}$, the inequality (1.5) is reversed;
3. If $\frac{2(\pi - 2)}{4-\pi} < a < 2\sqrt{2}$, the function $F_a(x)$ has a unique minimum, so

$$F_a(x) < \max \left\{ \frac{\pi}{2}(1+a), 2 + \sqrt{2}a \right\}$$

on $(0,1)$, which is equivalent to the right-hand side inequality (1.6).

Furthermore, the minimum point $x_0 \in (0,1)$ of the function $F_a(x)$ satisfies

$$\arccos x_0 = \frac{2(1-x_0)(a\sqrt{x_0+1} + x_0 + 1)}{\sqrt{1-x_0^2}(a\sqrt{x_0+1} + 2)},$$
and so
\[ F_a(x_0) = \frac{2(a + \sqrt{x_0 + 1})}{\sqrt{1 + x_0 + (a\sqrt{x_0 + 1} + 2)}} \triangleq \frac{2(a + u)^2}{au + 2} \geq 8 \left(1 - \frac{2}{a^2}\right), \]
where \( u = \sqrt{1 + x_0} \in (1, \sqrt{2}) \). The left-hand side inequality in (1.6) follows.

The proof of Theorem 1.2 is complete. \(\square\)

4. An open problem

Finally, we propose the following open problem.

4.1. Open Problem. For real numbers \( \alpha, \beta \) and \( \gamma \), let
\[ F_{\alpha,\beta,\gamma}(x) = \frac{\gamma + (1 + x)^\beta}{(1 - x)^\alpha} \arccos x, \quad x \in (0, 1). \]
Find the ranges of the constants \( \alpha, \beta \) and \( \gamma \) such that the function \( F_{\alpha,\beta,\gamma}(x) \) is monotonic on \((0, 1)\).

References